1. Integrating factors

2. Reduction of order
The reduction of order technique, which applies to second-order linear differential equations, allows us to go beyond equations with constant coefficients, provided that we already know one solution.

If our differential equation is

\[ y'' + a_1(x)y' + a_2(x)y = F(x), \]

and we know the solution, \( y_1(x) \), to the associated homogeneous equation, this method will furnish us with another, independent solution.

To accomplish the process, we will make use of *integrating factors*. 
Integrating factors are a technique for solving first-order linear differential equations, that is, equations of the form

\[ a(x) \frac{dy}{dx} + b(x)y = r(x). \]

Assuming \( a(x) \neq 0 \), we can divide by \( a(x) \) to put the equation in standard form:

\[ \frac{dy}{dx} + p(x)y = q(x). \]

The main idea is that the left-hand side looks almost like the result of the product rule for derivatives. If \( I(x) \) is another function then

\[ \frac{d}{dx}(Iy) = I \frac{dy}{dx} + \frac{dI}{dx}y. \]

The standard form equation is missing an \( I \) in front of \( \frac{dy}{dx} \), so let’s multiply it by \( I \).
When we multiply our equation by $I$, we get

$$I \frac{dy}{dx} + Ip(x)y = Iq(x),$$

so in order for the left-hand side to be $\frac{d}{dx}(Iy)$, we need to have

$$\frac{dI}{dx} = p(x)I.$$

Rearranging this into

$$\frac{dI}{I} = p(x) \, dx,$$

we can solve:

$$I(x) = c_1 e^{\int p(x) \, dx}.$$

Since we only need one function $I$, let’s set $c_1 = 1$. 
Using this $I$, we rewrite our equation as

$$\frac{d}{dx}(Iy) = q(x)I,$$

then integrate and divide by $I$ to get

$$y(x) = \frac{1}{I} \left( \int q(x)I \, dx + c \right).$$

Our $I$ is called an **integrating factor** because it is something we can multiply by (a factor) that allows us to integrate.
Example

Find a solution to

\[ y' + xy = xe^{x^2/2}. \]

1. Find the integrating factor

\[ I(x) = e^{\int x \, dx} = e^{x^2/2}. \]

2. Multiply it into the original equation:

\[ \frac{d}{dx} \left( e^{x^2/2} y \right) = e^{x^2/2} y' + xe^{x^2/2} y = xe^{x^2}. \]

3. Integrate both sides:

\[ e^{x^2/2} y = \frac{1}{2} e^{x^2} + c. \]

4. Divide by \( I \) to find the solution

\[ y(x) = e^{-x^2/2} \left( \frac{1}{2} e^{x^2} + c \right). \]
Example

Solve, for $x > 0$, the equation

$$xy' + 2y = \cos x.$$ 

1. Write the equation in standard form:

$$y' + \frac{2}{x} y = \frac{\cos x}{x}.$$ 

2. An integrating factor is

$$I(x) = e^{2 \ln x} = x^2.$$ 

3. Multiply by $I$ to get

$$\frac{d}{dx}(x^2 y) = x \cos x.$$ 

4. Integrate and divide by $x^2$ to get

$$y(x) = \frac{x \sin x + \cos x + c}{x^2}.$$
We now turn to second-order equations

\[ y'' + a_1(x)y' + a_2(x)y = F(x). \]

We know that the general solution to such an equation will look like

\[ y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x). \]

Suppose that we know \( y_1(x) \). We will guess the solution \( y(x) = u(x)y_1(x) \). Plugging it into our original equation yields

\[ u''y_1 + u'(2y'_1 + a_1(x)y_1) = F(x). \]

If we let \( w = u' \) then we have reduced our second-order equation to the first-order

\[ w' + \left( \frac{2y'_1}{y_1} + a_1 \right) w = \frac{F(x)}{y_1}. \]
We may solve
\[ w' + \left( \frac{2y_1'}{y_1} + a_1 \right) w = \frac{F(x)}{y_1} \]
using the integrating factor technique:
\[ I(x) = y_1^2(x)e^{\int^x a_1(s) \, ds} \]
and
\[ w(x) = \frac{1}{I(x)} \int^x \frac{I(s)F(s)}{y_1(s)} \, ds + \frac{c_1}{I(x)}. \]

Then integrate \( w \) to find \( u \):
\[ u(x) = \int^x \frac{1}{I(t)} \int^t \frac{I(s)F(s)}{y_1(s)} \, ds \, dt + c_1 \int^x \frac{1}{I(s)} \, ds + c_2. \]
Finally, we get

\[ y(x) = u(x)y_1(x) = c_1y_1(x) \int^x \frac{1}{I(s)} \, ds + c_2y_1(x) \]

\[ + y_1(x) \int^x \frac{1}{I(t)} \int^t \frac{I(s)F(s)}{y_1(s)} \, ds \, dt. \]

Using \( F = 0 \) gives us the two fundamental solutions

\[ y(x) = y_1(x) \text{ and } y(x) = y_1(x) \int^x \frac{1}{I(s)} \, ds. \]

And using \( c_1 = c_2 = 0 \), we get a particular solution

\[ y_p(x) = y_1(x) \int^x \frac{1}{I(t)} \int^t \frac{I(s)F(s)}{y_1(s)} \, ds \, dt. \]
Example

Determine the general solution to

\[ xy'' - 2y' + (2 - x)y = 0, \quad x > 0, \]
given that one solution is \( y_1(x) = e^x \).

1. Set up the equation for \( w \):

\[ w' + \frac{2(x - 1)}{x} w = 0. \]

2. Solve for \( w \):

\[ w(x) = c_1 x^2 e^{-2x}. \]

3. Integrate to find

\[ u(x) = \int w(x) \, dx + c_2 = -\frac{1}{4} c_1 e^{-2x}(1 + 2x + 2x^2) + c_2. \]

4. Multiply by \( y_1 \) for the general solution:

\[ y(x) = c_1 e^{-x}(1 + 2x + 2x^2) + c_2 e^x. \]
Example

Determine the general solution to

\[ x^2 y'' + 3xy' + y = 4 \ln x, \quad x > 0, \]

by first finding solutions to the associated homogeneous equation of the form \( y(x) = x^r \).

1. Find \( y_1(x) = x^{-1} \).
2. Put the equation in standard form by dividing by \( x^2 \):
   \[ y'' + 3x^{-1}y' + x^{-2}y = 4x^{-2} \ln x. \]
3. Set up the equation \( w' + x^{-1}w = 4x^{-1} \ln x. \)
4. Find \( w(x) = 4(\ln x - 1) + c_1 x^{-1} \).
5. Then \( u(x) = 4x(\ln x - 2) + c_1 \ln x + c_2. \)
6. Multiply by \( y_1(x) = x^{-1} \):
   \[ y(x) = 4(\ln x - 2) + c_1 x^{-1} \ln x + c_2 x^{-1}. \]