Please turn off and put away all electronic devices. No calculators, no notes, no books. Read the problems carefully. **Show all work.** Be as organized as possible. Please sign and date the pledge below to comply with the Code of Academic Integrity. Don’t forget to write your Name and PennID on the top of this page. Good luck!

<table>
<thead>
<tr>
<th>#</th>
<th>Points possible</th>
<th>Your score</th>
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My signature below certifies that I have complied with the University of Pennsylvania’s Code of Academic Integrity in completing this examination.

_____________________________  _______________________
Signature                              Date
Table 1: Boundary value problems for $\phi''(x) = -\lambda \phi(x)$

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>$\phi(0) = 0$</th>
<th>$\phi'(0) = 0$</th>
<th>$\phi(-L) = \phi(L)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi(L) = 0$</td>
<td>$\phi'(L) = 0$</td>
<td>$\phi'(-L) = \phi'(L)$</td>
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</tbody>
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<table>
<thead>
<tr>
<th>Eigenvalues</th>
<th>$\lambda_n = \left(\frac{n\pi}{L}\right)^2$</th>
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</tr>
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<tbody>
<tr>
<td>$n = 1, 2, 3, \ldots$</td>
<td>$n = 0, 1, 2, 3, \ldots$</td>
<td>$n = 0, 1, 2, 3, \ldots$</td>
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<table>
<thead>
<tr>
<th>Eigenfunctions</th>
<th>$\sin \frac{n\pi x}{L}$</th>
<th>$\cos \frac{n\pi x}{L}$</th>
<th>$\sin \frac{n\pi x}{L}$ and $\cos \frac{n\pi x}{L}$</th>
</tr>
</thead>
</table>

| Series               | $f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$ | $f(x) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}$ | $f(x) = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$ |

Table 2: Orthogonality relations for sines and cosines

\[
\int_{0}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} = \begin{cases} 0, & n \neq m \\ \frac{L}{2}, & n = m \neq 0 \end{cases}
\]

\[
\int_{0}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} = \begin{cases} 0, & n \neq m \\ \frac{L}{2}, & n = m \neq 0 \\ L, & n = m = 0 \end{cases}
\]

\[
\int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \end{cases}
\]

\[
\int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \\ 2L, & n = m = 0 \end{cases}
\]

\[
\int_{-L}^{L} \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} = 0
\]
**Problem 1 (20 pts):** Find a solution, \( u(x, t) \), to the heat equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \ t > 0,
\]

subject to the following boundary and initial conditions

\[
u(0, t) = u(\pi, t) = 0, \quad \text{and} \quad u(x, 0) = \sin(2x) - 3\sin(14x) + 2\sin(2017x).\]
Problem 2 (20 pts): Find a solution, $u(r, \theta)$, to the Laplace equation in the quarter of the disk of radius 2:

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 \leq r \leq 2, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

subject to the following conditions:

$$\frac{\partial u}{\partial \theta} (r, 0) = \frac{\partial u}{\partial \theta} (r, \frac{\pi}{2}) = 0,$$

$$|u(0, \theta)| < \infty, \quad u(2, \theta) = 4 - \cos(\theta).$$

(You may use that product solutions $u(r, \theta) = \phi(\theta) G(r)$ satisfy: $\frac{r}{\phi} \frac{d}{dr} \left( r \frac{dG}{dr} \right) = -\frac{1}{\phi} \frac{d^2 \phi}{d\theta^2}$)
Problem 3 (20 pts): Consider the heat equation with a source
\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2x, \quad 0 < x < 1, \; t > 0,
\]
with boundary conditions and initial condition given by
\[
u_x(0, t) = 2, \quad u_x(1, t) = \beta, \quad \text{and} \quad u(x, 0) = 4x^3.
\]
The total thermal energy is defined to be:
\[
E(t) = \int_0^1 u(x, t) \, dx.
\]
(a) Compute \( \frac{dE}{dt} \).
(b) Using part (a), find \( E(t) \).
(c) From part (b), determine the value of \( \beta \) for which an equilibrium exists. For this value of \( \beta \), determine the equilibrium temperature distribution.
Problem 4 (20 pts): Consider the eigenvalue problem

\[ \phi''(x) = -\lambda \phi(x), \quad 0 \leq x \leq L \]
\[ \phi(0) + 2\phi'(0) = 0 \]
\[ \phi(L) - \phi'(L) = 0. \]

Find the values of \( L \) for which \( \lambda = 0 \) is an eigenvalue.
Problem 5 (20 pts): Consider the function $f : [0, 1] \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} 
2x, & \text{for } 0 \leq x \leq \frac{1}{2} \\
1, & \text{for } \frac{1}{2} \leq x \leq 1.
\end{cases}$$

(a) Compute the Fourier Sine Series of $f(x)$. Simplify the coefficients.

(b) Draw the graph of the Fourier Sine Series of $f(x)$ on the range $-3 \leq x \leq 3$. Mark all points of discontinuity.

(c) For what values $0 \leq x \leq 1$ does the Fourier Sine Series converge to $f(x)$?
Scratch paper