This exam consists of 6 problems. Please write clearly, both in form and substance.

Good luck!

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My signature below certifies that I have complied with the University of Pennsylvania’s Code of Academic Integrity in completing this Math 241 Midterm Exam.

Name (printed):
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Problem 1. The odd periodic function of period $2\pi$ is defined on the interval $(\pi, 2\pi)$ by

$$f(x) = 1 + \sin^2\left(\frac{x}{3}\right), \quad x \in (\pi, 2\pi).$$

To what limit does the Fourier series of $f$ converge at the point $x = \pi$?

Proof. The Fourier series converges to

$$\frac{1}{2}(f(\pi_+) + f(\pi_-)),$$

and we know that $f(\pi_+) = 1 + \sin^2(\pi/3)$. To find $f(\pi_-)$, we note that by periodicity $f(\pi_-) = f(-\pi_-)$. Since the function is odd, we get that $f(x_-) = -f(-x_+)$, for every $x$, in particular $f(-\pi_-) = -f(\pi_+)$, so that the series converges to 0. $\square$
**Problem 2.** Consider the eigenvalue problem

\[ x^2 \frac{d^2 \phi}{dx^2} + x \frac{d \phi}{dx} + \lambda \phi = 0 \quad 1 \leq x \leq e \]

\[ \phi(1) = 0 \]

\[ \phi'(e) = 0 \]

(a) Find all eigenvalues and eigenfunctions.

(b) With respect to what weight function are the above eigenfunctions orthogonal?

**Proof.** This is a regular Sturm-Liouville problem, written in normal form as

\[ \frac{d}{dx} \left( x \frac{d \phi}{dx} \right) + \lambda \frac{1}{x} \phi = 0. \]

So the weight function in (b) is \( \sigma(x) = 1/x \). Moreover all eigenvalues are real.

Looking at the Rayleigh quotient we see that \( \lambda \geq 0 \), and that \( \phi = \text{constant} \) is the only possibility for an eigenfunction corresponding to \( \lambda = 0 \). Since it has to be zero at \( x = 1 \) it is zero everywhere. So \( \lambda > 0 \).

But the general solution of the equidimensional equation is \( \phi(x) = x^p \), so \( x \frac{d \phi}{dx} = px^p \), and

\[ x \frac{d}{dx} \left( x \frac{d \phi}{dx} \right) = p^2 x^p. \]

It follows that \( p^2 + \lambda = 0 \), so that

\[ p = \pm i \sqrt{\lambda}, \quad \text{and} \quad \phi(x) = x^{\pm i \sqrt{\lambda}} = e^{\pm i \sqrt{\lambda} \ln x}. \]

It follows that the general solution is

\[ \phi(x) = c_1 \cos(\sqrt{\lambda} \ln x) + c_2 \sin(\sqrt{\lambda} \ln x). \]

\( \phi(1) = 0 \) forces \( c_1 = 0 \), whereas \( \phi'(e) = 0 \), forces \( \cos(\sqrt{\lambda}) = 0 \). So \( \lambda = \left( \frac{2n - 1}{2} \pi \right)^2, \ n = 1, 2, \ldots \) with eigenfunctions \( \phi_n(x) = \sin \left( \frac{2n - 1}{2} \pi \ln x \right) \). \( \square \)
**Problem 3.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, and let $u(x,y,z,t)$ be a solution of

\[
\begin{cases}
\frac{\partial^2 u}{\partial t^2} = \nabla^2 u - u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Define the energy $E(t)$ of this solution by

\[ E(t) = \frac{1}{2} \iiint_{\Omega} (\frac{\partial u}{\partial t})^2 + |\nabla u|^2 + u^2 \, dV. \]

(a) Show that $E(t)$ is constant.

(b) Use (a) to prove that a solution to the above PDE with those boundary conditions and initial position $u(x,y,z,0)$ and initial velocity $\frac{\partial u}{\partial t}(x,y,z,0)$ is unique. (*Hint:* Suppose $u$ and $v$ are two such solutions, and study the energy of the difference.)

**Proof.** Let’s look at the derivative of $E(t)$.

\[
\frac{dE}{dt}(t) = \iiint_{\Omega} \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2} + \nabla u \cdot \nabla \left( \frac{\partial u}{\partial t} \right) + u \frac{\partial u}{\partial t} \, dV = \iiint_{\Omega} \nabla^2 u \frac{\partial u}{\partial t} + \nabla u \cdot \nabla \left( \frac{\partial u}{\partial t} \right) \, dV.
\]

This last integrand can be written as the divergence $\nabla \cdot \left( (\nabla u) \frac{\partial u}{\partial t} \right)$, so using the divergence theorem we get that

\[
\frac{dE}{dt} = \iint_{\partial \Omega} (\nabla u) \frac{\partial u}{\partial t} \cdot \vec{n} \, dA = 0.
\]

So the energy is constant. For point (b) if the energy at $t = 0$ is zero, then it is zero throughout. Since $u - v$ is obviously a solution (by linearity) it follows that all $(\frac{\partial(u - v)}{\partial t})^2 + |\nabla(u - v)|^2 + (u - v)^2 = 0$, so in particular $u - v = 0$. \qed
Problem 4. Let $u(\rho, \theta, \phi)$ be a solution of the Laplace equation $\nabla^2 u = 0$ inside a sphere of radius 2 centered at the origin, subject to the boundary condition $u(2, \theta, \phi) = 5P_2^0(\cos \phi)$, where $P_2^0(x)$ is the associated Legendre function of the first kind.

(a) Compute $u(1, \pi, 0)$.
(b) Compute $\lim_{\rho \to 0} u(\rho, \pi, 0)$.

Proof. Since the solutions obtained from separating the variables are $\rho^n \{\cos m\theta, \sin m\theta\} P_n^m(\cos \phi)$, it follows (from the uniqueness of the boundary value problem) that

$$u = c\rho^2 P_2^0(\cos \phi),$$

with $c = \frac{5}{4}$. It follows that $u(\rho, \theta, 0) = \frac{5}{4}\rho^2 P_2(0) = \frac{5}{4}\rho^2$. So $u(1, \pi, 0) = \frac{5}{4}$, and the limit is zero. □
Problem 5. Prove that if $f$ is a solution of the Bessel equation of order $\nu$:

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - \nu^2)f = 0,$$

then the function $g(z) = z^{-\alpha} f(z)$ satisfies the equation

$$z^{1-2\alpha} \frac{d}{dz}(z^{1+2\alpha} \frac{dg}{dz}) + (z^2 - (\nu + \alpha)(\nu - \alpha))g = 0$$

Proof. From $f(z) = z^\alpha g(z)$, we get first that

$$z \frac{df}{dz} = \alpha z^\alpha g + z^\alpha (\frac{dz}{dz} \frac{dg}{dz}),$$

and then

$$z \frac{d}{dz}(z \frac{df}{dz}) = \alpha^2 z^\alpha g + 2\alpha z^\alpha (\frac{dz}{dz} \frac{dg}{dz}) + z^\alpha (\frac{dz}{dz}(z \frac{dg}{dz})).$$

Since $f$ satisfies the equation

$$z \frac{d}{dz}(z \frac{df}{dz}) + (z^2 - \nu^2)f = 0,$$

it follows that $g$ satisfies

$$z \frac{d}{dz}(z \frac{dg}{dz}) + 2\alpha z \frac{dg}{dz} + (z^2 - \nu^2 + \alpha^2)g = 0.$$

Since $\nu^2 - \alpha^2 = (\nu - \alpha)(\nu + \alpha)$ and

$$z^{1-2\alpha} \frac{d}{dz}(z^{1+2\alpha} \frac{dg}{dz}) = z^2 \frac{d^2 g}{dz^2} + (1 + 2\alpha)z \frac{dg}{dz} = z \frac{d}{dz}(z \frac{dg}{dz}) + 2\alpha z \frac{dg}{dz},$$

the result follows.
Problem 6. Consider the functions $\phi_1, \phi_2$ and $\phi_3$ on the interval $[0, 1]$ defined by $\phi_i(x) = x^{i-1}$. Use the Gram-Schmidt method to find orthogonal functions $\psi_i$, $i = 1, 2, 3$, with respect to the weight function $\sigma(x) = x$.

Proof. $\psi_1 = \phi_1 = 1$. $\psi_2$ is obtained as

$$\psi_2 = \phi_2 - \frac{\langle \phi_2, \psi_1 \rangle}{\langle \psi_1, \psi_1 \rangle} \psi_1,$$

where $\langle f, g \rangle = \int_0^1 fg \, dx$. It follows that

$$\langle \phi_2, \psi_1 \rangle = \int_0^1 x \cdot 1 \, dx = \frac{1}{3},$$

$$\langle \psi_1, \psi_1 \rangle = \int_0^1 1 \, dx = \frac{1}{2},$$

so that $\psi_2 = x - \frac{2}{3}$. Similarly

$$\psi_3 = \phi_3 - \frac{\langle \phi_3, \psi_2 \rangle}{\langle \psi_2, \psi_2 \rangle} \psi_2 - \frac{\langle \phi_3, \psi_1 \rangle}{\langle \psi_1, \psi_1 \rangle} \psi_1,$$

where

$$\langle \phi_3, \psi_1 \rangle = \int_0^1 x^2 \cdot 1 \, dx = \frac{1}{4},$$

$$\langle \phi_3, \psi_2 \rangle = \int_0^1 x^2 (x - \frac{2}{3}) \, dx = \frac{1}{5} - \frac{1}{6} = \frac{1}{30},$$

$$\langle \psi_2, \psi_2 \rangle = \int_0^1 (x - \frac{2}{3})^2 \, dx = \frac{1}{36}.$$

It follows that

$$\psi_3 = x^2 - \frac{6}{5} (x - \frac{2}{3}) - \frac{1}{2} = x^2 - \frac{6}{5} x + \frac{3}{10}$$

$\square$