Problem 1. Consider Laplace’s equation $\nabla^2 u = 0$, in a three dimensional region $R$ (where $u$ is the temperature). Suppose that the heat flux is given on the boundary (not necessarily a constant).

(a) Explain physically why $\oint \nabla u \cdot \vec{n} dS = 0$.
(b) Show this mathematically.

Proof. (a) Physically, $u$ is the steady state temperature distribution and so the total heat energy has to be constant, so the total heat energy flowing out of $R$ (per unit time) has to be zero.

(b) The divergence theorem for the region $R$ gives:

$$0 = \iiint_R \nabla^2 u \, dV = \iiint_R \nabla \cdot \nabla u \, dV = \iiint_{\partial R} \nabla u \cdot \vec{n} dS.$$ 

Problem 2. Find the values of $a \in \mathbb{R}$ for which the function $\phi_a = r^a$ is an eigenfunction for the Laplacian in dimension $n$. (Recall that $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.)

What are the corresponding eigenvalues?

For what values of $a$ is $\phi_a$ harmonic?

Proof. A simple computation gives

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}} = \frac{x_i}{r},$$

so that $\frac{\partial r^a}{\partial x_i} = ar^{a-1} \frac{x_i}{r} = ar^{a-2} x_i$. Using the chain rule and the product rule we get

$$\frac{\partial^2 r^a}{\partial x_i^2} = ar^{a-2} + ax_i(a-2)r^{a-3}, \frac{\partial r}{\partial x_i} = ar^{a-2} + a(a-2)r^{a-4}x_i^2.$$ 

Suming from 1 to $n$ gives

$$\nabla^2 r^a = nar^{a-2} + a(a-2)r^{a-4}r^2 = a(n + a - 2)r^{a-2}.$$ 

It follows that the only way for $r^a$ to be an eigenfunction is when $a = 0$ or when $a = 2 - n$. So these are also the values for which $\phi_a$ is harmonic. Remark that the first case is the constant function 1. The other one depends on $n$. For $n = 2$ it is again the constant function 1, for $n = 1$ it is $r$, while for $n = 3$ it is $\frac{1}{r}$. 

\[\square\]
Problem 3. Consider the function \( v = \frac{1}{r} \) defined for \( r \neq 0 \) in \( \mathbb{R}^3 \). Suppose that \( u \) is a smooth harmonic function in an open neighborhood of the ball centered at the origin and radius \( \rho \).

Use the divergence theorem (and the related integration by parts) for \( v \nabla^2 u \) and \( u \nabla^2 v \) on the spherical shell \( \varepsilon < r < \rho \), to show that the average value of \( u \) on \( r = \rho \) is the same as the average value of \( u \) on \( r = \varepsilon \).

Can you tell what happens when \( \varepsilon \to 0 \)?

(Hint: If you haven’t shown in the previous problem that \( v \) is harmonic, you may use spherical coordinates to do so.

You might have to use the first problem.)

Proof. Let’s use integration by parts for \( v \nabla^2 u \) and \( u \nabla^2 v \):

\[
\iiint_{D_{\varepsilon,\rho}} v \nabla^2 u \, dV = \iiint_{D_{\varepsilon,\rho}} \nabla \cdot (\nabla u) v \, dV = \iint_{\partial D_{\varepsilon,\rho}} (\nabla u \cdot \vec{n}) v \, dA - \iiint_{D_{\varepsilon,\rho}} \nabla u \cdot \nabla v \, dV,
\]

\[
\iiint_{D_{\varepsilon,\rho}} u \nabla^2 v \, dV = \iiint_{D_{\varepsilon,\rho}} \nabla \cdot (\nabla v) u \, dV = \iint_{\partial D_{\varepsilon,\rho}} (\nabla v \cdot \vec{n}) u \, dA - \iiint_{D_{\varepsilon,\rho}} \nabla u \cdot \nabla v \, dV.
\]

Since both \( u \) and \( v \) are harmonic, subtracting the two equalities gives:

\[
\iint_{\partial D_{\varepsilon,\rho}} (\nabla u \cdot \vec{n}) v \, dA = \iint_{\partial D_{\varepsilon,\rho}} (\nabla v \cdot \vec{n}) u \, dA.
\]

Note that the boundary \( \partial D_{\varepsilon,\rho} \) consists of two spheres, and that \( v = r^{-1} \) is constant on them. Using now Problem 1, we see that the left hand side of the above equality is zero. We also see that \( \nabla v = -r^{-2} \vec{n} \), where \( \vec{n} \) is the outward unit normal on the sphere. It follows that

\[
0 = -\rho^{-2} \iint_{r=\rho} u \, dA + \varepsilon^{-2} \iint_{r=\varepsilon} u \, dA.
\]

Dividing both sides by \( 4\pi \) gives the desired result.

When \( \varepsilon \to 0 \), then \( u \) is approximately \( u(0) \) on \( r = \varepsilon \), so that the second term converges to \( u(0) \). It follows that

\[
u(0) = \frac{1}{A(r = \rho)} \iint_{r=\rho} u \, dA,
\]

which is the mean value theorem for harmonic functions in three dimensions. \( \Box \)

Problem 4. Find the harmonic function \( u(r, \theta) \) in the disc \( r^2 \leq 1 \) that satisfies the boundary condition

\[
u(1, \theta) = 1 + \sin(2\theta).
\]

Explain why the solution is always less than or equal to 2.
Proof. Since
\[ u(r, \theta) = \sum_{n=0}^{\infty} A_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^n \sin(n\theta) \]
is the general solution, it follows that
\[ u(r, \theta) = 1 + r^2 \sin(2\theta). \]
The maximum principle implies that the maximum occurs on the circle \( r = 1 \), and since \( \sin(2\theta) \leq 1 \), the result follows.

Problem 5. Consider the functions \( \phi_1, \phi_2 \) and \( \phi_3 \) on the interval \([0, 1]\) defined by \( \phi_i(x) = x^{i-1} \). Use the Gram-Schmidt method to find orthogonal functions \( \psi_i, i = 1, 2, 3 \), with respect to the weight function \( \sigma(x) = x^2 \).

Proof. \( \psi_1 = \phi_1 = 1 \). \( \psi_2 \) is obtained as
\[
\psi_2 = \phi_2 - \frac{\langle \phi_2, \psi_1 \rangle}{\langle \psi_1, \psi_1 \rangle} \psi_1,
\]
where \( \langle f, g \rangle = \int_0^1 fg x^2 dx \). It follows that
\[
\langle \phi_2, \psi_1 \rangle = \int_0^1 x \cdot 1 \cdot x^2 dx = \frac{1}{4},
\]
\[
\langle \psi_1, \psi_1 \rangle = \int_0^1 1 \cdot x^2 dx = \frac{1}{3},
\]
so that \( \psi_2 = x - \frac{3}{4} \). Similarly
\[
\psi_3 = \phi_3 - \frac{\langle \phi_3, \psi_2 \rangle}{\langle \psi_2, \psi_2 \rangle} \psi_2 - \frac{\langle \phi_3, \psi_1 \rangle}{\langle \psi_1, \psi_1 \rangle} \psi_1,
\]
where
\[
\langle \phi_3, \psi_1 \rangle = \int_0^1 x^2 \cdot 1 \cdot x^2 dx = \frac{1}{5},
\]
\[
\langle \phi_3, \psi_2 \rangle = \int_0^1 x^2 (x - \frac{3}{4}) \cdot x^2 dx = \frac{1}{6} - \frac{3}{20} = \frac{1}{60},
\]
\[
\langle \psi_2, \psi_2 \rangle = \int_0^1 (x - \frac{3}{4})^2 x^2 dx = \frac{1}{80}.
\]
It follows that
\[
\psi_2 = x^2 - \frac{4}{3} (x - \frac{3}{4}) - \frac{3}{5} = x^2 - \frac{4}{3} x + \frac{2}{5}.
\]
Problem 6. Prove that if \( f(z) \) is a solution of Bessel’s Differential Equation of order zero, then \( \frac{df}{dz} \) satisfies Bessel’s D.E. of order one.

Proof. \( f \) is a solution of Bessel’s D.E of order 0 means that

\[
z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + z^2 f = 0.
\]

Taking derivatives implies that

\[
z^2 \frac{d^3 f}{dz^3} + z \frac{d^2 f}{dz^2} + z^2 \frac{df}{dz} + 2z \frac{d^2 f}{dz^2} + \frac{df}{dz} + 2zf = 0.
\]

The second line is easily seen to be \(-\frac{df}{dz}\), so that

\[
z^2 \frac{d^2 f'}{dz^2} + z \frac{df'}{dz} + (z^2 - 1)f' = 0.
\]

\(\square\)