Applications of Differential Calculus

Maxima, Minima, and Stationary Points

February 22, 2018
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Theorem (Bolzano-Weierstrass)

Suppose $S \subset \mathbb{R}^N$ is a closed and bounded set. Then every sequence $\{x_n\}_{n \in \mathbb{N}}$, with $x_n \in S$ for every $n \in \mathbb{N}$, has a convergent subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$, such that

$$\lim_{k \to \infty} x_{n_k} \in S.$$
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Theorem

Suppose $S \subset \mathbb{R}^N$ is a closed and bounded nonempty set. Then every continuous function $f : S \to \mathbb{R}$ is bounded and attains both its maximum and minimum value. That is there exist $x_m, x_M \in S$, such that

$$f(x_m) \leq f(x) \leq f(x_M), \text{ for every } x \in S.$$
Proof:
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Write $y_n = f(x_n)$, with $x_n \in S$, and use the Bolzano-Weierstrass theorem to get a convergent subsequence

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This implies both that \( \sup \{ f(S) \} \in \mathbb{R} \) and that

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Definition

A function \( f : S \to \mathbb{R}, \ S \subset \mathbb{R}^N \) is said to have a relative (local) maximum at \( a \in S \), if there exists \( r > 0 \) such that

\[ f(x) \leq f(a), \quad \text{for every } x \in S \cap B(a, r). \]
Definition (cont.):

Relative (local) minimum if \( f(x) \geq f(a) \), for every \( x \in S \cap B(a, r) \).

Either relative minimum or relative maximum is called a (relative, local) extremum.

All definitions have a variant with strict.
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The Fundamental Theorem of Algebra

Theorem.

Suppose

\[ p(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0 \in \mathbb{C}[z] \]

is a polynomial with complex coefficients of degree \( n \geq 1 \). (It is not constant)
Then there exists \( a \in \mathbb{C} \) such that

\[ p(a) = 0. \]
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**Proof:**

Note first that

\[ |p(z)| = |c_n||z|^n \left|1 + \frac{c_{n-1}}{c_n} \frac{1}{z} + \cdots + \frac{c_0}{c_n} \frac{1}{z^n}\right| \geq \]


Proof (cont.):

\[ \geq |c_n||z|^n \left( 1 - \frac{|c_{n-1}|}{|c_n|} \frac{1}{|z|} - \cdots - \frac{|c_0|}{|c_n|} \frac{1}{|z^n|} \right) \rightarrow \infty. \]
Proof (cont.):

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In particular there exists \( M > 0 \), such that \( |p(z)| > |p(0)| \), for every \( z \in \mathbb{C}, |z| = M \).
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In particular there exists \( M > 0 \), such that \( |p(z)| > |p(0)| \), for every \( z \in \mathbb{C}, \ |z| = M \).

The set \( S = B[0, M] = \{ z \in \mathbb{C} \ | \ |z| \leq M \} \) is closed and bounded, and the function \( |p(z)| \) is continuous.

So it attains an absolute minimum at some point \( a \) in \( B(0, M) \).
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**CLAIM:** If \( |p(z)| \) has a local minimum at some point \( a \in \mathbb{C} \), then \( p(z) = 0. \) \(( n \geq 1)\)
Proof (cont.):

By making the change of variables $z' = z - a$, we may (and will) assume that $a = 0$.  

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By contradiction:
Suppose $p(0) \neq 0$, that is $c_0 \neq 0$. Dividing by it, we may (and will) assume that $c_0 = 1$. 
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Suppose $p(0) \neq 0$, that is $c_0 \neq 0$. Dividing by it, we may (and will) assume that $c_0 = 1$.
Write

$$p(z) = 1 + c_k z^k + c_{k+1} z^{k+1} + \cdots + c_n z^n,$$

with $c_k \neq 0$. ($n \geq 1$)
The Fundamental Theorem of Algebra

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By making the change of variables $z' = z - a$, we may (and will) assume that $a = 0$.

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Then

$$\left| p(z) \right| \leq \left| 1 + c_k z^k \right| + \left| c_{k+1} \right| \left| z \right|^{k+1} + \cdots + \left| c_n \right| \left| z \right|^n.$$
Proof (cont.):

Note that we can find $z' \in \mathbb{C}$ such that $|z'| = |z|$, and

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It follows that

$$|p(z')| \leq 1 - |c_k||z|^k + |c_{k+1}||z|^{k+1} + \cdots |c_n||z|^n =$$

$$1 - |c_k||z|^k \left(1 - \frac{|c_{k+1}|}{|c_k|}|z| - \cdots - \frac{|c_n|}{|c_k|}|z|^{n-k} \right).$$

$$\phi(z)$$

Note that as $z \to 0$, $\phi(z) \to 1$. 
Proof (cont.):

So as soon as $\phi(z) > 0$, it follows that

$$|p(z')| < 1 = |p(0)|,$$

which was supposed to be a local minimum. \textbf{Contradiction!} Q.E.D.
Proof (cont.):

So as soon as \( \phi(z) > 0 \), it follows that

\[ |p(z')| < 1 = |p(0)|, \]

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Proposition

Suppose the function \( f : S \to \mathbb{R}, \ S \subset \mathbb{R}^N \) has an extremum at the point \( a \in \text{int} \ S \).
If \( f \) is differentiable at \( a \), then

\[ Df(a) = 0. \]
Applications of Differential Calculus

Review

The Fundamental Theorem of Algebra

Stationary Points

$$(Df(a) = \nabla f(a) = df(a) \in \mathcal{L}(\mathbb{R}^N, \mathbb{R}).)$$
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Proof:

Follows from the one dimensional case:
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The function

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has an extremum at \( t = 0 \), so its derivative is zero.
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The function

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has an extremum at \( t = 0 \), so its derivative is zero. But its derivative is the directional derivative of \( f \) in the direction of \( v \in \mathbb{R}^N \), so

\[ Df(a)v = 0, \quad \text{for every } v \in \mathbb{R}^N, \]

so

\[ Df(a) = 0. \]

Q.E.D.
Second Derivative

Definition
Suppose that $S \subset \mathbb{R}^N$ and $f : S \rightarrow \mathbb{R}$ is differentiable at the point $a \in \text{int } S$.
If
$$Df(a) = 0,$$
then $a$ is called a stationary (or a critical) point of $f$. 
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1. $f''(a) > 0$, $\implies$ $f$ has a strict local \textit{minimum} at $a$. 
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So for differentiable maps, all extrema occur at critical points. Recall that in the 1-dimensional case ($N = 1$), we looked at the second derivative, and

1. $f''(a) > 0$, $\implies$ $f$ has a strict local **minimum** at $a$.
2. $f''(a) < 0$, $\implies$ $f$ has a strict local **maximum** at $a$.
3. $f''(a) = 0$, $\implies$ **undetermined** (almost anything can happen).
The Hessian

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Definition

If the function $f : S \to \mathbb{R}$, $(S \subset \mathbb{R}^N)$ is twice differentiable at the point $a \in \text{int } S$, then the Hessian (matrix) of $f$ at $a$ is the $N \times N$ matrix

$$H(a) = \left[ \frac{\partial^2 f}{\partial x_i \partial x_j}(a) \right]$$
The Hessian

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$$H(a) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_i \partial x_j}(a) \end{bmatrix}$$

**Recall** that the second derivative of $f$ at $a$ is the map

$$D^2 f(a) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R},$$

given by

$$D^2 f(a)(v, w) = \sum_{i,j=1}^{N} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)v_i w_j.$$
The Hessian

**Remark**

\[ D^2 f(a)(v, w) = v^T H(a) w. \]

*(column vectors!)*
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We want to prove that if the Hessian is non singular (invertible) at a stationary point, then the behaviour of \( f \) at \( a \) is completely determined:
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We want to prove that if the Hessian is non singular (invertible) at a stationary point, then the behaviour of \( f \) at \( a \) is completely determined:

It is the same as the behaviour of the quadratic form:

\[ Q(x) = x^T H(a) x = \sum h_{i,j} x_i x_j, \]

at zero.
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at zero.

So we need to understand this behaviour, and then see that \( f \) behaves in the same way around \( a \).
Remark:

Since every symmetric matrix is diagonalizable by means of an orthogonal matrix, it follows that after an orthogonal change of variables, the quadratic form $Q$ can be written as:

$$Q(y_1, \ldots, y_N) = \sum_{i=1}^{N} \lambda_i y_i^2,$$

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ are the eigenvalues of $H$. 

If $0 \leq \lambda_1 \leq \cdots \leq \lambda_N$, then $Q$ has a minimum at zero (strict minimum if $0 < \lambda_1$).

If $\lambda_1 \leq \cdots \leq \lambda_N \leq 0$, then $Q$ has a maximum at zero (strict if $\lambda_N < 0$).
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Quadratic Forms

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So

1. If $0 \leq \lambda_1 \leq \cdots \leq \lambda_N$, then $Q$ has a minimum at zero (strict minimum if $0 < \lambda_1$).
2. If $\lambda_1 \leq \cdots \leq \lambda_N \leq 0$, then $Q$ has a maximum at zero (strict if $\lambda_N < 0$).
If $\lambda_1 \leq \cdots \leq \lambda_i < 0 < \lambda_j \leq \cdots \leq \lambda_N$, all other ones being zero, then $Q$ has a saddle point of type $(i, N - j + 1)$. 