MATH 361 — HOMEWORK 10.

due on Friday, April 17.

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Topics:
• Chapter 6: Differentiable Mappings
  – 6.1 Definition of the Derivative
  – 6.2 Matrix Representation
  – 6.3 Continuity of Differentiable Mappings; Differentiable Paths
  – 6.4 Conditions for Differentiability
  – 6.5 The Chain Rule
  – 6.6 Product Rules and Gradients
  – 6.7 The Mean Value Theorem
• Differentiability of Multilinear Maps and Inverses. Operations on Functions (The Lectures)
• Higher Derivatives (The Lectures)

Tenth Homework Assignment.

Reading:
• Read the slides (or/and watch the videos).

Exercises: (In what follows $E$ and $F$ are Banach Spaces).

Problem 1. Compute the second derivative of a continuous map $T \in L(E, F)$.

Problem 2. Prove that every continuous $k$–linear map $\phi \in L^{(k)}(E_1 \times \cdots \times E_k; F)$ is twice differentiable and compute its second derivative. What about higher derivatives?

Problem 3. Prove that the inverse map $\text{Inv} : \text{GL}(E) \to L(E)$ is twice differentiable and compute its second derivative. What about higher derivatives?

Problem 4. Prove that the determinant map $\det : M_N(\mathbb{R}) \to \mathbb{R}$ is differentiable, and prove that $D\det(I_N) = Tr$. ($I_N$ is the identity matrix and $Tr$ is the trace.).

Can you see what the derivative of the determinant is at a general point $A \in M_N(\mathbb{R})$?
**Problem 5.** Suppose that \( U \subset E \) is an open set and that \( f : U \to F \) is \( n \)-times differentiable. Suppose \( x \in U \) and \( h \in E \) and consider the function
\[
\varphi(t) = f(x + th)
\]
defined for all values of \( t \in \mathbb{R} \) for which \( x + th \in U \). Prove that the domain of definition of \( \varphi \) is an open set, that \( \varphi \) is \( n \)-times differentiable and that
\[
\frac{d^n \varphi}{dt^n}(t) = D^n f(x + th)(h, h, \ldots, h).
\]

**Problems:** (These are from the book. I wrote them here for the benefit of those who do not have the textbook handy!)

**Problem 6.** (Page 338-5) Find the tangent vector to the curve \( c(t) = (3t^2, e^t, t + t^2) \) at the point corresponding to \( t = 1 \).

**Problem 7.** (Page 344-4) Find the equation of the tangent plane to \( z = x^3 + y^4 \) at \( x = 1, y = 3 \).

**Problem 8.** (Page 362-1) Verify the equality of the mixed partials for \( f(x, y) = (e^{x^2+y^2})xy^2 \).

**Problem 9.** (Page 362-5) Compute the first and second derivatives for \( f(x, y) = e^x \cos y \) at the point \((0,0)\).

**Problem 10.** (Page 388-31) Let \((K,d)\) be a compact metric space and consider the Banach space \((C(K,\mathbb{R}), \|\|_{\infty})\). Define for \( x_0 \in K, \delta_{x_0} : C(K,\mathbb{R}) \to \mathbb{R}; f \mapsto f(x_0) \). Prove that \( \delta_{x_0} \) is differentiable.

**Problem 11.** (Page 388-32) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by
\[
f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}
\]
if \((x, y) \neq (0,0)\) and \( f(0,0) = 0 \). Show that \( \partial^2 f / \partial x \partial y \) and \( \partial^2 f / \partial y \partial x \) exist at \((0,0)\) but are not equal.