MATH 361 — HOMEWORK 4.

due on Friday, October 2.

Textbook: *"Elementary Classical Analysis"* , second edition
by J. E. Marsden and M. J. Hoffman

Topics:

- **Review of Math 360**
- **5. Uniform Convergence**
  - 5.1 Pointwise and Uniform Convergence
  - 5.2 The Weierstrass M Test
  - 5.5 The Space of Continuous Functions
  - 5.6 The Arzela-Ascoli Theorem
  - 5.7 The Contraction Mapping Principle and Its Applications

Fourth Homework Assignment.

Reading:

- Read section 5.7 of Chapter 5., paying close attention to the examples. Read your notes.

Exercises:

Problem 1. Write down the formula for the Euler Approximation of the solution to the initial value problem

$$\begin{aligned} \frac{dx}{dt} &= F(t,x) \\ x(t_0) &= x_0 \end{aligned}$$

for values $t < t_0$.

Problem 2. Prove that if (N_i, ρ_i), $i = 1, 2$, are metric spaces and $F : N_1 \to N_2$ is a uniformly continuous function that maps bounded sets into bounded sets, then the map:

$$\mathcal{F}_b(S, N_1) \ni f \mapsto F \circ f \in \mathcal{F}_b(S, N_2)$$

is uniformly continuous. (Here S is any nonempty set, and $\mathcal{F}_b(S, N_i)$ is a metric space with the metric $\rho_{i,\infty}$.)

Problem 3. Prove that if (N_i, ρ_i), $i = 1, 2$, are metric spaces and $F : N_1 \to N_2$ is a continuous function, then the map:

$$C(K, N_1) \ni f \mapsto F \circ f \in C(K, N_2)$$

is continuous, for every compact metric space (K, d).

Problem 4. (Dini's Theorem) Let (K, d) be a compact metric space and $\{f_k\}_k$ a sequence of continuous functions

$$f_k : K \to \mathbb{R},$$

such that

(a) $f_k(x) \geq 0$, for every $x \in K$,

(b) $f_k(x) \leq f_\ell(x)$ for every $x \in K$ and every $k \geq \ell$.

Prove that if $\{f_k\}_k$ converges pointwise to zero then it converges uniformly to zero.

Problem 5. For what intervals $[0, r]$, $r \leq 1$, is $f : [0, r] \to [0, r]$, $f(x) = x^2$, a contraction, a strict contraction and a uniform strict contraction respectively?

How about $f(x) = x^a$, $a \geq 0$?

(Recall that $T : M \to M$ is a:

-*contraction* if $d(T(x), T(y)) \leq d(x, y)$, for every $x, y \in M$,

-*strict contraction* if $d(T(x), T(y)) < d(x, y)$, for every $x, y \in M, x \neq y$,

-*uniform strict contraction* if there exists $c \in [0, 1)$ such that $d(T(x), T(y)) \leq cd(x, y)$, for every $x, y \in M$.

The Contraction Mapping Principle is really about uniform strict contractions.)

Problem 6. Suppose f is a solution to the initial value problem

$$\frac{dx}{dt} = x^2 + t^2$$
$$x(0) = 1$$

What is the third derivative of f at $t = 0$?

***Problems*:**

- Page 283: problems $5, 7$
- Page 316: problems $25, 27$.