MATH 361 — HOMEWORK 8.

due on Friday, November 6.

Textbook: *"Elementary Classical Analysis"* , second edition
by J. E. Marsden and M. J. Hoffman

Topics:

- **5. Uniform Convergence**
  - 5.10 Power Series
- **Multilinear Maps, Functional Calculus (with power series)**

Eighth Homework Assignment.

Reading:

- Read your notes. Read the slides (or/and watch the videos).
- You can find some of the Multiliear Maps facts in the "Worked Examples for Chapter 4".

Exercises:

Problem 1. Suppose $(E, \| \|)$ is a Banach space, $T \in L(E)$ and p_1, p_2 power series with radius of convergence (strictly) greater than $\|T\|$.

Prove that

$$p_1(T)p_2(T) = (p_1p_2)(T).$$

Problem 2. Use **Problem 1.** to show that

$$\exp(z_1T)\exp(z_2T) = \exp(z_1 + z_2)T,$$

for every $T \in L(E)$ and every $z_1, z_2 \in k$ $(= \mathbb{R}, \mathbb{C})$.

Problem 3. Let $(E, \| \|)$ and $(F, \| \|)$ are normed vector spaces, $f : (a, b) \to L(E, F)$ ($a, b \in \mathbb{R}$, $a < b$) and $v \in E$.

Prove that if f is differentiable at $t_0 \in (a, b)$, then the function $g : (a, b) \to F$, defined by

$$g(t) = f(t)v \in F$$

is differentiable at t_0 and that

$$\frac{dg}{dt}(t_0) = \frac{df}{dt}(t_0)v.$$

Problem 4. Write up the analogue of **Problem 3.** for a complex-differentiable f and prove it.

Problem 5. Suppose $(E_i, \| \|)$ and $(F, \| \|)$ are vector spaces, that all the E_i's are finite dimensional of dimension m_i and that $\{e_{i\ell_i} \mid 1 \leq \ell_i \leq m_i\} \subset E_i$ are bases.

Prove that any $\varphi \in \mathscr{L}^{(k)}(E_1, \ldots, E_k; F)$ is uniquely determined by the values

$$\varphi(e_{1\ell_1}, e_{2\ell_2}, \ldots, e_{k\ell_k}) \in F.$$

Conclude that $\mathscr{L}^{(k)}(E_1, \ldots, E_k; F)$ ia isomorphic to $F^{m_1 m_2 \cdots m_k}$.

Prove that $\mathscr{L}^{(k)}(E_1, \ldots, E_k; F) = L^{(k)}(E_1, \ldots, E_k; F)$.

Problem 6. Write down the proof of the fact that if $(E_i, \| \|)$ and $(F, \| \|)$ are N.V.Spaces, then

$$L^{(k)}(E_1, \ldots, E_k; F) \cong L(E_1, L(E_2, \ldots, L(E_k, F) \ldots)).$$

Problem 7. Let $E \neq \{0\}$ be a normed vector space. Prove that it is impossible to have $S, T \in L(E)$ such that

$$ST - TS = I_E,$$

where I_E is the identity operator on E.

(Hint: Prove that this would imply $ST^{n+1} - T^{n+1}S = (n+1)T^n$, and therefore the inequality $(n+1)\|T^n\| \leq 2\|S\|\|T\|\|T^n\|$, which leads to $T^n = 0$ for large n, so finally $T = 0$.)