Sample Midterm Exam 2

Name:
Signature:
Student ID:

This exam consists of 6 problems. Please write clearly, both in form and substance.

Good luck!

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>9=3+2+2+2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>5</td>
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<tr>
<td>4</td>
<td>6</td>
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</tr>
<tr>
<td>5</td>
<td>6=3×2</td>
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</tr>
<tr>
<td>6</td>
<td>6=1+5</td>
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Total 37

My signature below certifies that I have complied with the University of Pennsylvania’s Code of Academic Integrity in completing this Math 420 Midterm Exam.

Name (printed):
Signature:
Problem 1. Consider the differential equation

\[ \frac{dY}{dt} = A \cdot Y, \]

where \( A \) is a 5 \times 5 (real) matrix. Suppose we know that \( v = (1, i, 1 + i, 1 - i, 2 + 2i) \) is a generalized eigenvector of order 2, corresponding to the eigenvalue \( \lambda = -1 - i \). (That is \((A - \lambda I)^2 v = 0\).)

What is the solution of the system that satisfies the initial condition \( Y(0) = (1, 1, 2, 0, 4) \)? (Hint: \( Y(0) = \mathbb{R}v + \mathbb{I}v \).)
Problem 2. Consider the system

\[
\begin{align*}
\frac{dx}{dt} &= x(4 - x^2 - y^2) - y \\
\frac{dy}{dt} &= y(4 - x^2 - y^2) + x.
\end{align*}
\]

(a) Write the system in polar coordinates.
(b) What is the long term behaviour of the solution that passes through the point \(x_0 = 1, y_0 = 1\)?
(c) What is the long term behaviour of the solution to the initial value problem \(x_0 = 5, y_0 = 5\)?
(b) What is the long term behaviour of the solution to the initial value problem \(x_0 = 0, y_0 = 2\)?
Problem 3. Draw the phase portrait for the system:

\[
\frac{dY}{dt} = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} Y.
\]
**Problem 4.** Check whether the system

\[
\begin{align*}
\frac{dx}{dt} &= -x \sin y + 2y \\
\frac{dy}{dt} &= -\cos y
\end{align*}
\]

is Hamiltonian. If so find a Hamiltonian function.

On what level-curves of the Hamiltonian do the equilibrium points lie?

*Proof.* We first check \(\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y} :\)

\[
\frac{\partial f}{\partial x} = -\sin y,
\]

\[
\frac{\partial g}{\partial y} = \sin y.
\]

Next we solve \(\frac{\partial H}{\partial y} = -x \sin y + 2y, \frac{\partial H}{\partial x} = \cos y.\) From the second equation we get

\[
H = x \cos y + c(y)
\]

and plugging this into the first one we get

\[-x \sin y + \frac{dc}{dy} = -x \sin y + 2y.\]

It follows that \(c = y^2,\) so

\[
H = x \cos y + y^2
\]

is a Hamiltonian function.

The equilibrium points are obtained by solving

\[-x \sin y + 2y = 0\]

\[
\cos y = 0.
\]

So at these points the hamiltonian function is \(H = y^2.\) Moreover since \(y = k\pi + \frac{\pi}{2},\) it follows that the critical points lie on the level curves

\[
x \cos y + y^2 = \left(k\pi + \frac{\pi}{2}\right)^2,
\]

where \(k \in \mathbb{Z}\) is any integer. \(\square\)
**Problem 5.** Consider the system

\[
\begin{align*}
\frac{dx}{dt} &= x^2 - 2x + y^2 - 3 \\
\frac{dy}{dt} &= -x^2 - 2x - y^2 + 3.
\end{align*}
\]

(a) Sketch the \(x\)- and \(y\)-nullclines and find the equilibrium points.
(b) Classify the equilibrium points.
(c) Describe the possible fate of the solution with initial condition \(x_0 = 0, y_0 = 0\).

**Proof.** The \(x\)-nullcline is given by

\[(x - 1)^2 + y^2 = 4,
\]
which is the circle centered at \((-1, 0)\) and radius 2.

The \(y\)-nullcline is given by

\[(x + 1)^2 + y^2 = 4,
\]
which is the circle centered at \((1, 0)\) and radius 2.

Their intersection are the two points \((0, \sqrt{3}), (0, -\sqrt{3})\).

\[
\begin{bmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{bmatrix} = \begin{bmatrix} 2x - 2 & 2y \\
-2x - 2 & -2y
\end{bmatrix},
\]

so at \((0, \sqrt{3})\), the matrix of the linearization is

\[
2 \begin{bmatrix} -1 & \sqrt{3} \\
-1 & -\sqrt{3}
\end{bmatrix},
\]

so this is a spiral sink. \((T < 0, D > 0, T^2 - 4D < 0)\).

At \((0, -\sqrt{3})\), the matrix of the linearization is

\[
2 \begin{bmatrix} -1 & -\sqrt{3} \\
-1 & \sqrt{3}
\end{bmatrix},
\]

so this is a saddle point. \((D < 0)\).

If you draw the direction field on the nullclines you see that the solution starting at \((0, 0)\) is most likely ending up at \((0, \sqrt{3})\). \(\square\)
Problem 6. Find the general solution of the differential equation:

\[
\frac{dy}{dt} = \left(2t + \frac{1}{t}\right)y - y^2 - t^2.
\]

(a) Check that \(y_1(t) = t\) is a solution of the equation.
(b) Using (a) find the general solution of the equation.