Topology

Let (X, d) be a metric space. A *ball* centered at $x \in X$ of radius $r > 0$ is the subset $B(x, r) = \{x' \in X : d(x, x') < r\}$.

If S is a subset of X, then a point $x \in X$ is an *interior point* of S if there exists $r > 0$ such that the ball $B(x, r)$ is completely contained in S. The set of all interior points of S is called the *interior* of S and is denoted by $\mathring{S}$.

A point $x \in X$ is a *limit point* of S if every ball centered at x intersects the set $S - \{x\}$. The set S together with all its limit points forms the *closure* of S which is denoted by $\bar{S}$.

A subset S of X is called *open* if $S = \mathring{S}$. A subset S of X is called *closed* if $S = \bar{S}$.

Proposition. *Let (X, d) be a metric space. Then every ball $B(x, r)$ is an open subset of X.*

Proposition. *Let (X, d) be a metric space and let S be a subset of X. The following statements are equivalent:*

- *S is open*
- *S^c is closed*
- *every point of S is an interior point of S*
- *S is the union of a (possibly empty) collection of balls*

Proposition. *Let (X, d) be a metric space and let S be a subset of X. The following statements are equivalent:*

- *S is closed*
- *S^c is open*
- *S contains all its limit points*

Proposition. *Any union of open sets is an open set. A finite intersection of open sets is an open set.*

Proposition. *Any intersection of closed sets is a closed set. A finite union of closed sets is a closed set.*

We can characterize the interior and the closure of a subset S of X in the following way:

- $\mathring{S}$ is the union of all open subsets of S (i.e. the largest open subset of S)
- $\bar{S}$ is the intersection of all closed subsets of X which contain S (i.e. the smallest closed superset of S)

Let (X, d) be a metric space. The distance between a point $x \in X$ and a non-empty subset $S \subseteq X$ is defined by

$$d(x, S) := \inf\{d(x, s) : s \in S\}$$

Let S be a proper subset of X which has more than one element. The following statements hold:

- $x \in \mathring{S} \Leftrightarrow d(x, S^c) > 0$
- $x \in \bar{S} \Leftrightarrow d(x, S) = 0$
- x is a limit point of $S \Leftrightarrow d(x, S - \{x\}) = 0$
- $x \in S$ is an isolated point of $S \Leftrightarrow d(x, S - \{x\}) > 0$