Continuity

1. Let $x_0 \in \mathbb{R}$ be a real number. Check if the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by the rule
$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ x_0 & \text{if } x = 0 \end{cases}$$
is continuous.

2. Check if the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by the rule
$$f(x) = \begin{cases} x \cdot \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
is continuous.

3. Check if the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by the rule
$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
is continuous.

4. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous at every real number.

5. Show that if a function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and periodic, then it attains its supremum and infimum.

6. Let f and g be continuous functions defined on $S \subseteq \mathbb{R}$. Show that the function $\max\{f, g\} : S \to \mathbb{R}$ defined by the rule
$$\max\{f, g\}(x) := \max\{f(x), g(x)\}$$
is also continuous.

7. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Prove that there exists a fixed point $x \in [0, 1]$ of f, which means that $f(x) = x$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial of odd degree. Show that there exists $x \in \mathbb{R}$ such that $f(x) = 0$.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, does $\{f(x_n)\}_{n\in\mathbb{N}}$ have to be a Cauchy sequence? What if we have a function $f : \langle 0, 1\rangle \to \mathbb{R}$ instead?

10. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies $f(x + y) = f(x) + f(y)$ for every $x, y \in \mathbb{R}$. Show that the function f has to be of the form $f(x) = ax$ for some $a \in \mathbb{R}$.

11. Investigate which continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfy the equality $f(|z|) = f(\operatorname{Re} z) + f(\operatorname{Im} z)$ for every $z \in \mathbb{C}$.