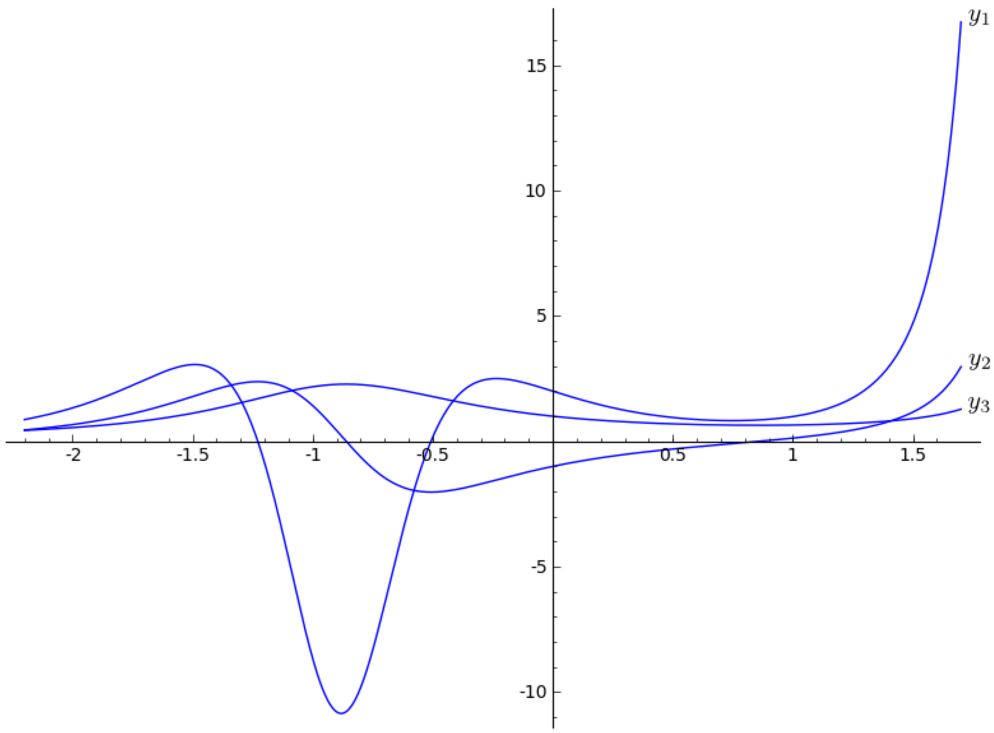
Extra problems

1. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by the rule $f(x) = |x| \sin x$. Show that f is differentiable on $\mathbb{R}$ and that its first derivative is a continuous function. Is f twice differentiable?

2. (a) Let $f_n : [-1, 1] \to \mathbb{R}$ be the sequence of functions $f_n(x) = \cos \frac{x}{n}$. Does the sequence f_n converge uniformly?

   (b) Let $g_n : \mathbb{R} \to \mathbb{R}$ be the sequence of functions $g_n(x) = \cos \frac{x}{n}$. Does the sequence g_n converge uniformly?

3. The curves y_1, y_2, and y_3 in the graph below are the graphs af a function f and its first and second derivatives. Which curve is the graph f, which one is the graph of f', and which one is the graph of f''? Justify your answer.



   Sketch the graph of the antiderivative $g(x) = \int_0^x f(t)\,dt$ of the function f.

4. Let $f_0 : \mathbb{R} \to \mathbb{R}$ be a continuous function. For $n \in \mathbb{N}$ we define the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ recursively:

$$f_n(x) := \int_0^x f_{n-1}(t)dt$$

   Assume that the sequence of functions f_n converges uniformly to a function f. Show that $f(x) = 0$ for every $x \in \mathbb{R}$.

5. Let the function $f : [0, 1] \to \mathbb{R}$ be defined by the following rule:

$$f(x) = \begin{cases} 1 & \text{if } x \in \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \\ 0 & \text{otherwise} \end{cases}$$

   Show that f is integrable.

6. Let the function $f : [0, 1] \to \mathbb{R}$ be defined by the following rule:

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   Show that f is integrable.

7. For which values of a does the series $1 + \sin x + \sin x^2 + \sin x^3 + \sin x^4 + \cdots$ converge uniformly on $[-a, a]$?

8. (a) Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions which coverges uniformly to $f : \mathbb{R} \to \mathbb{R}$. Let $g \colon \mathbb{R} \to \mathbb{R}$ be any function. Show that the sequence $f_n \circ g : \mathbb{R} \to \mathbb{R}$ converges uniformly.

   (b) For which values of a does the series $1 + \sin x + \sin^2 x + \sin^3 x + \sin^4 x + \cdots$ converge uniformly on $[-a, a]$?

9. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a uniformly convergent sequence of continuous functions and let $\{x_n\}_{n \in \mathbb{N}}$ be a convergent sequence of real numbers. Show that the sequence of real numbers $\{f_n(x_n)\}_{n \in \mathbb{N}}$ converges. Does the converse hold?

10. Show that the conclusion of the previous problem does not hold if we only assume that the sequence $f_n : \mathbb{R} \to \mathbb{R}$ converges pointwise.