Limit superior and inferior

The real numbers have two structures which we need in order to define the limit superior and limit inferior of a sequence:

- $\mathbb{R}$ has a metric which enables us to talk about convergence of sequences and continuity of functions
- $\mathbb{R}$ has an order which enables us to compare different real numbers

However, limit superior and limit inferior can attain infinite values so their codomain should be a set bigger than $\mathbb{R}$:

$$\liminf : \mathbb{R}^{\mathbb{N}} \to [-\infty, +\infty] \qquad \limsup : \mathbb{R}^{\mathbb{N}} \to [-\infty, +\infty]$$

The set $[-\infty, +\infty]$ is the set consisting of all real numbers together with two extra elements $\pm\infty$. We have to introduce the structures on this set which parallel the ones we have on $\mathbb{R}$. The metric on the set $[-\infty, +\infty]$ can be defined using the formula:

$$d(x, y) = |\arctan y - \arctan x|$$

To introduce the order we only need to define the comparison of $\pm\infty$ with other elements. We set $+\infty$ to be the maximum of $[-\infty, +\infty]$ and $-\infty$ to be the minimum of $[-\infty, +\infty]$. This order has the property that any nonempty subset of $[-\infty, +\infty]$ has the supremum and infimum. Moreover, the map

$$\tan : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-\infty, +\infty]$$

is a bijective isometry, so the metric space $[-\infty, +\infty]$ is compact. Hence, any sequence of elements in $[-\infty, +\infty]$ has a convergent subsequence. Given a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of $[-\infty, +\infty]$, let Λ be the set of all the limits of convergent subsequences of $\{x_n\}_{n\in\mathbb{N}}$. Since $[-\infty, +\infty]$ is compact, Λ is nonempty so it has a supremum and infimum in $[-\infty, +\infty]$. We define:

$$\liminf_{n\to\infty} x_n := \inf \Lambda \qquad \limsup_{n\to\infty} x_n := \sup \Lambda$$

Note that the set $\mathbb{R}$ is a subset of $[-\infty, +\infty]$ and hence it inherits the metric d. The usual metric space $\mathbb{R}$ with the distance given by $|y - x|$ and the metric space $(\mathbb{R}, d)$ have the same open sets. This means that any sequence $\{x_n\}_{n\in\mathbb{N}}$ which converges to $x_0 \in \mathbb{R}$ in the usual sense, also converges to x_0 in $(\mathbb{R}, d)$ and vice versa.

Some important properties:

- $\liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n$ and the equality holds if and only if the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges.
- If the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges, then its limit inferior and limit superior both equal to $\lim_{n\to\infty} x_n$.
- If the inequality $x_n \le y_n$ holds for all but finitely many indices $n \in \mathbb{N}$, then
  - $\limsup_{n\to\infty} x_n \le \limsup_{n\to\infty} y_n$
  - $\liminf_{n\to\infty} x_n \le \liminf_{n\to\infty} y_n$
- The following inequalities are true unless the expressions of the form $\pm\infty \mp \infty$ crop up:
  - $\limsup_{n\to\infty}(x_n + y_n) \le \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$
  - $\liminf_{n\to\infty}(x_n + y_n) \ge \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$
- If $\lambda \le x_n$ holds only for finitely many indices $n \in \mathbb{N}$, then $\limsup_{n\to\infty} x_n \le \lambda$.
- If $\lambda \ge x_n$ holds only for finitely many indices $n \in \mathbb{N}$, then $\liminf_{n\to\infty} x_n \ge \lambda$.
- We have the following formulas:
  - $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k \ge n\}$
  - $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k : k \ge n\}$