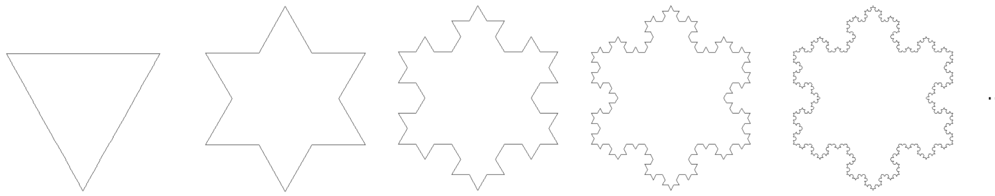
Sets and real numbers

1. Can you make sense of the following expressions? Which numbers do they represent?

- $\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$
- $\frac{2}{2+\frac{2}{2+\frac{2}{\ddots}}}$
- $2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}$
- $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}$
- $2.222\cdots$
- $9.999\cdots$
- $99999\cdots$

2. Does the following sequence of shapes converge to something? What is the perimeter and the area of the limit?



3. Let $\mathbb{R}^+$ be the set of positive real numbers. Show that the operation on the set $\mathbb{R}^+$ given by the rule
$$x \oplus y := xy$$
satisfies all the axioms for addition operation.

4. Show that the following two operations on the set $\mathbb{R}^+$ satisfy the axioms for multiplication:
$$x \otimes y := x^{\ln y} \qquad\qquad x \otimes y := y^{\ln x}$$

5. Does the set $\mathbb{R}^+$ together with the operations defined above have a structure of an ordered field? What about the least-upper-bound property?

6. Is the number $\sqrt{2}+\frac{1}{\sqrt{2}}$ rational?

7. Show that between any two real numbers there is an irrational number.

8. Determine the value of the supremum and infimum of the following sets if they are nonempty and appropriately bounded:

- $\{x \in \mathbb{R} : x < 2\}$
- $\{x \in \mathbb{N} : x < 2\}$
- $\{x \in \mathbb{Q} : x < 2\}$
- $\{x \in \mathbb{R} - \mathbb{Q} : x < 2\}$
- $\{x \in \mathbb{R} : x^2 \leq 2\}$
- $\{x \in \mathbb{N} : x^2 \leq 2\}$
- $\{x \in \mathbb{Q} : x^2 \leq 2\}$
- $\{x \in \mathbb{R} - \mathbb{Q} : x^2 \leq 2\}$

9. Show that for any nonempty subset S of real numbers which is bounded above and any positive real number a we have $\sup\{a \cdot s : s \in S\} = a \sup S$.

10. Let a be a real number greater than 1 and let b be any positive real number. Prove that there is a unique real number x such that $a^x = b$.

11. Let a be a real number greater than 1. Show that for any $x \in \mathbb{R}$, the number a^x is equal to:

- $\sup\{a^r : r \in \mathbb{R} \text{ and } r < x\}$
- $\sup\{a^r : r \in \mathbb{Q} \text{ and } r < x\}$
- $\inf\{a^r : r \in \mathbb{R} \text{ and } r > x\}$
- $\inf\{a^r : r \in \mathbb{R} - \mathbb{Q} \text{ and } r > x\}$

12. How can we define the number a^x when $0 < a < 1$ using a formula analogous to the ones from the previous problem? Check if the following equality holds:
$$a^{-x} = \left(\frac{1}{a}\right)^x$$

13. Let a and b be real numbers greater than 1. Show that for any $x \in \mathbb{R}$ we have $(ab)^x = a^x b^x$.

14. We denote by Y^X the set of all functions $f : X \to Y$ with domain X and codomain Y. Determine which of the following sets have the same cardinality:

- $\varnothing$
- X
- $X^{\{0,1\}}$
- $X^{\varnothing}$
- $\varnothing^X$
- $\mathcal{P}(X)$
- $\mathcal{P}(\varnothing)$
- $X \times X$
- $\varnothing \times X$
- $\{0,1\}^X$
- $\{0\}^X$
- $X^{\{0\}}$

15. Let $\Bbbk$ be a field. For every pair of functions $f, g : X \to \Bbbk$, we define the function $f + g : X \to \Bbbk$ by the rule:

$$(f + g)(x) = f(x) + g(x)$$

For a function $f : X \to \Bbbk$ and a number $\lambda \in \Bbbk$, we define the function $\lambda \cdot f : X \to \Bbbk$ by the rule:

$$(\lambda \cdot f)(x) = \lambda \cdot f(x)$$

Show that the set $\Bbbk^X$ together with the operations defined above forms a vector space over the field $\Bbbk$.

16. What is the vector space $\mathbb{R}^X$ when $X = \{1, \ldots, n\}$?

17. Show how to define addition and multiplication on the set $\{0, 1\}$ so that it becomes a field. We denote this field by $\mathbb{Z}/2\mathbb{Z}$.

18. Is it possible to make $\mathbb{Z}/2\mathbb{Z}$ an ordered field?

19. Fix a set X and let $\mathcal{P}(X)$ be the power set of X. Show that the operation on the set $\mathcal{P}(X)$ given by the rule

$$A \oplus B := (A - B) \cup (B - A)$$

satisfies all the axioms for addition operation.

20. Show that the operation on $\mathcal{P}(X)$ given by the rule $A \otimes B := A \cap B$ satisfies all the axioms for multiplication operation.

21. Is the set $\mathcal{P}(X)$ together with the operations defined above a field? What are the solutions of the equation $x = -x$?

22. Show how the set $\mathcal{P}(X)$ together with the operation of addition defined above can be made into a vector space over the field $\mathbb{Z}/2\mathbb{Z}$.