Sequences and series

1. Show that if the sequence of real or complex numbers $\{x_n\}_{n\in\mathbb{N}}$ converges, than the sequence $\{|x_n|\}_{n\in\mathbb{N}}$ also converges.
2. Show that if the sequence of vectors $\{x_n\}_{n\in\mathbb{N}}$ converges, than the sequence $\{\|x_n\|\}_{n\in\mathbb{N}}$ also converges.
3. Let $\{x_n\}_{n\in\mathbb{N}}$ be a convergent sequence of real numbers. Show that the sequence $\{|x_{n+1}-x_n|\}_{n\in\mathbb{N}}$ is also convergent.
4. For which complex numbers $z \in \mathbb{C}$ does the sequence $\{z^n\}_{n\in\mathbb{N}}$ converge? What are the possible limits?
5. Let $\{x_n\}_{n\in\mathbb{N}}$ be an enumeration of the set of rational numbers $\mathbb{Q}$. Describe the set of limits of all convergent subsequences of $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$.
6. Let $\{x_n\}_{n\in\mathbb{N}}$ be a convergent sequence in a metric space (X, d). Show that the set $\{x_n : n \in \mathbb{N}\}$ is bounded.
7. Let S be a subset of the metric space (X, d). Prove that $x \in X$ is inside the closure of S if and only if there is a sequence $\{s_n\}_{n\in\mathbb{N}}$ of elements of S such that $x = \lim_{n\to\infty} s_n$.
8. Show that a subset S of a metric space (X, d) is closed if and only if every convergent sequence $\{s_n\}_{n\in\mathbb{N}}$ of elements of S has its limit in S.
9. Let (X, d) be a metric space with discrete metric. Which sequences in (X, d) are convergent?
10. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be Cauchy sequences in a metric space (X, d). Show that the sequence $\{d(x_n, y_n)\}_{n\in\mathbb{N}}$ of real numbers is convergent.
11. Give an example of a metric space (X, d) and a Cauchy sequence in (X, d) which is not convergent.
12. Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence. Show that $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$. Does the converse hold?
13. Find a collection of real numbers $a_{mn} \in \mathbb{R}$ where $m, n \in \mathbb{N}$ such that the following are all true:
    - For every $m \in \mathbb{N}$, the sequence $\{a_{mn}\}_{n\in\mathbb{N}}$ is convergent. The sequence $\{\lim_{n\to\infty} a_{mn}\}_{m\in\mathbb{N}}$ is convergent.
    - For every $n \in \mathbb{N}$, the sequence $\{a_{mn}\}_{m\in\mathbb{N}}$ is convergent. The sequence $\{\lim_{m\to\infty} a_{mn}\}_{n\in\mathbb{N}}$ is convergent.
    - We have $\lim\limits_{m\to\infty} \lim\limits_{n\to\infty} a_{mn} \neq \lim\limits_{n\to\infty} \lim\limits_{m\to\infty} a_{mn}$.
14. Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence of real numbers. Does the sequence $\{\frac{1}{n}\sum_{k=1}^{n} x_k\}_{n\in\mathbb{N}}$ have to converge?
15. Let x be a real number. Does the sequence $\dfrac{\lfloor x\rfloor + \lfloor 2x\rfloor + \cdots + \lfloor nx\rfloor}{n^2}$ converge?
16. Does the series $\sum_{n=0}^{\infty} e^{in}$ converge?
17. Suppose that $\sum_{n=1}^{\infty} x_n$ is a series of positive real numbers which is convergent. Show that the following series are convergent:
    - $\sum_{n=1}^{\infty} x_n^2$
    - $\sum_{n=1}^{\infty} \sqrt{x_n \cdot x_{n+1}}$
    - $\sum_{n=1}^{\infty} \frac{1}{n}\sqrt{x_n}$
18. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that the subsequences $\{x_{2k}\}_{k\in\mathbb{N}}$, $\{x_{2k+1}\}_{k\in\mathbb{N}}$, and $\{x_{3k}\}_{k\in\mathbb{N}}$ are all convergent. Show that $\{x_n\}_{n\in\mathbb{N}}$ is convergent.
19. Suppose that $\sum_{n=1}^{\infty} x_n$ is a series of positive real numbers which is convergent. Show that $\sum_{n=1}^{\infty} \frac{1}{x_n}$ is divergent. Does the converse hold?
20. Does there exist a sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers such that $\sum_{n=1}^{\infty} x_n^m = m$ for every $m \in \mathbb{N}$?
21. Find the limit superior and the limit inferior of the sequence $\left\{1 + (-1)^n + \frac{1}{n}\right\}_{n\in\mathbb{N}}$.
22. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Assume that $\{x_{n_k}\}_{k\in\mathbb{N}}$ and $\{x_{m_k}\}_{k\in\mathbb{N}}$ are two subsequences whose combined indices cover the entire set $\mathbb{N}$. Show the equality:
$$\limsup_{n\to\infty} x_n = \max\left\{\limsup_{k\to\infty} x_{n_k}, \limsup_{k\to\infty} x_{m_k}\right\}$$
    What happens if the two subsequences converge to the same number? What if we have three or more subsequences? What if we have infinitely many subsequences?