Sequences

1. Show that if the sequence of real or complex numbers \(\{x_n\}_{n \in \mathbb{N}}\) converges, than the sequence \(\{|x_n|\}_{n \in \mathbb{N}}\) also converges.

2. Show that if the sequence of vectors \(\{x_n\}_{n \in \mathbb{N}}\) converges, than the sequence \(\{|x_n|\}_{n \in \mathbb{N}}\) also converges.

3. Let \(\{x_n\}_{n \in \mathbb{N}}\) be a convergent sequence of real numbers. Show that the sequence \(\{|x_n+1-x_n|\}_{n \in \mathbb{N}}\) is also convergent.

4. For which complex numbers \(z \in \mathbb{C}\) does the sequence \(\{z^n\}_{n \in \mathbb{N}}\) converge? What are the possible limits?

5. Let \(\{x_n\}_{n \in \mathbb{N}}\) be an enumeration of the set of rational numbers \(\mathbb{Q}\). Describe the set of limits of all convergent subsequences of \(\{x_n\}_{n \in \mathbb{N}}\) in \(\mathbb{R}\).

6. Let \(\{x_n\}_{n \in \mathbb{N}}\) be a convergent sequence in a metric space \((X, d)\). Show that the set \(\{x_n : n \in \mathbb{N}\}\) is bounded.

7. Let \(S\) be a subset of the metric space \((X, d)\). Prove that \(x \in X\) is inside the closure of \(S\) if and only if there is a sequence \(\{s_n\}_{n \in \mathbb{N}}\) of elements of \(S\) such that \(x = \lim_{n \to \infty} s_n\).

8. Show that a subset \(S\) of a metric space \((X, d)\) is closed if and only if every convergent sequence \(\{s_n\}_{n \in \mathbb{N}}\) of elements of \(S\) has its limit in \(S\).

9. Let \((X, d)\) be a metric space with discrete metric. Which sequences in \((X, d)\) are convergent?

10. Let \(\{x_n\}_{n \in \mathbb{N}}\) and \(\{y_n\}_{n \in \mathbb{N}}\) be Cauchy sequences in a metric space \((X, d)\). Show that the sequence \(\{d(x_n, y_n)\}_{n \in \mathbb{N}}\) of real numbers is convergent.

11. Give an example of a metric space \((X, d)\) and a Cauchy sequence in \((X, d)\) which is not convergent.

12. Find a collection of real numbers \(a_{mn} \in \mathbb{R}\) where \(m, n \in \mathbb{N}\) such that the following are all true:
   - For every \(m \in \mathbb{N}\), the sequence \(\{a_{mn}\}_{n \in \mathbb{N}}\) is convergent. The sequence \(\lim_{n \to \infty} a_{mn}\) is convergent.
   - For every \(n \in \mathbb{N}\), the sequence \(\{a_{mn}\}_{m \in \mathbb{N}}\) is convergent. The sequence \(\lim_{m \to \infty} a_{mn}\) is convergent.
   - We have \(\lim_{m \to \infty} \lim_{n \to \infty} a_{mn} = \lim_{n \to \infty} \lim_{m \to \infty} a_{mn}\).

13. Does the series \(\sum_{n=0}^{\infty} e^{in}\) converge?

14. Let \(\{x_n\}_{n \in \mathbb{N}}\) be a bounded sequence of real numbers. Does the sequence \(\{\frac{1}{n} \sum_{k=1}^{n} x_k\}_{n \in \mathbb{N}}\) have to converge?

15. Suppose that \(\sum_{n=1}^{\infty} x_n\) is a series of positive real numbers which is convergent. Show that the following series are convergent:
   - \(\sum_{n=1}^{\infty} x_n^2\)
   - \(\sum_{n=1}^{\infty} \sqrt{x_n \cdot x_{n+1}}\)
   - \(\sum_{n=1}^{\infty} \frac{1}{n} \sqrt{x_n}\)

16. Suppose that \(\sum_{n=1}^{\infty} x_n\) is a series of positive real numbers which is convergent. Show that \(\sum_{n=1}^{\infty} \frac{1}{x_n}\) is divergent. Does the converse hold?

17. Does there exist a sequence \(\{a_n\}_{n \in \mathbb{N}}\) of real numbers such that \(\sum_{n=1}^{\infty} a_n^m = m\) for every \(m \in \mathbb{N}\)?