Sequences

1. Show that if the sequence of real or complex numbers \( \{x_n\}_{n \in \mathbb{N}} \) converges, than the sequence \( \{|x_n|\}_{n \in \mathbb{N}} \) also converges.
2. Show that if the sequence of vectors \( \{x_n\}_{n \in \mathbb{N}} \) converges, than the sequence \( \{||x_n||\}_{n \in \mathbb{N}} \) also converges.
3. Let \( \{x_n\}_{n \in \mathbb{N}} \) be a convergent sequence of real numbers. Show that the sequence \( \{|x_{n+1} - x_n|\}_{n \in \mathbb{N}} \) is also convergent.
4. For which complex numbers \( z \in \mathbb{C} \) does the sequence \( \{z^n\}_{n \in \mathbb{N}} \) converge? What are the possible limits?
5. Let \( \{x_n\}_{n \in \mathbb{N}} \) be an enumeration of the set of rational numbers \( \mathbb{Q} \). Describe the set of limits of all convergent subsequences of \( \{x_n\}_{n \in \mathbb{N}} \) in \( \mathbb{R} \).
6. Let \( \{x_n\}_{n \in \mathbb{N}} \) be a convergent sequence in a metric space \((X, d)\). Show that the set \( \{x_n : n \in \mathbb{N}\} \) is bounded.
7. Let \( S \) be a subset of the metric space \((X, d)\). Prove that \( x \in X \) is inside the closure of \( S \) if and only if there is a sequence \( \{s_n\}_{n \in \mathbb{N}} \) of elements of \( S \) such that \( x = \lim_{n \to \infty} s_n \).
8. Show that a subset \( S \) of a metric space \((X, d)\) is closed if and only if every convergent sequence \( \{s_n\}_{n \in \mathbb{N}} \) of elements of \( S \) has its limit in \( S \).
9. Let \((X, d)\) be a metric space with discrete metric. Which sequences in \((X, d)\) are convergent?
10. Let \( \{x_n\}_{n \in \mathbb{N}} \) and \( \{y_n\}_{n \in \mathbb{N}} \) be Cauchy sequences in a metric space \((X, d)\). Show that the sequence \( \{d(x_n, y_n)\}_{n \in \mathbb{N}} \) of real numbers is convergent.
11. Give an example of a metric space \((X, d)\) and a Cauchy sequence in \((X, d)\) which is not convergent.
12. Let \( \{x_n\}_{n \in \mathbb{N}} \) be a Cauchy sequence. Show that \( \lim_{n \to \infty} |x_{n+1} - x_n| = 0 \). Does the converse hold?
13. Find a collection of real numbers \( a_{mn} \in \mathbb{R} \) where \( m, n \in \mathbb{N} \) such that the following are all true:
   - For every \( m \in \mathbb{N} \), the sequence \( \{a_{mn}\}_{n \in \mathbb{N}} \) is convergent. The sequence \( \lim_{n \to \infty} a_{mn} \) is convergent.
   - For every \( n \in \mathbb{N} \), the sequence \( \{a_{mn}\}_{m \in \mathbb{N}} \) is convergent. The sequence \( \lim_{m \to \infty} a_{mn} \) is convergent.
   - We have \( \lim_{m \to \infty} \lim_{n \to \infty} a_{mn} \neq \lim_{n \to \infty} \lim_{m \to \infty} a_{mn} \).
14. Let \( \{x_n\}_{n \in \mathbb{N}} \) be a bounded sequence of real numbers. Does the sequence \( \left\{ \frac{1}{n} \sum_{k=1}^{n} x_k \right\}_{n \in \mathbb{N}} \) have to converge?
15. Let \( x \) be a real number. Does the sequence \( \frac{|x| + |2x| + \cdots + |nx|}{n^2} \) converge?
16. Does the series \( \sum_{n=0}^{\infty} e^{in} \) converge?
17. Suppose that \( \sum_{n=1}^{\infty} x_n \) is a series of positive real numbers which is convergent. Show that the following series are convergent:
   - \( \sum_{n=1}^{\infty} x_n^2 \)
   - \( \sum_{n=1}^{\infty} \sqrt{x_n \cdot x_{n+1}} \)
   - \( \sum_{n=1}^{\infty} \frac{1}{n} \sqrt{x_n} \)
18. Suppose that \( \sum_{n=1}^{\infty} x_n \) is a series of positive real numbers which is convergent. Show that \( \sum_{n=1}^{\infty} \frac{1}{x_n} \) is divergent. Does the converse hold?
19. Does there exist a sequence \( \{x_n\}_{n \in \mathbb{N}} \) of real numbers such that \( \sum_{n=1}^{\infty} x_n^m = m \) for every \( m \in \mathbb{N} \)?
20. Find the limit superior and the limit inferior of the sequence \( \{1 + (-1)^n + \frac{1}{n}\}_{n \in \mathbb{N}} \).
21. Let \( \{x_n\}_{n \in \mathbb{N}} \) be a bounded sequence of real numbers. Prove that \( \lim_{n \to \infty} \sup x_n = \lim_{n \to \infty} \sup \{x_k : k \geq n\} \).