Topology

1. Determine the interior, closure, limit points, and isolated points of the following subsets of the plane:

   (a) $\{(x,y) : xy = 0\}$
   (b) $\{(x,y) : x^2 + y^2 \neq 1\}$
   (c) $\left\{\left(\frac{1}{n}, -\frac{1}{n}\right) : n \in \mathbb{N}\right\}$
   (d) $\mathbb{Z} \cup \{(x,y) : x > 0\}$
   (e) $\varnothing$
   (f) $\mathbb{R}^2$
   (g) $[0,1] \times [0,1\rangle$
   (h) $\mathbb{Q} \times \mathbb{R}$
   (i) $\mathbb{Q} \times [0,1\rangle$

   Which of the sets above are open? Which ones are closed? Bounded?

2. Let (X,d) be a metric space. The distance between a point $x \in X$ and a non-empty subset $S \subseteq X$ is defined by

   $$d(x,S) := \inf\{d(x,s) : s \in S\}$$

   Let S be a proper subset of X which has more than one element. Prove the following statements:

   (a) $x \in \bar{S} \Leftrightarrow d(x,S) = 0$
   (b) $x \in \mathring{S} \Leftrightarrow d(x,S^c) > 0$
   (c) x is a limit point of $S \Leftrightarrow d(x, S - \{x\}) = 0$
   (d) $x \in S$ is an isolated point of $S \Leftrightarrow d(x, S - \{x\}) > 0$

3. Show that the complement of $\mathring{S}$ is equal to $\overline{S^c}$.

4. Show that in $\mathbb{R}^n$ the closure of the open ball $B(x,r)$ is the closed ball $\bar{B}(x,r)$. Is this true in any metric space?

5. Show that in $\mathbb{R}^n$ the interior of the closed ball $\bar{B}(x,r)$ is the open ball $B(x,r)$. Is this true in any metric space?

6. Investigate which of the following are true:

   - $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$
   - $\overline{A \cap B} \supseteq \bar{A} \cap \bar{B}$
   - $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$
   - $\overline{A \cup B} \supseteq \bar{A} \cup \bar{B}$

   What happens if we replace the closure operation with the interior operation?

7. Is the set of all algebraic numbers dense in $\mathbb{R}$?

8. Investigate if there exists a countably infinite subset S of the Euclidean space $\mathbb{R}^n$ such that:

   (a) S is open and S is not closed
   (b) S is neither open nor closed
   (c) S is compact
   (d) S is not compact
   (e) S is closed and S is not open
   (f) S is open and closed
   (g) $\bar{S}$ is compact
   (h) $\bar{S}$ is not compact
   (i) $\bar{S}$ is countable
   (j) $\bar{S}$ is uncountable
   (k) $\mathring{S}$ is not empty
   (l) $\bar{S} - S$ is countably infinite

   Do the same if the set $\mathbb{R}^n$ is equipped with the discrete metric.

9. Let (X,d) be a metric space such that the set X is finite. Show that any subset of X is open.

10. Show that the set of all open subsets of $\mathbb{R}$ has the same cardinality as the set $\mathbb{R}$.

11. Show that any finite subset of a metric space is compact.

12. Show that the set $\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is a compact subset of $\mathbb{R}$.

13. Show that a finite union of compact subsets of a metric space is compact.

14. Show that a compact subset of any metric space is bounded.

15. Is every closed and bounded subset of a metric space compact?

16. Let d be the discrete metric on a set X. Determine which subsets of the metric space (X,d) are compact.

17. Let (X,d) be a metric space and let Y be a subset of X. Show that the set Y equipped with the mapping $d|_{Y \times Y} : Y \times Y \to \mathbb{R}$ forms a metric space.

18. Let (X, d) be a metric space and let Y be a subset of X. Show that the subset S of Y is open relative to Y if and only if S is an open subset of the metric space $(Y, d|_{Y\times Y})$.

19. Let $\mathbb{k}$ be a field and consider the set of all formal power series with coefficients in the field $\mathbb{k}$:

$$\mathbb{k}[[x]] = \left\{\sum_{k=0}^{\infty} a_k x^k : a_0, a_1, a_2, \ldots \text{ is a sequence of elements of } \mathbb{k}\right\}$$

Prove that the set $\mathbb{k}[[x]]$ can be made into a metric space by defining the distance between $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} b_k x^k$ to be either zero if these series are identical, or $\frac{1}{2^k}$ if k is the smallest index such that $a_k \neq b_k$.

20. Determine the interior and the closure of the set $\mathbb{k}[x]$ of all polynomials in $\mathbb{k}[[x]]$.