Example: S_4

The representatives are 1, (12)(34), (12), (1234), (123)

	(1)	(3)	(6)	(6)	(8)
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	?				
χ_4	?				
χ_5	?				

← the sign representation

From $1^2+1^2+?^2+?^2+?^2=24$ we get that the dimensions of the irreducible representations are 1,1,2,3,3.

	(1)	(3)	(6)	(6)	(8)
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	2				
χ_4	3	?	?	?	?
χ_5	3				

The representation of S_4 as a full symmetry group of a tetrahedron is irreducible, so we can compute its character to determine the fourth row of the table. To compute the trace of a linear operator we can always choose a basis in which to represent the operator as a matrix and we should try to choose the most suitable basis.

	(1)	(3)	(6)	(6)	(8)
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	2				
χ_4	3	-1	1	-1	0
χ_5	3	?	?	?	?

← e.g. for (1234) we pick vertices $\{v_1,v_2,v_3\}$ to be the basis and then we have $v_1\to v_2$, $v_2\to v_3$, $v_3\to v_4=-v_1-v_2-v_3$

If $\rho:G\to GL(V)$ is a (irreducible) representation with character χ and $\varphi:G\to\mathbb{C}^*$ is a group homomorphism, then $\varphi\cdot\rho:G\to GL(V)$ is also a (irreducible) representation whose character is equal to $\varphi\cdot\chi$.

Hence, $\chi_2\cdot\rho_4$ is a 3-dimensional irreducible representation. Since its character is $\chi_2\cdot\chi_4$ we see that this representation is different from ρ_4 so this is the missing 3-dimensional irreducible representation:

	(1)	(3)	(6)	(6)	(8)
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	2	?	?	?	?
χ_4	3	-1	1	-1	0
χ_5	3	-1	-1	1	0

For the unique two dimensional irreducible character we can use the orthogonality relations which will give as a system of 4 equations with 4 unknowns:

$$\begin{aligned} 3a + 6b + 6c + 8d &= -2 \\ 3a - 6b - 6c + 8d &= -2 \\ -3a + 6b - 6c + 0d &= -6 \\ -3a - 6b + 6c + 0d &= -6 \end{aligned}$$

← since 2,3,6,6,8 are real numbers their conjugates are the numbers themselves

Note that the unknowns a, b, c, d are COMPLEX numbers. We can solve this system (add last two then add first two etc.) and complete the table:

	(1)	(3)	(6)	(6)	(8)
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	2	2	0	0	-1
χ_4	3	-1	1	-1	0
χ_5	3	-1	-1	1	0

The two dimensional representation can be realized as $S_4\to S_3\simeq D_3\to GL(\mathbb{C}^2)$ where the first map $S_4\to S_3$ is induced by the conjugation action of the group S_4 on the set {(12)(34),(13)(24),(14)(23)}.