Groups naturally arise as collections of functions which preserve certain property of the system under consideration. These types of collections happen to be equipped with an operation which make them groups. An analogy can be made with the collection of all natural numbers {1,2,3,…} which is not just a set but it has additional structure - we can add and even multiply its elements.

Now how do the aforementioned collections of functions arise? This is related with the notion of symmetry. Let $S=\{(x,y)\in\mathbb{R}^2 : -1\leq x\leq 1 \text{ and } -1\leq y\leq 1\}$ be a square. We would like to say that S has a certain rotational and bilateral symmetry, but that it doesn't have translational symmetry. How to encode this mathematically?

Let $R(\vartheta):\mathbb{R}^2\to\mathbb{R}^2$ be the function which rotates the whole plane about the origin by the angle ϑ. One can explicitly write this in the following way $R(\vartheta)(x,y) = (x\cdot\cos\vartheta - y\cdot\sin\vartheta,\ x\cdot\sin\vartheta + y\cdot\cos\vartheta)$. This function transforms the whole plane $\mathbb{R}^2$, for example it sends:

- the point (0,0) to (0,0)
- the point (1,0) to $(\cos\vartheta, \sin\vartheta)$
- the point (0,2) to $(-2\cdot\sin\vartheta, 2\cdot\cos\vartheta)$

The set $R(\vartheta)(S)$ is the set of all points of the square S which are rotated by ϑ - so it is the square S rotated by the angle ϑ. Note that:

- $R(30°)(S) \neq S$
- $R(90°)(S) = S$
- $R(180°)(S) = S$

We say that R(90°) is a symmetry of the square S since the 90° rotation tranforms the square S back to itself. R(30°) is not a symmetry of S since it transforms S to something else.

Now let's abstract. Fix a subset $X\subseteq\mathbb{R}^n$ (we're interested in the cases n=2 and n=3). X can be a triangle, square, sphere, circle, random bunch of points… We define:

$$\text{Sym } X := \{f:\mathbb{R}^n\to\mathbb{R}^n \mid f \text{ is a distance preserving bijection such that } f(X)=X\}$$

Here distance preserving means $d(f(x),f(y))=d(x,y)$ where the Euclidean distance $d(x,y)$ is given by the usual formula $\sqrt{\sum(y_i-x_i)^2}$.

Sym X is a subset of the group of all permutations of the set $\mathbb{R}^n$. It is easy to check that Sym X is a subgroup of the group of permutations of $\mathbb{R}^n$.

In the case of the square S it can be shown that Sym S = $\{1,r,r^2,r^3,s,rs,r^2s,r^3s\}$ where $r:\mathbb{R}^2\to\mathbb{R}^2$ is the rotation R(90°) and $s:\mathbb{R}^2\to\mathbb{R}^2$ is the reflection about the x-axis.

…………………

More generally, for $n\geq 3$ let P_n be the standard polygon with n vertices and n edges. We define the group D_n as the symmetry group Sym P_n. It turns out that D_n has 2n elements:

$$D_n = \{1,r,r^2,\dots,r^{n-1},s,rs,r^2s,\dots,r^{n-1}s\}$$

Here, $r:\mathbb{R}^2\to\mathbb{R}^2$ is the rotation by the angle $2\pi/n$ and $s:\mathbb{R}^2\to\mathbb{R}^2$ is the reflection about the x-axis. The element r^ks is the reflection about the line which passes through the origin and forms the angle $k\cdot\pi/n$ with the x-axis. The relations satisfied by r and s are:

- $r^n=1$
- $s^2=1$
- $sr=r^{-1}s$ (so D_n is not commutative)

We see that an elements f of Sym D_n is actually a function $f:\mathbb{R}^2\to\mathbb{R}^2$ which transforms the plane. We say that f acts on the plane. This is how groups appear naturally - as bunches of functions acting on some set. We'd like to abstract this. There are two equivalent definitions of what a group action is. So, let $(G,*)$ be a group and X a set.

DEF 1) A group action is a group homomorphism $\Phi:G\to S_X$.
DEF 2) A group action is a rule $R:G\times X\to X$ which assigns to every pair (g,x) consisting of $g\in G$ and $x\in X$, an element $R(g,x)=g\cdot x$ of the set X such that the following holds:
  (a) $e\cdot x=x$ for every $x\in X$
  (b) $g\cdot(h\cdot x) = (g*h)\cdot x$ for every $g,h\in G$ and $x\in X$

Let's prove that these two definitions are equivalent. We'll define two bijections $\alpha:E_1\to E_2$ and $\beta:E_2\to E_1$ where E_1:={all homomorphisms $\Phi:G\to S_X$} and E_2:={all group action rules $R:G\times X\to X$}. First we define $\alpha:E_1\to E_2$. Assume we have a homomorphism $\Phi:G\to S_X$. Then we can construct out of it the following rule:

$$R(g,x) := \Phi(g)(x)$$

One can check that R satisfies conditions (a) and (b) and hence we can take this R to be the output $\alpha(\Phi)$.

Now we construct a map $\beta:E_2\to E_1$. Given $g\in G$ we can define the function m_g:X→X (multiplication by g) which operates $m\_g(x):=g\cdot x$. The function m_g is a bijection, the inverse being m_g^{-1}. Now define the homomorphism $\Phi:G\to S_x$ by saying $\Phi(g):=m\_g$. One can check that $\Phi:G\to S_x$ is indeed a homomorphism and hence we can take this Φ to be the output $\beta(R)$.

Now we have maps in both directions and it is straightforward to check that α and β are inverses of each other.

Examples:
E1) Let $V=\{v_1,v_2,v_3,v_4\}$ be the set of all the vertices of the square S. Then every symmetry of S maps a vertex v_i to some vertex v_j and this induces a group action of Sym S on the set V. Example calculations:

$r\cdot v_1=v_2$	$s\cdot v_1=v_4$
$r\cdot v_2=v_3$	$s\cdot v_2=v_3$
$r\cdot v_3=v_4$	$s\cdot v_3=v_2$
$r\cdot v_4=v_1$	$s\cdot v_4=v_1$

This generalizes to any polygon P_n in the plane.

E2) Let $D=\{d_1,d_2\}$ be the set of all the diagonals of the square S. Then every symmetry of S maps a diagonal d_i to some diagonal d_j so we get an action of Sym S on the set D.

E3) Let $(G,*)$ be a group and let $X=G$. G acts on itself by the left multiplication $g\cdot x:=g*x$. G also acts on itself by right multiplication $g\cdot x:=x*g^{-1}$.

E4) Let $(G,*)$ be a group and let $X=G$. G acts on itself by the conjugation action $g\cdot x=g*x*g^{-1}$.

Some important definitions:

Let $(G,*)$ be a group which acts on a set X. We can define an equivalence relation on X by saying that $x'\sim x''$ if there exists $g\in G$ such that $x''=g\cdot x'$. For $x\in X$, then the equivalence class $[x]$ is denoted by O_x and is called the ORBIT of x. We have:

$$O_x = \{x' \mid x'\sim x\} = \{x' \mid \text{there exists } g\in G \text{ such that } x'=g\cdot x\} = \{g\cdot x \mid g\in G\}.$$

Beacuse of the last equality we sometimes use the notation $G\cdot x$ for the orbit O_x. As usual with the equivalence classes, the orbits partition the set X. Moreover, the group operates independently on each orbit.

We say that a group G acts TRANSITIVELY on a set X if there exists only one orbit of X under the action of G. This orbit is then the whole set X. In that case, for any $x\in X$ we have $X=O_x$. Transitivity is equivalent to the fact that for every pair of elements $x',x''\in X$ there exists $g\in G$ such that $x''=g\cdot x'$.

A STABILIZER of an element $x\in X$ is a subset $G_x\subseteq G$ defined as $G_x:=\{g\in G \mid g\cdot x=x\}$. It is easy to check that G_x is a actually a subgroup of G.

The action of G on X is FAITHFUL if the homomorphism $\Phi:G\to S_x$ is injective. This means that if the element $g\in G$ acts as the identity, so that $g\cdot x=x$ for all $x\in X$, then g must be the identity $e\in G$. Note that even when the action is faithful, there still might exist $g\neq e$ such that $g\cdot x=x$ for some (but not all) $x\in X$.

Examples (these are the continuation of the previous examples E1,…,E4)
E1) G acts transitively and faithfully on V. The stabilizer of v_1 is the subgroup $\{1,rs\}$ which is not normal.
E2) G acts transitively but not faitfully on D because r^2 acts as the identity. The stabilizer of d_1 is the subgroup $\{1,r^2,rs,r^3s\}$ which is normal (it is of index 2).
E3) G acts transitively and faithfully on G. The stabilizer of g is the subgroup $\{e\}$. More generally, for H a subgroup of G we can define two actions of the group H on the set $X=G$:
- $h\cdot x=h*x$
- $h\cdot x=x*h^{-1}$

The orbits of the first actions are the right cosets, and of the second are the left cosets of H. Stabilizer of every point is trivial and the action is faithful.
E4) The orbits of this action are called CONJUGACY CLASSES. The conjugation action is faithful if and only if the center of the group G is trivial. The action is not transitive (unless the group G is trivial) as $\{e\}$ is always one orbit. As an example let's take S_3. We have the following orbits:
- $\{1\}$
- $\{(123),(132)\}$
- $\{(12),(13),(23)\}$

Stabilizers are as follows:
- the stabilizers of 1 is the whole S_3
- the stabilizer of both (123) and (132) is {1,(123),(132)}.
- the stabilizer of (12) is {1,(12)}, (13) is {1,(13)}, and (23) is {1,(23)}

PROP. 6.7.7. Let G act on X. Fix $x \in X$, and let G_x be the stabilizer of x. Then:

(a) for $g,h \in G$ we have $g \cdot x = h \cdot x \Leftrightarrow g^{-1}h \in G_x$
$\Leftrightarrow h \in gG_x$

(b) if $x' \in O_x$ so that $x' = g \cdot x$, then the stabilier G_x' of x' is the conjugate group gG_xg^{-1}

Example: The group (G,∗) operates on the set of all left cosets G/H in the following way. If C is a left coset with representative $c \in C$, so that C=[c]=cH, then we define $g \cdot C$ to be the coset represented by $g*c$:

$g \cdot cH := (g*c)H$

This is a well defined operation, i.e. it doesn't depend on the choice of the representative $c \in C$ of the coset C.

PROP. 6.8.1 Let H be a subgroup of G.

(a) The action of G on the set G/H of left cosets of H is transitive

(b) The stabilizer of the coset $[e]=H \in G/H$ is the subgroup H of G

Note that multiplication of H by an element $h \in H$ generally doesn't act as an identity on H, but it still maps H precisely to H, i.e. $h \cdot H = H$ but for a given $h' \in H$ we might have $h*h' \neq h'$.

Example: Let $G=S_3$ and H={1,(12)}. Then the set of left cosets is the following:

$G/H = \{H, \{(13),(123)\}, \{(23),(132)\}\} = \{C_1,C_2,C_3\}$

Example calculations: $(12)\cdot C_1=(12)\cdot H=H=C_1 \Rightarrow$ (12) is in the stablizer of C_1
$(13)\cdot C_3=(13)\cdot\{(23),(132)\}=\{(132),(23)\}=C_3 \Rightarrow$ (13) is in the stabilizer of C_3
$(123)\cdot C_3=(123)\cdot\{(23),(132)\}=\{(12),1\}=C_1 \Rightarrow$ (123) is not in the stabilizer of C_3

PROP 6.8.4 Let G act on X, and let $x \in X$. Let G_x be the stabilizer of x, and let O_x be the orbit of x. Then there is a bijective map:

$\varepsilon: G/G_x \to O_x$

defined by the rule $\varepsilon(gG_x) := g\cdot x$. This map is compatible with the action of G on G/G_x and on O_x, i.e. $\varepsilon(g\cdot C)=g\cdot\varepsilon(C)$ where $C \in G/G_x$ is a left coset of G_x.

Note that G/G_x is a set of all left cosets and it might not have a natural structure of a group since G_x might not be normal.

If H is a subgroup of G, the number of elements of G/H is called the INDEX of H in G and is denoted by [G:H]. For example $[\mathbb{Z}:n\mathbb{Z}]=n$ since there are n left cosets of $n\mathbb{Z}$ in $\mathbb{Z}$. The number of left cosets is always the same as the number of right cosets.

If G is finite, we know that $|G|=|H||G/H|=|H|[G:H]$.

PROP 6.9.2 (THE COUNTING FORMULA). Let a finite group G act on a finite set X. Then $|G|=|G_x||O_x|$.
This follows from the proposition 6.8.4: $|G|=|G_x||G/G_x|=|G_x||O_x|$.

This means that $|O_x|=[G:G_x]$. The equality $|O_x|=[G:G_x]$ is true even if G is infinite since $|O_x|=|G/G_x|=[G:G_x]$.

Another useful formula uses the fact that the orbits of the action of G on X partition the set X, so that if X is finite, we have finitely many orbits $O_1,O_2,\dots,O_k$ where $O_i \neq O_j$ if $i \neq j$ and then

$|X|=|O_1|+|O_2|+\dots+|O_k|$

Dodecahedron example (https://www.mathsisfun.com/geometry/dodecahedron.html):
Sym D acts on the set of faces F of the dodecahedron. The stabilizer of a particular face $f \in F$ consists of 5 rotations by $k\cdot 2\pi/5$ as well as 5 reflections about planes which pass through a vertex $v_i \in f$ of the face f and the midpoint of the side sitting opposite to v_i and are perpendicular to f. So, the stablizer has 10 elements. Since the dodecahedron has 12 faces and Sym D operates transitively on the set of faces, we conclude $|\text{Sym } D|=10\cdot 12=120$.

Note: The calculation in the textbook is a bit different. The reason is that the group which is calculated there is not Sym D, but the group of rotational symmetries which is a subgroup of the full group of symmetries Sym D. The stablizer in this case only includes 5 rotations and no reflections. Since the group of rotational symmetries also acts transitively on the set of faces, the order of the group of rotational symmetries of the dodecahedron is 60.

Now we can count the number of vertices. Note that Sym D acts transitively on the set of vertices and that $|G_v|=6$ (three rotations and three reflections), hence $|V|=|G|/|G_v|=20$. For the edges we get $|E|=|G|/|G_e|$ and $|G_e|=4$ (2 rotations and 2 reflections) so $|E|=30$.

Here we used symmetry to solve the problem, i.e. the fact that dodecahedron had certain symmetry helped us to calculate $|V|$ and $|E|$. You can try to do the analogous calculations for:

- cube (https://www.mathsisfun.com/geometry/hexahedron.html)
- tetrahedron (https://www.mathsisfun.com/geometry/tetrahedron.html)
- octahedron (https://www.mathsisfun.com/geometry/octahedron.html)
- icosahedron (https://www.mathsisfun.com/geometry/icosahedron.html)

Extra examples: some actions of S_3

Let S_3 act on two sets U and V of order 3. Decompose the U×V into orbits for the diagonal action when:

(a) operation on U and V is transitive
(b) operation of U is transitive, while the orbits of the action on V are $\{v_1\}$ and $\{v_2,v_3\}$.

For (a) let's look at the action of S_3 on U. For $u\in U$ we have that $|O_u|=3 \Rightarrow |G_u|=2$. Hence $G_u=\{1,s\}$ where $s\in\{(12),(13),(23)\}$. Since the elements of the same orbit have conjugate stabilizers, we can assume that we've chosen u such that $G_u=\{1,(12)\}$. Similarly, choose $v\in V$ such that $G_v=\{1,(12)\}$. Then the stablizer of (u,v) contains 1 and (12).

Let $r=(123)$ and $r^2=(132)$. Then $ru\neq u$ and $r^2u\neq u$ since these elements are not in G_u. Moreover, we claim that $r^2u\notin\{u,ru\}$. We only need to check $r^2u\neq ru$ and this holds because the equality would imply $ru=u$. Hence, $U=\{u,ru,r^2u\}$ and in the same way $V=\{v,rv,r^2v\}$.

Note that the orbit of (u,v) contains at least three different elements, namely (u,v), (ru,rv), and (r^2u,r^2v). Since its stabilizer has at least 2 elements, we conclude by the orbit stabilizer theorem that the orbit has to be equal to $\{(u,v), (ru,rv), (r^2u,r^2v)\}$.

To calculate the stabilizer of (ru,v) set $g(ru,v)=(ru,v)$. This implies $gv=v$ hence g is either 1 or (12). In the latter case we would have $(12)ru=ru \Rightarrow (12)(123)u=ru \Rightarrow (132)(12)u=ru \Rightarrow r^2u=ru \Rightarrow ru=u$ which is a contradiction. Hence, the stabilizer of (ru,v) is trivial, and hence the orbit of (ru,v) has 6 elements. So there are two orbits:

$\{(u,v), (ru,rv), (r^2u,r^2v)\}$
$\{(ru,v), (r^2u,v), (u,rv), (r^2u,rv), (u,r^2v), (ru,r^2v)\}$

For (b) let's look at the stabilizers. We again pick $u\in U$ such that $U=\{u,ru,r^2u\}$ and $G_u=\{1,(12)\}$. We calculate the stabilizer of (u,v_1).
$g(u,v_1)=(u,v_1)$ is equivalent to $gu=u$ since every g acts trivially on v_1. Hence, the stabilizer of (u,v_1) is actually $\{1,(12)\}$. This means that the orbit has order 3. Since we can easily find three distinct elements in the orbit, we're done in this case:

$\{(u,v_1), (ru,v_1), (r^2u,v_1)\}$

Now let's look at the stabilizer of v_2. Since the orbit has order 2, the stabilizer has order 3. Hence it must be $\{1,(123),(132)\}$ since that is the only subgroup of S_3 of order 3. Now we calculate the stabilizer of (u,v_2):

$g(u,v_2)=(u,v_2)$ implies $gu=u$ and $gv_2=v_2 \Rightarrow g\in\{1,(12)\}$ and $g\in\{1,(123),(132)\}$ so we conclude $g=1$

Hence, the stabilizer is trivial and so the orbit has order 6. So we have two orbits:

$\{(u,v_1), (ru,v_1), (r^2u,v_1)\}$
$\{(u,v_2), (ru,v_2), (r^2u,v_2), (u,v_3), (ru,v_3), (r^2u,v_3)\}$

Example: operations on subsets

If G acts on X, and if S is a subset of X, then $gS:=\{g\cdot s \mid s\in S\}$ is a subset such that $|S|=|gS|$ since the function $s\mapsto g\cdot s$ is a bijection between S and gS. This allows us to define the action of G on the set of all subsets of X of order r.

Example: Sym C where C is a cube. This group has 48 elements. This group acts on the set of faces F of C, hence also on the set of all unordered pairs of faces of C (case r=2). There are 15 pairs and they form two orbits

{pairs of opposite faces} ~ easily can rotate one into another, there are 3 such pairs
{pairs of adjacent faces} ~ rotate first one face to match, and then rotate the other one, there are 15-3=12 elements in this orbit

The stabilizer of a subset S of order r is the set of all g such that $gS=S$. Note that this doesn't mean that $g\cdot s=s$ for every $s\in S$, just that the elements of S get mapped back to S and they will likely get permuted.