Let $f:V\to V$ be a linear operator on a vector space V. If we pick a basis $\{e_1,\dots,e_n\}$ we can represent f by a matrix A. This representation depends on the choice of the basis. If we choose another basis $\{f_1,\dots,f_n\}$, and if T is the transition matrix, then the operator f in the new basis will have the matrix $B=T^{-1}AT$ - the new matrix is a conjugate of A.

Now let $\rho:G\to GL(V)$ be a representation of a finite group G. If we pick a basis $\{e_1,\dots,e_n\}$ of V we can represent a chosen operator $\rho(g)$ by a matrix A. However, the matrix A can be big and may contain many non-zero entries suggesting that it encodes a lot of data. This is misleading - since G is finite we have $g^n=1$ for $n=|G|$. Hence, $\rho(g)$ satisfies the polynomial equation $x^n-1=0$. Since the minimal polynomial $\mu(x)$ of the operator $\rho(g)$ has to divide x^n-1 and since $x^n-1=(x-1)(x-\zeta)\cdots(x-\zeta^{n-1})$ we conclude that $\mu(x)$ is a product of distinct linear factors which means that $\rho(g)$ diagonalizes in some basis. So, if we choose a suitable basis, the matrix corresponding to $\rho(g)$ will be diagonal. Furthemore, the entries on the diagonal are some n-th roots of unity because $\rho(g)^n=1$. So we see that ultimately every operator $\rho(g)$ can be represented by a diagonal matrix whose diagonal entries are highly restricted.

Note that this doesn't mean that there is a basis in which all the operators $\rho(g)$ diagonalize simultaneously as the operators $\rho(g)$ and $\rho(h)$ may not commute!

It turns out that the information stored in $\rho(g)$ can be quite accurately compressed to a single complex number, namely its trace.

The character χ_ρ of a representation ρ on a vector space V is the function $\chi_\rho:G\to\mathbb{C}$ such that $\chi_\rho(g)$ is the trace of the linear operator $\rho(g)$.

The dimension of the character is the dimension of V. The character of an irreducible representation ρ is called irreducible character.

Properties (PROP. 10.4.2):
- $\chi_\rho(1)$ is the dimension of χ_ρ
- the character is constant on conjugacy classes
- if χ_ρ has dimension d, then the complex number $\chi_\rho(g)$ is a sum of d roots of unity (e.g. it cannot be π)
- $\chi_\rho(g^{-1})$ is the complex conjugate of $\chi_\rho(g)$ ~ take the inverse of the diagonalized $\rho(g)$
- the character of $\rho'\oplus\rho''$ is the sum $\chi'+\chi''$
- isomorphic representations have the same characters

Let $n=|G|$ and let $\mathbb{C}^G$ be the set of all functions $G\to\mathbb{C}$. Every such a function can be considered as a n-tuple of complex numbers so the vector space $\mathbb{C}^G$ is isomorphic to $\mathbb{C}^n$. On $\mathbb{C}^G$ we introduce the structure of an unitary space using the standard Hermitian product $(\cdot,\cdot)$ on $\mathbb{C}^n$ scaled by the factor $|G|$:

$\langle f,g\rangle=(f,g)/|G|$

The set of all characters Ch(G) is a subset of $\mathbb{C}^G$ (it is not a subgroup or a subspace) and we can calculate $\langle\chi',\chi''\rangle$ using the inner product defined on $\mathbb{C}^G$.

Examples of characters of Z/4Z:

	χ_1	χ_2	χ_3	χ_4	χ_5	
1	1	1	1	1	2	
r	1	i	-1	-i	0	← this is NOT a character table
r^2	1	-1	1	-1	-2	
r^3	1	-i	-1	i	0	
					↳rotations in $\mathbb{C}^2$	

Some calculations:
- $\langle\chi_2,\chi_3\rangle=0$ - $\langle\chi_2,\chi_4\rangle=0$ - $\langle\chi_2,\chi_2\rangle=1$
- $\langle\chi_2,\chi_5\rangle=1$ - $\langle\chi_3,\chi_2\rangle=0$ - $\langle\chi_5,\chi_5\rangle=2$

Note that $\chi_5=\chi_2+\chi_4$

Main results:
[1] Representation theory of a finite group is completely encoded by its characters since two representations ρ' and ρ'' are isomorphic if and only if their characters are equal.

[2] There are only finitely many irreducible characters, therefore there are only finitely many isomorphism classes of irreducible representations. The number of irreducible characters is equal to the number of conjugacy classes of G.

[3] The set of irreducible characters χ_i and χ_j is orthonormal:

$$\sum_g \chi_i(g)\overline{\chi_j(g)} = \begin{cases} 0 & \text{if } i\neq j \\ |G| & \text{if } i=j \end{cases}$$

[4] Let $\chi_1,\dots,\chi_k$ be all the irreducible characters of a finite group G and let d_i be the dimension of χ_i. Then:

- d_i divides the order $|G|$
- $|G|=d_1^2+\dots+d_k^2$

[5] Column orthogonality:

$\sum_{\chi_i} \chi_i(g)\overline{\chi_i(h)}$ is $|G|/|C|$ if g and h belong to the same conjugacy class C, and 0 otherwise
↳ this is the sum over all IRREDUCIBLE characters

These results allow us to decompose any character χ as a linear combination of the irreducible characters χ. Namely, if we set $n_i=\langle\chi,\chi_i\rangle$ then $\chi=n_1\chi_1+\dots+n_k\chi_k$. If ρ_i is a representation corresponding to χ_i, then $\rho=n_1\rho_1\oplus\dots\oplus n_k\rho_k$.

Furthermore, if χ_ρ is the character of a representation ρ, then $\langle\chi_\rho,\chi_\rho\rangle=1$ implies that ρ is an irreducible representation. Otherwise we would have in the decomposition $\rho=n_1\rho_1\oplus\dots\oplus n_k\rho_k$ at least two numbers $n_i\geq1$ and hence $\langle\chi_\rho,\chi_\rho\rangle\geq2$. This fact can be used to check whether a given representation is irreducible.

It turns out that the span of the characters in the vector space $\mathbb{C}^G$ is the set of all functions $f:G\to\mathbb{C}$ with the property that f maps conjugate elements of G to the same complex value. Hence, irreducible characters form a orthonormal basis for that subspace.

Character tables:
The irreducible characters can be assembled in a table, for example D_3 has 3 conjugacy classes:

	(1)	(2)	(3)	
	1	r	s	← number of elements in the conjugacy classes / ← representative element of the conjugacy class
χ_1	1	1	1	
χ_2	?			← value of the character on each conjugacy class $\{1\}$, $\{r,r^2\}$ and $\{s,rs,r^2s\}$
χ_3	?			

The first row corresponds to the trivial representation $\rho:G\to GL(\mathbb{C})$ which maps every $g\in G$ to the identity operator. The first column consists of values $\chi_i(1)$ which are dimensions d_i of the characters. Using the fact that d_i divides $|G|$ and that $|G|=1^2+?^2+?^2$ we can easily determine the first column:

	(1)	(2)	(3)
	1	r	s
χ_1	1	1	1
χ_2	1	?	?
χ_3	2		

If N is a normal subgroup of G, and $\rho:G/N\to GL(V)$ is an irreducible representation of the group G/N, then we can compose it with the projection $G\to G/N$ to get an irreducible representation of G. So $G\to G/N\to GL(V)$ is an irreducible representation whose character is just the composition of the projection $G\to G/N$ with the character $\chi:G\to\mathbb{C}$ of ρ. In our case we can take $N=C_3$ and then $G/N\simeq\mathbb{Z}/2\mathbb{Z}$. Character table for $\mathbb{Z}/2\mathbb{Z}$ can be easily determined:

	(1)	(1)
	0	1
ϑ_1	1	1
ϑ_2	1	?

The ? value has to be -1 by the orthogonality relations. If ϑ_2 corresponds to the representation ρ_2, we get a representation of G by composing $D_3\to D_3/C_3\simeq\mathbb{Z}/2\mathbb{Z}$ with $\rho_2:G\to GL(\mathbb{C})$. Its character is just the composition of the projection $D_3\to D_3/C_3$ with ϑ_2. So we get a irreducible character mapping $1\mapsto1$, $r\mapsto1$, and $s\mapsto-1$. So this character is a new one and we can fill the second row.

	(1)	(2)	(3)
	1	r	s
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	a	b

Note that if we consider the trivial character ϑ_1 of D_3/C_3, then the procedure above yields the character χ_1 of G which was already in the table.

For the last row you can use several approaches.
1) Using row orthogonality conditions we have $\langle\chi_i,\chi_j\rangle=0$ for $i\neq j$ which is a system of linear equations with variables being mising values:

$$0=\langle\chi_3,\chi_1\rangle = 2 + 2a + 3b$$
$$0=\langle\chi_3,\chi_2\rangle = 2 + 2a - 3b$$

This system can be easily to get $a=-1,b=0$. When calculating the inner product of rows, don't forget to take into account how many times each value in the character table appears in the sum (it apppears as many as there are elements in the corresponding conjugacy class). Also, don't forget the conjugates.

Note that in principle you can also use equation $\langle\chi_3,\chi_3\rangle=1$ which yields $4+2|a|^2+3|b|^2=6$. However, this equation is quadratic in absolute values of complex numbers a, b and it may be difficult to extract usable information from it.

If we have two missing rows, then the linear system might not uniquely determine the missing values, but it will give some relationships between them. Consider the following table:

	(1) 1	(2) r	(3) s
χ_1	1	1	1
χ_2	1	c	d
χ_3	2	a	b

We can form the following equations:

$$0=\langle\chi_3,\chi_1\rangle = 2 + 2a + 3b$$
$$0=\langle\chi_2,\chi_1\rangle = 1 + 2c + 3d$$
$$0=\langle\chi_3,\chi_2\rangle = 2 + 2a\bar{c} + 3b\bar{d} \quad \text{(don't forget the conjugates)}$$

Note that the last equation is not linear.

2) We can use the column orthogonality too, and the same caveats as in the previous case apply.

3) If we know an irreducible representation of D_3 on the plane we can just read off traces. In this case you have to prove that the corresponding representation is irreducible over $\mathbb{C}$, for example by calculating $\langle\chi,\chi\rangle$. If you happen to have a representation which is reducible, then you can break it down into irreducible ones and you can use those factors to fill your table (although you may just get the ones already present in the table).

Finally:

	(1) 1	(2) r	(3) s
χ_1	1	1	1
χ_2	2	1	-1
χ_3	2	-1	0

One-dimensional characters:
A one dimensional character is the character of a representation of G on a one-dimensional vector space. Such characters must be irreducible because the corresponding representation ρ is one-dimensional. This implies that such a character is a group homomorphism $G\to\mathbb{C}^*$ since $tr(AB)=tr(A)tr(B)$ holds if A and B are 1x1 matrices.

A character of dimension greater than 1 is NOT necessarily a homomorphism since $tr(AB)\neq tr(A)tr(B)$ in general.

Regular representations come from group actions by composing the action $G\to S_x$ with the standard representation of the symmetric group $S_x\to Gl(\mathbb{C}^n)$ where $n=|X|$.