Additional problems

1. Show that a noncommutative group of order p^3 has exactly $p+4$ normal subgroups.
2. Let H be a subgroup of index n in a group G. Show that there is a homomorphism $f\colon G \to S_n$ such that $H = f^{-1}(S_{n-1})$.
3. Let G be the set of all invertible $n \times n$ matrices each of whose rows and columns sums to 1. Show that G is a subgroup of $\mathrm{GL}(n)$ isomorphic to $\mathrm{GL}(n-1)$.
4. Let G be a commutative finite group which contains two distinct elements of order 2. Show that then 4 divides $|G|$. Is this true if G is not commutative?
5. Let G be a finite group and let $\varphi\colon G \to \mathbb{C}^\times$ be a nontrivial homomorphism. Calculate $\sum_{g \in G} \varphi(g)$.
6. Show that the group $\mathbb{Z}/4\mathbb{Z}$ is not isomorphic to a product of simple groups. Do the same for S_3.
7. Show that if G is a noncommutative finite group, then $|Z(G)| \leq \frac{1}{4}|G|$.
8. For $\sigma \in S_m$ and $\tau \in S_n$ calculate the parity of the permutation of $\{1, \ldots, m\} \times \{1, \ldots, n\}$ which maps (i, j) to $(\sigma(i), \tau(j))$.
9. Let σ be a product of all the elements of S_n in some order. Is σ even or odd?
10. Let G be a p-group. Show that for every divisor d of $|G|$ there exists a normal subgroup of G of order d.
11. Let G be a finite group such that for every divisor d of $|G|$ there exists precisely one subgroup of G of order d. Show that G is cyclic.
12. Let G be a finite group such that $g^2 = 1$ for every $g \in G$. Prove that $G \cong \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$.
13. For a finite set S determine the structure of the group $\mathcal{P}(S)$ under the operation of symmetric difference.
14. Show that there cannot exist an action of the group $\mathbb{Z}$ on the set of all smooth functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $1 \cdot f = f'$.
15. Let G be a finite group such that the action of $\mathrm{Aut}(G)$ on G has only two orbits. Prove that G is abelian.
16. Show that only the trivial group and the group $\mathbb{Z}/2\mathbb{Z}$ have the identity map as their sole automorphism.
17. Prove that there is no group G such that $\mathrm{Aut}(G) \cong \mathbb{Z}$.
18. Show that the representation of the group $\mathbb{Z}$ on $\mathbb{C}^2$ such that $1 \in \mathbb{Z}$ acts by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ cannot be decomposed in a direct sum of irreducible representations.
19. Let V be a vector space of finite dimension. Show that the group $\mathbb{Z}$ has infinitely many nonisomorphic representations on V.
20. Show that for every finite group G of order n there is a subset $X \subseteq \mathbb{R}^{n-1}$ such that $G \cong \mathrm{Sym}\, X$.