Groups and homomorphisms

1. Investigate which of the following pairs $(S, *)$ consisting of a set S and an operation $*$ on the elements of the set S have a structure of a group:

(a) $(\mathbb{N}, +)$	(e) $(\mathbb{Z}, \cdot)$	(i) $(\mathbb{Q}^\times, \cdot)$	(m) $(\mathbb{R}^+, \cdot)$	(q) $(\mathbb{C}^\times, \cdot)$
(b) $(\mathbb{N}, \cdot)$	(f) $(\mathbb{Q}, +)$	(j) $(\mathbb{Q}^\times, /)$	(n) $(\mathbb{R}^+, /)$	(r) $(\mathbb{C}^\times, /)$
(c) $(\mathbb{Z}, +)$	(g) $(\mathbb{Q}, -)$	(k) $(\mathbb{R}^\times, \cdot)$	(o) $(\mathbb{C}, +)$	(s) $(S^1, \cdot)$
(d) $(\mathbb{Z}, -)$	(h) $(\mathbb{Q}, \cdot)$	(l) $(\mathbb{R}^+, +)$	(p) $(\mathbb{C}, \cdot)$	(t) $(\mathbb{R}^3, \times)$

2. Find the elements of finite order in additive groups $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.

3. Find the elements of finite order in multiplicative groups $\mathbb{Q}^\times$, $\mathbb{R}^\times$, $\mathbb{C}^\times$.

4. Let G be a commutative group. Show that the set $\{g \in G \mid g \text{ has finite order}\}$ is a subgroup of G.

5. Find a counterexample to the statement that if G is any group, then the set $\{g \in G \mid g \text{ has finite order}\}$ is a subgroup of G.

6. Let G be a group and g an element in G. Show that the set $\langle g \rangle := \{g^n : n \in \mathbb{Z}\}$ is a subgroup of G.

7. Show that $\langle g \rangle$ is infinite if and only if the order of g is infinite. Show that if g has finite order, then its order is equal to the order of $\langle g \rangle$.

8. Which of the following maps are group homomorphisms? Which of them are injective? What are their images?

(a) $\exp : \mathbb{R} \to \mathbb{R}^\times$	(d) $g \mapsto g^{-1} : G \to G$	(g) $g \mapsto g^2 : G \to G$
(b) $\exp : \mathbb{C} \to \mathbb{C}^\times$	(e) $g \mapsto 1 : G \to G$	(h) $A \mapsto A^t : \mathrm{GL}(2) \to \mathrm{GL}(2)$
(c) $\det : \mathrm{GL}(2) \to \mathbb{R}^\times$	(f) $z \mapsto \bar{z} : \mathbb{C}^\times \to \mathbb{C}^\times$	(i) $\delta_t : G^T \to G$

9. Let G be a group and T a set. Show how to define an injective group homomorphism $G \to G^T$.

10. Let G be a group and $g \in G$. Show that if $g^n = 1$, then the order of g divides the number n. Find an example when these two numbers are different.

11. Let $f \colon G \to H$ be a group homomorphism and let the element $g \in G$ have finite order. Show that $f(g)$ has finite order and that the order of $f(g)$ divides the order of g.

12. Show that the matrices of the form $\left(\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right)$ where $x, y \in \mathbb{R}$ and $x \neq 0$ or $y \neq 0$ form a group under the operation of matrix multiplication. Show that this group is isomorphic to the group $\mathbb{C}^\times$. Determine which matrix operations correspond to $z \mapsto \bar{z}$ and $z \mapsto |z|$.

13. Show that the matrices of the form $\left(\begin{smallmatrix} z & w \\ -\bar{w} & \bar{z} \end{smallmatrix}\right)$ where $z, w \in \mathbb{C}$ and $z \neq 0$ or $w \neq 0$ form a group under the operation of matrix multiplication. We denote this group by $\mathbb{H}^\times$.

14. For the following pairs of groups G and H determine if G is isomorphic to H:

(a) $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$	(d) $(\mathbb{Q}, +)$ and $(\mathbb{Q} \times \mathbb{Q}, +)$	(g) $(\mathbb{C}^\times, \cdot)$ and $(\mathbb{R}, +)$
(b) $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$	(e) $(\mathbb{R}^\times, \cdot)$ and $(\mathbb{R}, +)$	(h) $(\mathbb{C}^\times, \cdot)$ and $(\mathbb{R}^\times, \cdot)$
(c) $(\mathbb{Z}, +)$ and $(\mathbb{Z} \times \mathbb{Z}, +)$	(f) $(\mathbb{R}^\times, \cdot)$ and $(\mathbb{C}, +)$	(i) $(\mathbb{H}^\times, \cdot)$ and $(\mathbb{C}^\times, \cdot)$

15. Let G be a group. Is G isomorphic to G°?

16. Define the following matrices:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Prove that the eight matrices $\pm 1, \pm I, \pm J, \pm K$ form a finite subgroup of $\mathbb{H}^\times$. We denote this group by Q.

17. Let S be a set. Prove that the power set $\mathcal{P}(S)$ with the operation $A \Delta B := (A - B) \cup (B - A)$ has a structure of a group (you may assume associativity).

18. Prove that if a and b are elements of a group G, then ab and ba have the same order.

19. Let G be a group with elements $a, b \in G$ which satisfy $a^2 = b^2$ and $abab = 1$. Show that a and b must have finite order.

20. Let G be a finite group. Show that there exists a natural number n such that $g^n = 1$ for all $g \in G$.

21. Let G be a finite group. Show that the only homomorphism $\mathbb{Q} \to G$ is the trivial one.

22. Pick any two groups from the following list[1] and classify all the homomorphisms between them: $\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$, S_3, $\mathbb{Q}$. Determine which ones are isomorphisms.

23. Prove that an infinite group is cyclic, if and only if it is isomorphic to all of its subgroups except the trivial one $\{e\}$.

24. Does there exist a proper subgroup of $\mathbb{Q}$ which is isomorphic to $\mathbb{Q}$? Does there exist a proper subgroup of $\mathbb{R}$ which is isomorphic to $\mathbb{R}$?

25. Let G be a group and $g \in G$ an element of order 15. Show that the equation $x^7 = g$ has a solution in G.

[1] There are 25 possibilities

Normal subgroups and quotients

1. Show that any subgroup of index 2 has to be normal.

2. For the following pairs of groups G and H draw the cosets of H in G:

   (a) $\mathbb{Z}$ and $6\mathbb{Z}$
   (b) $\mathbb{R}^\times$ and $\langle -1 \rangle$
   (c) $\mathbb{C}^\times$ and S^1
   (d) $\mathbb{C}^\times$ and $\mathbb{R}^+$
   (e) $\mathbb{C}^\times$ and $\mathbb{R}^\times$
   (f) $\mathbb{C}^\times$ and C_7

3. For the following pairs of groups G and H check if H is a normal subgroup of G and in the case it is try to determine the quotient G/H:

   (a) $\mathrm{GL}(2)$ and $\mathrm{SL}(2)$
   (b) $\mathbb{R}$ and $\mathbb{Z}$
   (c) $\mathbb{R}$ and $\mathbb{Q}$
   (d) $\mathbb{Q}$ and $\mathbb{Z}$
   (e) $\mathbb{C}$ and $\mathbb{R}$
   (f) $\mathbb{C}$ and $\mathbb{Z}$
   (g) $\mathbb{R}^\times$ and $\langle -1 \rangle$
   (h) $\mathbb{R}^+$ and $\langle 2 \rangle$
   (i) $\mathbb{C}^\times$ and S^1
   (j) $\mathbb{C}^\times$ and $\mathbb{R}^+$
   (k) $\mathbb{C}^\times$ and $\mathbb{R}^\times$
   (l) $\mathbb{C}^\times$ and C_n
   (m) $\mathbb{C}^\times$ and $\langle 2 \rangle$
   (n) S_3 and any subgroup of S_3
   (o) Q and any subgroup of Q

4. Find a few examples of groups G with a normal subgroup $N \subseteq G$ such that $G \ncong N \times G/N$.

5. Let H be a subgroup of G. Show that H is normal if and only if the sets of left and right cosets of H coincide.

6. Let N be a normal subgroup of G, and let $\pi : G \to G/N$ be the canonical homomorphism. Fix a homomorphism $f : G \to H$:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] & H \\ G/N & \end{tikzcd}$$

   Show that there exists a homomorphism $\bar{f} : G/N \to H$ such that $f = \bar{f} \circ \pi$ if and only if $N \subseteq \ker f$:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] & H \\ G/N \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

   Is it possible to find two different functions $G/N \to H$ both of which when composed with π give f?

7. Using the previous problem, show that there are n homomorphisms $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$.

8. If every subgroup of a finite group G is normal, must G be abelian?

Group actions

1. Fix a set S. Show that there is a bijection $\{\text{partitions of the set } S\} \to \{\text{equivalence relations on the set } S\}$ which maps a partition $\{P_\alpha\}_{\alpha \in A}$ into the relation $x \sim y$ if x and y belong to the same P_α.

2. Let S be a set and $\{P_\alpha\}_{\alpha \in A}$ a partition of S. Show that there is a group action on S whose orbits are precisely the sets P_α.

3. Let S be a set with an equivalence relation. Show that there is a group G and an action of G on S such that $x \sim y$ if and only if there exists $g \in G$ such that $g \cdot x = y$.

4. Let the group G act on the set X and denote by Y^X the set of all functions from X to Y. Show that the rule $(g \cdot f)(x) := f(g^{-1} \cdot x)$ defines an action of G on Y^X.

5. Let G be a group. Show that $\mathrm{Aut}(G)$ acts on G by the rule $f \cdot g := f(g)$. Does G act on $\mathrm{Aut}(G)$?

6. Prove that the rule $f \cdot S := f(S)$ defines an action of the group S_X on the set $\mathcal{P}(X)$. Determine the orbits and stabilizers.

7. Determine the orbits and stabilizers of the following group actions:

   (a) $\mathbb{R}$ acts on $\mathbb{R}^2$ by rotations
   (b) $\mathbb{Z}$ acts on $\mathbb{R}$ by translations
   (c) $\mathbb{R}$ acts on $\mathbb{C}$ by addition
   (d) $\mathbb{R}^+$ acts on $\mathbb{C}$ by multiplication
   (e) S^1 acts on $\mathbb{C}$ by multiplication
   (f) $\mathrm{GL}(2)$ acts on $\mathbb{R}^2$ by left multiplication
   (g) $\mathrm{SL}(2)$ acts on $\mathbb{R}^2$ by left multiplication
   (h) $\mathrm{SL}(2)$ acts on $\mathrm{GL}(2)$ by left multiplication

8. Let S be the set of solutions of the differential equation $\ddot{x}(t) = -x(t)$ which models the motion of a simple harmonic oscillator. Every solution is a linear combination of the functions $\cos t$ and $\sin t$ and it is completely determined by two numbers $x(0)$ and $\dot{x}(0)$ which correspond to the inital position and the initial velocity of the particle. Explicitly write down the function $F\colon \mathbb{R}^2 \to S$ which maps $(x_0, v_0) \in \mathbb{R}^2$ to the unique solution $F_{x_0,v_0} \in S$ with initial position x_0 and initial velocity v_0. Show that F is an isomorphism of vector spaces.

9. Show that the rule $(\lambda \cdot x)(t) = x(t + \lambda)$ defines an action of the additive group $\mathbb{R}$ on S. Using the isomorphism F we can translate this action to an action of $\mathbb{R}$ on $\mathbb{R}^2$. Describe explicitly what we get.

10. Determine the orbits and stabilizers of the previously defined action of $\mathbb{R}$ on S.

11. Let S be the set of solutions of the differential equation $\ddot{x}(t) = 1$ which models the motion of a particle in acceleration. Determine the orbits and stabilizers of the time translation action of $\mathbb{R}$ on S.

12. Let G be a group with a subgroup H of finite index. Prove that G has a normal subgroup N of finite index contained in H. What can you say about the index of N?

13. If G is a finitely generated group, prove that there are at most finitely many subgroups of index n in G.

14. Let G be a group of odd order and $g \in G$ an element which is not the identity. Show that g and g^{-1} are not conjugate.

15. Find all finite groups that have exactly two conjugacy classes.

Sylow theorems

1. Show that $|G| = p^k$ for some prime $p \iff$ order of every element of G is a power of p.
2. Show that there is no simple group of order 200.
3. Show that every group of order 340 has a normal cyclic subgroup of order 85.
4. Compute the number of elements of order 7 in a simple group of order 168.
5. Prove there is no simple group of order 351.
6. Calculate the number of Sylow 3-subgroups and the number of Sylow 5-subgroups of S_5. Check that the numbers you obtain are consistent with Sylow theorems.
7. If p is a prime number, find all Sylow p-subgroups of S_p.
8. Prove that if p is a prime number, then $(p-1)! \equiv -1 \pmod{p}$.
9. Prove Wilson's theorem: n is a prime number $\iff (n-1)! \equiv -1 \pmod{n}$.
10. Show that every group of order 48 has a normal subgroup of order 8 or 16.
11. Let H be a proper subgroup of G. If $|G/H| \leq 4$, show that G is not simple unless G is $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$.
12. Let $|G| = p$ where p is a prime number $\implies G$ is simple.
13. Let $|G| = pq$ where p, q are prime numbers $\implies G$ is not simple.
14. Let $|G| = pqr$ where p, q, r are prime numbers $\implies G$ is not simple. Break this into cases:
    (a) $|G| = p^3$
    (b) $|G| = p^2q$, where $p \neq q$
    (c) $|G| = pqr$, where p, q, r are all different
15. Find all simple groups of order $|G| < 60$.
16. Assume that the class equation of G is $60 = 1 + 20 + 15 + 12 + 12$. Prove that G has to be simple.
17. Prove there is no simple group of order 132.
18. Prove there is no simple group of order 495.
19. Prove there is no simple group of order 90.
20. Prove that if N is a normal subgroup of G that contains a Sylow p-subgroup of G, then the number of Sylow p-subgroups of N is the same as that of G.
21. Show there is only one group of order 1001 up to isomorphism.
22. Let H be a Sylow p-subgroup and let K be any p-subgroup. Show that if K is contained in the normalizer of H, then $K \subseteq H$.
23. Prove there is no simple group of order 520.
24. Show that a group of order 108 has a normal subgroup of order 9 or 27.
25. Prove there is no simple group of order 144.
26. Assume that no Sylow subgroup of G is normal. Is G simple?
27. Suppose that G is an infinite simple group. Show that for every proper subgroup H of G, the index $[G : H]$ is infinite.

Permutations

1. Show that if X and Y have the same cardinality, then the groups S_X and S_Y are isomorphic.
2. Show that the transpositions (12), (13), $\ldots$, $(1n)$ generate S_n.
3. Show that the transpositions (12), (23), (34), $\ldots$, $(n-1\,n)$ generate S_n.
4. Show that (12) and $(12\ldots n)$ generate S_n if $n \geq 2$.
5. Is every power of a cycle in S_n again a cycle?
6. Is the subgroup of S_{2n} generated by the transpositions switching $2k-1$ and $2k$ commutative?
7. Let σ be an odd permutation in S_n. Determine when the equation $\sigma x = x\sigma^4$ can be solved.
8. Prove that (12345) and (12354) are conjugate in S_5, but not in A_5.
9. Prove that the symmetric group S_n is a maximal subgroup of S_{n+1}.
10. Show that if G is a subgroup of S_n which contains an odd permutation, then $G \cap A_n$ is of index 2 in G.
11. Show that if G is a subgroup of S_n of index 2, then $G = A_n$.
12. Show that S_{m+n} has a subgroup of order mn.
13. Show that the number of elements of order 2 in S_n is odd.
14. For every $\tau \in S_n$ calculate the parity of the permutation of the set S_n given by $\sigma \mapsto \tau\sigma$.
15. For every $\tau \in S_n$ calculate the parity of the permutation of the set S_n given by $\sigma \mapsto \tau\sigma\tau^{-1}$.
16. Is there a subgroup of order 15 in A_5?
17. Is there an integer $n > 1$ such that every group of order at most n can be embedded in S_{n-1}?
18. Prove that every finite group G of order n can be embedded in A_{n+2}.
19. Can S_n be embedded in A_{n+1} if $n > 2$?
20. Show that the number $(a+b+c)!$ is divisible by $a! \cdot b! \cdot c!$.
21. Show that the 15 puzzle is unsolvable:

1	2	3	4
5	6	7	8
9	10	11	12
13	**14**	**15**	

$\xrightarrow{?}$

1	2	3	4
5	6	7	8
9	10	11	12
13	**15**	**14**	

22. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. For every permutation $\sigma \in S_n$ we denote by A_σ the matrix in the standard basis of the linear operator which sends e_i to $e_{\sigma(i)}$. Show that $\sigma \mapsto A_\sigma$ is an injective group homomorphism $S_n \to \mathrm{GL}(n)$. What is $\det A_\sigma$? What is $\operatorname{Tr} A_\sigma$?
23. Does there exist an embedding $A_n \to \mathrm{SL}(n)$?
24. Show that the permutations $\sigma\tau$ and $\tau\sigma$ have the same number of fixed points.

Representations

1. When is the representation of the group $\mathbb{Z}/n\mathbb{Z}$ by rotations of the plane $\mathbb{R}^2$ irreducible?
2. Decompose the standard representation of the group $\mathbb{Z}/n\mathbb{Z}$ acting on $\mathbb{C}^2$ into irreducible representations.
3. Is the standard representation of the group D_n irreducible?
4. Show that the representation of S_4 as the rotational symmetry group of a cube is irreducible.
5. Show that the representation of A_4 as the rotational symmetry group of a tetrahedron is irreducible.
6. Show that the representation of the group Q by matrices

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

   is irreducible.
7. Determine the conjugacy classes and the character tables of the following groups:
   (a) $\mathbb{Z}/n\mathbb{Z}$
   (b) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
   (c) S_4
   (d) A_4
   (e) Q
   (f) D_4
   (g) D_5
   (h) D_6
8. Calculate the character table of a noncommutative group of order 21.
9. Calculate the character table of a noncommutative group of order 55.
10. Determine the decomposition of the following complex representations into irreducible ones:
    (a) the standard representation of $\mathbb{Z}/n\mathbb{Z}$ acting on $\mathbb{C}^2$
    (b) the standard representations of A_4 and S_4
    (c) the representation of S_4 as the full symmetry group of a tetrahedron
11. Decompose the restriction of each irreducible character of S_4 into irreducible characters of A_4. Do the same for the cyclic subgroups of Q as well as the rotational subgroups of D_4, D_5, and D_6.
12. Let χ be a character of dimension d. Show that the modulus of $\chi(g)$ is at most d. When is it equal to d?
13. Show that every p-group has a 1-dimensional representation which is not trivial.
14. Calculate the character table of a noncommutative group of order 27.
15. Let ρ be an irreducible representation of a group G which is not trivial. Show that $\sum_{g \in G} \rho(g) = 0$.
16. What can be said about a group that has exactly three irreducible characters, of dimensions 1, 2, and 3, respectively?
17. Let ρ be a representation of a group G on a vector space V. Show that the linear span of an orbit $G \cdot v$ is an invariant subspace of V.
18. Is the restriction of ρ to the linear span of an orbit $G \cdot v$ always irreducible?
19. Let ρ be an irreducible representation of a group G on a vector space V. Show that V is the linear span of some orbit $G \cdot v$.
20. Decompose the standard representation of the group S_n into irreducible ones.
21. What are the one-dimensional characters of the group S_n?

Additional problems

1. Show that a noncommutative group of order p^3 has exactly $p+4$ normal subgroups.
2. Let H be a subgroup of index n in a group G. Show that there is a homomorphism $f\colon G \to S_n$ such that $H = f^{-1}(S_{n-1})$.
3. Let G be the set of all invertible $n \times n$ matrices each of whose rows and columns sums to 1. Show that G is a subgroup of $\mathrm{GL}(n)$ isomorphic to $\mathrm{GL}(n-1)$.
4. Let G be a commutative finite group which contains two distinct elements of order 2. Show that then 4 divides $|G|$. Is this true if G is not commutative?
5. Let G be a finite group and let $\varphi\colon G \to \mathbb{C}^\times$ be a nontrivial homomorphism. Calculate $\sum_{g\in G} \varphi(g)$.
6. Show that the group $\mathbb{Z}/4\mathbb{Z}$ is not isomorphic to a product of simple groups. Do the same for S_3.
7. Show that if G is a noncommutative finite group, then $|Z(G)| \leq \frac{1}{4}|G|$.
8. For $\sigma \in S_m$ and $\tau \in S_n$ calculate the parity of the permutation of $\{1,\dots,m\} \times \{1,\dots,n\}$ which maps (i,j) to $(\sigma(i), \tau(j))$.
9. Let σ be a product of all the elements of S_n in some order. Is σ even or odd?
10. Let G be a p-group. Show that for every divisor d of $|G|$ there exists a normal subgroup of G of order d.
11. Let G be a finite group such that for every divisor d of $|G|$ there exists precisely one subgroup of G of order d. Show that G is cyclic.
12. Let G be a finite group such that $g^2 = 1$ for every $g \in G$. Prove that $G \cong \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$.
13. For a finite set S determine the structure of the group $\mathcal{P}(S)$ under the operation of symmetric difference.
14. Show that there cannot exist an action of the group $\mathbb{Z}$ on the set of all smooth functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $1 \cdot f = f'$.
15. Let G be a finite group such that the action of $\mathrm{Aut}(G)$ on G has only two orbits. Prove that G is abelian.
16. Show that only the trivial group and the group $\mathbb{Z}/2\mathbb{Z}$ have the identity map as their sole automorphism.
17. Prove that there is no group G such that $\mathrm{Aut}(G) \cong \mathbb{Z}$.
18. Show that the representation of the group $\mathbb{Z}$ on $\mathbb{C}^2$ such that $1 \in \mathbb{Z}$ acts by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ cannot be decomposed in a direct sum of irreducible representations.
19. Let V be a vector space of finite dimension. Show that the group $\mathbb{Z}$ has infinitely many nonisomorphic representations on V.
20. Show that for every finite group G of order n there is a subset $X \subseteq \mathbb{R}^{n-1}$ such that $G \cong \mathrm{Sym}\, X$.