Automorphisms of $C^*$-algebras and von Neumann algebras

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Lemma 1

If $\alpha$ is an automorphism of a $C^*$-algebra $\mathcal{A}$ acting on a Hilbert space and $\alpha$ is weak-operator bicontinuous on the unit ball of $\mathcal{A}$ (i.e. $\alpha$ is ultra-weakly bicontinuous on $\mathcal{A}$) then $\alpha$ has an extension $\bar{\alpha}$ which is an automorphism of $\mathcal{A}^-$, $\bar{\alpha}$ is ultra-weakly bicontinuous on $\mathcal{A}^-$, and $\|\bar{\alpha} - \iota\| = \|\alpha - \iota\|$.

*-representations of self-adjoint operator algebras which have no unitarily equivalent non-zero subrepresentations are called disjoint representations.

Lemma 2

If $\{\phi_{\alpha}\}$ are *-representations of the self-adjoint operator algebra $\mathcal{A}$ then $\{\phi_{\alpha}\}$ consists of mutually disjoint representations if and only if $\phi(\mathcal{A})^- = \bigoplus (\phi_{\alpha}(\mathcal{A})^-)$, where $\phi = \bigoplus \phi_{\alpha}$. 
Lemma 3

If \( \alpha \) is an automorphism of a \( C^* \)-algebra \( \mathfrak{A} \) acting on a Hilbert space, and \( \| \alpha - \iota \| < 2 \), then \( \alpha \) extends to an automorphism \( \tilde{\alpha} \) of \( \mathfrak{A}^- \), leaving each element of the center of \( \mathfrak{A}^- \) fixed, such that \( \| \tilde{\alpha} - \iota \| = \| \alpha - \iota \| \).

With \( E' \) a projection in \( \mathfrak{A}' \), and \( \phi \) defined by \( \phi(A) = \alpha(A)E' \), for \( A \in \mathfrak{A} \), \( (\phi \oplus \iota)(\mathfrak{A}) \) acting on \( E'(\mathcal{H}) \oplus \mathcal{H} \) does not have strong-operator closure \( \phi(\mathfrak{A})^- \oplus \mathfrak{A}^- \).
Lemma 4

Let $\alpha$ be an inner automorphism of a von Neumann algebra $\mathcal{R}$, for which $\|\alpha - \iota\| < 2$. Then there is a unitary operator $U$ in $\mathcal{R}$, with spectrum $\sigma(U)$ in the half-plane $\{z : \text{Re}z \geq \frac{1}{2} \left( 4 - \|\alpha - \iota\|^2 \right)^{\frac{1}{2}} \}$, such that $\alpha(U) = UAU^*$ for all $A$ in $\mathcal{R}$.

Step 1 : Prove the result when $\mathcal{R} = \mathcal{M}_n$, the space of all operators on an $n$-dimensional Hilbert space, $n$ being an integer. This involves proving that the convex hull of the eigenvalues of $U$ is at a distance at least $\frac{1}{2} \left( 4 - \|\alpha - \iota\|^2 \right)^{\frac{1}{2}}$ from the origin.

Step 2 : For any $k$ such that $0 < k < \frac{1}{2} \left( 4 - \|\alpha - \iota\|^2 \right)^{\frac{1}{2}}$, prove that there is a $U$ implementing $\alpha$ such that $\sigma(U)$ is in the half-plane $\{z : \text{Re}z \geq k \}$. This is achieved by suitably approximating $U$ by linear combinations of its spectral projections and appealing to Step 1. (Proof by contradiction)

Step 3 : Use a limiting argument to construct $U$ satisfying the required properties in the theorem.
For $a \in [0, \frac{1}{2}\pi)$, define $S_a = \{\exp it : -a \leq t \leq a\}$. Let $b$ be such that $2 \sin b = \|\alpha - \iota\|$. Choose real numbers $c, \delta$ such that $b < c < \frac{1}{2}\pi$ and $0 < \delta < \frac{1}{2} \cos c$, and let $\epsilon_n = (c - b)(1 - \delta)^{n-1}$. Let $E$ and $F$ be spectral projections for $U_n$ corresponding to the Borel sets

$\{e^{it} : b + (1 - 2\delta)\epsilon_n \leq t \leq b + \epsilon_n\},$

$\{e^{it} : -b - \epsilon_n \leq t \leq -b - (1 - 2\delta)\epsilon_n\}.$

$C_E C_F = 0$ and thus, there is a central projection $Q \in \mathcal{R}$ such that $E \leq Q$ and $F \leq I - Q$.

The unitary operator $U_{n+1} := \{e^{-i\delta\epsilon_n}Q + e^{i\delta\epsilon_n}(I - Q)\}U_n$ has spectrum in $S_{b+\epsilon_{n+1}}$ and implements the automorphism.
Definition:
An automorphism $\alpha$ of a $C^*$-algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ is said to be:

(i) **extendable** if there is an automorphism of the weak-operator closure of $\mathcal{A}$ equal to it on $\mathcal{A}$.

(ii) **spatial** if there is a unitary operator $U$ on $\mathcal{H}$ such that $\alpha(A) = UAU^*$ for each $A \in \mathcal{A}$.

(iii) **weakly-inner** if it is spatial and $U$ can be chosen in the weak-operator closure of $\mathcal{A}$.
If $\phi$ is a faithful representation of $\mathcal{A}$ on a Hilbert space, $\epsilon_\phi(\mathcal{A})$, $\sigma_\phi(\mathcal{A})$, and $\iota_\phi(\mathcal{A})$, denote the groups of those elements $\alpha$ of the automorphism group of $\mathcal{A}$ for which $\phi\alpha\phi^{-1}$ is extendable, spatial, and weakly-inner, respectively.

$\pi(\mathcal{A})$ denotes the intersection of all the subgroups $\iota_\phi(\mathcal{A})$ and refer to its elements as permanently weakly (for brevity, $\pi-$) inner automorphisms of $\mathcal{A}$. 
Lemma 5

If \( t \rightarrow \alpha(t) \) is a norm-continuous one-parameter group of automorphisms of a \( C^* \)-algebra \( \mathcal{A} \) acting on a Hilbert space \( \mathcal{H} \) then each \( \alpha(t) \) is weakly-inner.

\[
\alpha(t)[AB] = AB + t\delta(AB) + O(t^2) = \alpha(t)[A]\beta(t)[B] = AB + t(A\delta(B) + \delta(A)B) + O(t^2).
\]
Thus \( \delta \), the generator of the one-parameter group is a derivation.

By the Derivation Theorem, we have that \( \delta = \text{ad } iA|\mathcal{A} \) with \( A \in \mathcal{A}^- \) (and \( A = A^* \) as \( \delta(B^*) = \delta(B)^* \)).

\[
\alpha(t)[B] = U_t B U_{-t}, \text{ with } U_t (= \exp iA) \text{ a unitary operator in } \mathcal{A}^-.
\]
Lemma 6

If $\mathcal{A}$ is a $C^*$-algebra and $U$ a unitary operator acting on a Hilbert space $\mathcal{H}$ such that $\alpha(A) = UAU^*$ lies in $\mathcal{A}$ for all $A$ in $\mathcal{A}$ and $\Re a > 0$ for each $a$ in $\sigma(U)$, then $\alpha$ lies on a norm-continuous one-parameter subgroup of $\alpha(\mathcal{A})$ and is $\pi$-inner.

Let $\bar{\alpha}$ be the extension of $\alpha$ to $B(\mathcal{H})$ defined by $\bar{\alpha}(B) = UBU^*$. We may choose $H$ self-adjoint with $\sigma(H)$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$ such that $U = \exp iH$. Then, $\bar{\alpha} = \exp(i\text{ad}H)$.

$\text{ad} iH$ has spectrum in $\{it : |t| \leq r\}$, where $2\|H\| = r < \pi$, by choice of $H$. Thus, $\bar{\alpha}$ has spectrum in $\{\exp it : |t| \leq r\}$.

$\bar{\alpha}^s = \exp(\text{ad} isH)$ is an automorphism of $B(\mathcal{H})$, for all real $s$.

$$\bar{\alpha}^s = \frac{1}{2\pi i} \int_C g_s(z)(z - \bar{\alpha})^{-1} \, dz$$

so that $\bar{\alpha}^s$ leaves $\mathcal{A}$ invariant. (use Runge’s theorem to approximate $(z_0 - z)^{-1}$ by polynomials where $z_0 \in C$)
Theorem 7

If $\alpha$ is an automorphism of a $C^*$-algebra $\mathcal{A}$ and $\|\alpha - \iota\| < 2$, then $\alpha$ lies on a norm-continuous one-parameter subgroup of $\alpha(\mathcal{A})$. Such subgroups generate $\gamma(\mathcal{A})$, the connected component of $\iota$ in $\alpha(\mathcal{A})$ with its norm topology, as a group; and $\gamma(\mathcal{A})$ is an open subgroup of $\alpha(\mathcal{A})$. Each element of $\gamma(\mathcal{A})$ is $\pi$-inner.

Pass to the reduced atomic representation of $\mathcal{A}$. $\mathcal{A}^-$ is a type $I$ von Neumann algebra. $\bar{\alpha}$ is a $*$-automorphism which preserves the center and hence is implemented by a unitary in $\mathcal{A}^-$. Use previous theorems to conclude that $\alpha$ lies on a norm-continuous one-parameter subgroup of $\alpha(\mathcal{A})$. 
Corollary 8

Each norm-continuous representation of a connected topological group by automorphisms of a C*-algebra has range consisting of $\pi$-inner automorphisms.

Corollary 9

If $\mathcal{A}$ is a C*-algebra which has a faithful representation $\varphi$ as a von Neumann algebra then $\iota_0(\mathcal{A}) = \gamma(\mathcal{A}) = \pi(\mathcal{A}) = \iota_\varphi(\mathcal{A})$; and each element of $\gamma(\mathcal{A})$ lies on some norm-continuous one-parameterr subgroup of $\alpha(\mathcal{A})$.

Let $\mathcal{A}$ be a C*-algebra, $\varphi$ a faithful representation of $\mathcal{A}$. Then $\gamma(\mathcal{A}) \subseteq \pi(\mathcal{A}) \subseteq \iota_\varphi(\mathcal{A}) \subseteq \sigma_\varphi(\mathcal{A}) \subseteq \epsilon_\varphi(\mathcal{A}) \subseteq \alpha(\mathcal{A})$. Thus each of the above groups contains the open ball, with center $\iota$ and radius 2, in $\alpha(\mathcal{A})$. Each of these groups is open, hence closed, and the quotient of any of them by a smaller one is discrete.

The subgroups $\gamma(\mathcal{A}), \pi(\mathcal{A}), \iota_0(\mathcal{A})$ of $\alpha(\mathcal{A})$ are normal.
Let $\mathfrak{A} := C(X) \otimes \mathcal{M}_n$, where $X$ is a compact Hausdorff space and $\mathcal{M}_n$ is the algebra of $n \times n$ complex matrices. The center $\mathfrak{C}$ of $\mathfrak{A}$ is the set of matrices whose only non-zero entries consist of a single $A$ in $C(X)$ and as continuous functions on $X$ with values in $\mathcal{M}_n$.

$\pi(\mathfrak{A})$ consists of precisely those automorphisms of $\mathfrak{A}$ which leave each element of $\mathfrak{C}$ fixed.

With $\alpha \in \pi(\mathfrak{A})$ and $\rho$ a point of $X$, a homomorphism $\varphi_\rho$ of $C(X) \otimes \mathcal{M}_n$ onto $\mathcal{M}_n$ is determined by $\varphi_\rho(A \otimes B) = \rho(A)B$.

Define $\alpha(\rho)(B) := \varphi_\rho(\alpha(I \otimes B))$. Then $\alpha(\rho)$ is an isomorphism of $\mathcal{M}_n$ into $\mathcal{M}_n$. $\rho \to \alpha(\rho)$ is norm-continuous. Conversely, a norm-continuous map $\rho \to \alpha(\rho)$ from $X$ to $\alpha(\mathcal{M}_n)$ gives rise to an element of $\pi(\mathfrak{A})$.

The correspondence between elements of $\pi(\mathfrak{A})$ and continuous mappings of $X$ into $\alpha(\mathcal{M}_n)$ is a group isomorphism when this second set is provided with pointwise multiplication through the group structure of $\alpha(\mathcal{M}_n)$.
\[ \alpha(\mathcal{M}_n) \approx U(n)/T_1, \] where \( U(n) \) is the group of unitary operators in \( \mathcal{M}_n \) and \( T_1 \) is the circle group.

**Theorem 10 (Covering Homotopy Theorem)**

Let \( \mathcal{B}' \) be a bundle over \( X' \). Let \( X \) be a \( C_\sigma \) space (any covering has a countable subcovering), let \( f_0 : X \to B' \) be a map, and let \( \bar{f} : X \times I \to X' \) be a homotopy of \( p'f_0 = \bar{f}_0 \). Then there is a homotopy \( f : X \times I \to B' \) of \( f_0 \) covering \( \bar{f} \) (i.e. \( p'f = \bar{f}_0 \)), and \( f \) is stationary with \( f \).

\[ \gamma(A) \subseteq \iota_0(A) \subseteq \pi(A). \]

\( \pi(A)/\gamma(A) \) is the group of homotopy classes of mappings of \( X \) into \( U(n)/T_1 \).

\( \iota_0(A)/\gamma(A) \) is the group of homotopy of classes of mappings of \( X \) into \( U(n)/T_1 \) which can be lifted to \( U(n) \).
If $X$ is contractible, then each continuous map of $X$ is homotopic to a constant mapping. Thus $\gamma(\mathcal{U}) = \iota_0(\mathcal{U}) = \pi(\mathcal{U})$.

If $X = U(n)/T_1$, then $\gamma(\mathcal{U}) \subsetneq \iota_0(\mathcal{U}) \subsetneq \pi(\mathcal{U})$.

$\iota_0(\mathcal{U}) \subsetneq \pi(\mathcal{U})$ as $\pi_1(U(n)/T_1) \approx \mathbb{Z}_n$ has torsion but $\pi_1(U(n)) \approx \mathbb{Z}$ has no torsion. Thus, the identity map from $U(n)/T_1$ to itself has no lifting to $U(n)$.

$\gamma(\mathcal{U}) \subsetneq \iota_0(\mathcal{U})$ as $p \circ i \circ r$ is not nulhomotopic where $U(n)/T_1 \approx SU(n)/\mathbb{Z}_n \xrightarrow{r} SU(n) \xrightarrow{i} U(n) \xrightarrow{p} U(n)/T_1$, $r$ being the map that takes $U\mathbb{Z}_n$ to $U^n$, $i$ being the inclusion map and $p$ being the projection map.
References:


- *The Topology of Fibre Bundles* - Norman Steenrod