MATH 114-004: INTEGRATION

1. INTEGRALS OVER REGIONS, POLAR COORDINATES AND JACOBIANS

Exercise 1. Compute
\[ \int_1^2 \int_1^y \frac{y^3}{\sqrt{x}} \, dx \, dy + \int_2^4 \int_{y/4}^4 \frac{y^3}{\sqrt{x}} \, dx \, dy \]
by drawing the regions and then changing the order of integration.
Answer: \( \frac{93}{2} \).

Exercise 2. Compute the integral
\[ \int_0^8 \int_1^{y^2/3} 5e^{x^2/2} \, dx \, dy. \]
Answer: \( 16(e - 1) \).

Exercise 3. Compute
\[ \int_{1/2}^{2\pi} \int_0^{\pi/y} \sin(xy) \, dx \, dy + \int_1^{2\pi} \int_1^{\pi/y} \sin(xy) \, dx \, dy \]
Hint: Draw the regions.
Answer: \(-\ln(4)\).

Exercise 4. (Polar) Evaluate \( \iint_R y \, dA \) where \( R \) is the upper half of the unit circle.
Answer: \( 2\pi/3 \).

Exercise 5. Find the average value of \( f(x, y) = x + y \) on \( 0 \leq x, y \leq 8 \). This is

darts-on-a-chessboard problem.
Answer: 8

Exercise 6. (Polar) Integrate \( x^2 \) in the region \( x \geq \frac{1}{2}, \ x^2 + y^2 \leq 1 \).
Answer: \( \frac{\pi}{12} + \frac{\sqrt{3}}{32} \).

Exercise 7. (Polar) Find \( A = \int_{-\infty}^{\infty} e^{-x^2} \, dx \) by first noticing that
\[ A^2 = \iint_{\mathbb{R}^2} e^{-x^2-y^2} \, dx \, dy. \]
Answer: \( A = \sqrt{\pi} \).

Exercise 8. Compute
\[ \iint_D xy e^{x^2+y^2} \, dx \, dy \]
where \( D \) is the region \( 0 \leq x \leq y \) and \( x^2 + y^2 \leq 4 \).
Answer: \( \frac{2e^{1/4}}{8} \).

Exercise 9. Compute
\[ \iint_R y^2 - x^2 \, dx \, dy \]
over \( R \) defined via \( 0 \leq x, y \geq 2x \) and \( x^2 + y^2 \leq 2 \).
Hint: Suppose the angle between $y = 2x$ and the $x$-axis is $\alpha$. We do not know the angle exactly but we can compute $\sin \alpha$ and $\cos \alpha$ by drawing a triangle with sides 1, 2 and $\sqrt{5}$.
Answer: $\frac{2}{\sqrt{5}}$.

Exercise 10. (Polar/Cylindrical) Find the volume under the curve $z = 4 - x^2 - y^2$ and $z \geq 0$.
Answer: $8\pi$.

Exercise 11. Compute
\[
\int_0^1 \int_0^1 \int_y^1 4z \cos(x^2) \, dx \, dy \, dz
\]
Answer: $\sin(1)$.

Exercise 12. (Polar) Compute the volume of the set $(x, y, z)$ that satisfies $e^x + x^2 + y^2 \leq z \leq e^x + 9 - 3x^2 - 3y^2$.
Answer: $81\pi/8$.

Exercise 13. The region in the positive quadrant is enclosed by the curves $xy = 1$, $xy = 3$, $x^2 - y^2 = 1$ and $x^2 - y^2 = 2$, as shown in the picture.

Find
\[
\iint_R xy(x^2 + y^2) \, dx \, dy
\]
Hint: Use $u = xy$, $v = x^2 - y^2$ as new coordinates and use the fact that
\[
J(u, v) = \begin{vmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{vmatrix}^{-1}
\]
Answer: 2.

Exercise 14. Consider the region $D$ defined by the inequalities $\frac{1}{2} \leq y \leq \frac{2}{x}$ and $x \leq y \leq 3x$. Compute the integral
\[
\iint_D 2y \cdot e^\frac{y^2}{x} \, dx \, dy
\]
by introducing $u = y/x$, $v = xy$.
Answer: $e^3 - e$.

Exercise 15. (Cylindrical coordinates) We have a sphere with radius 2, with two cylinders of radius 1 removed, see picture of the sphere from above. Find the volume of this set.
Exercise 16. Compute the generalized integral
\[ \iint_{\mathbb{R}^2} \frac{4x^2 + y^2}{1 + (4x^2 + y^2)^4} \, dx \, dy. \]
*Hint:* Use elliptical coordinates \( x = r \cos(t), \ y = 2r \sin(t) \). Do not forget to compute the Jacobian!
Answer: \( \frac{128}{9} \).

Exercise 17. (Final) Compute
\[ \iint_D \frac{1 + x + y}{2 + x - y} \, dx \, dy. \]
where \( D \) is the region bounded by the lines \( x + y = -1, \ x + y = 3, \ x - y = -1 \) and \( x - y = 3 \).
Answer: \( \pi/8 \).

Exercise 18. Compute
\[ \iint_D \frac{1}{\sqrt{y^2 - x^2}} \, dx \, dy. \]
where \( D \) is the unbounded region \( y^2 \leq x \leq y \).
*Hint:* Use the substitution \( x = y \cos(t) \), and then you will need
\[ \int \arccos(y) \, dy = y \arccos(y) - \sqrt{1 - y^2} + C. \]
Answer: 1.

Exercise 19. Compute
\[ \iint_D \frac{1}{xy} \, dx \, dy. \]
where \( D \) is the region \( x \leq y \leq 3x, \ 1 \leq xy \leq 4 \) and \( x, y \geq 0 \).
Answer: \( \ln(2) \cdot \ln(3) \).

Exercise 20. Compute
\[ \iint_D xy \, dx \, dy. \]
where \( D \) is the region in the first quadrant bounded by the curves \( x^2 + y^2 = 1, \ x^2 + y^2 = 2 \ y = x^2 \) and \( y = x^2 + 1 \).
*Hint:* Use \( u = x^2 + y^2, \ v = y - x^2 \). The calculations are scary but doable.
Answer: \( \frac{26 - 9\sqrt{5}}{16} \cdot \frac{\pi}{16} \).

Exercise 21. Show that the integral
\[ \int_0^\infty \int_0^\infty \frac{1}{1 + x^3 + y^3} \, dx \, dy \]
converges.
Hint: First show that for $r \geq 0$ and all $t$, we have that 
\[ r^3(\cos^3 t + \sin^3 t) \leq r^3(\cos^2 t + \sin^2 t). \]
Switch to polar and find some other function $g(r, t)$ such that it is clear that
\[
\int_0^\infty \int_0^\infty \frac{1}{1 + x^3 + y^3} dx dy \leq \int_0^\infty \int_0^{\pi/2} g(r, \theta) r d\theta dr
\]
where the latter integral can be computed easily in polar coordinates.

Exercise 22. Let $D$ be the region $(x + y + z)^2 + 4(y + z)^2 + 4z^2 \leq 1$. Compute
\[
\iiint_D x^2 + y^2 + z^2 \, dx dy dz
\]
Hint: Find a change of coordinates that makes the region into a sphere. Then use spherical coordinates. Some simplification can be done by considering certain symmetries.
Answer: $2\pi/15$.

Exercise 23. Compute the volume of the region $(x + y)^2 + 4y^2 + z^2 \leq 1$.
Answer: $2\pi/3$.

Exercise 24. Compute
\[
\int_0^2 \int_{3y}^{3y+6} \frac{y^2}{(1 + x - 3y)^3} dx dy
\]
by first doing an appropriate change of variables.
Answer: $64/49$.

Exercise 25. Compute the integral
\[
\int_1^e \int_{1/x}^x \frac{\ln(x/y) \ln(xy)}{xy} dy dx
\]
by introducing $u = \ln(xy)$, $v = \ln(x/y)$.
Hint: Draw the $xy$-region and think about what $uv$-region this is mapped to.
Answer: $1/3$.

Exercise 26. Compute the following sum
\[
\int_0^\sqrt{3} \int_0^\sqrt{x^2+y^2} \arctan \left( \frac{y}{x} \right) dy dx + \int_2^4 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{x^2+y^2}} \arctan \left( \frac{y}{x} \right) dz dy dx
\]
by first rewriting it as a single integral in cylindrical coordinates.
Answer: The integral becomes $\int_0^\pi \int_0^1 \theta \cdot r \cdot d\theta dr = \frac{32\pi^2}{27}$.

Exercise 27. (Jacobian, Centroid) Find the centroid of the triangle with vertices $(0, 0)$, $(4, 3)$, $(7, 0)$ in the $xy$-plane, $0 \leq z \leq 1$ and density $\delta(x, y, z)$ given by $1 + z$.
Hint: Introduce new coordinates such that $u = 0$ on the line $(0, 0)$ to $(4, 3)$, and $v = 0$ on the line from $(0, 0)$ to $(7, 0)$.
Answer: $\left( \frac{11}{3}, 1, \frac{5}{9} \right)$.

Exercise 28. (Spherical) Compute the integral
\[
\iiint_D e^{-x^2+y^2+z^2/2} \, dV
\]
where $D$ is the infinite cone given by $x^2 + y^2 \leq z^2$ and $z \geq 0$.
Answer: $\frac{2\pi(1-\sqrt{2})}{3}$.
Exercise 29. A region $E$ is enclosed by the curves $y = 1/x$, $y = 3/x$, $y = x$ and $y = 3x$. Find the centroid of this region, assuming uniform density.
Answer: \( \left( \frac{4(5\sqrt{3} - 6)}{9\ln(3)}, \frac{20 - 8\sqrt{3}}{3\ln(3)} \right) \)

2. Line integrals, surface integrals, flow and flux

Exercise 30. We have a vector field $F = (M, N)$ defined in the plane, and it has continuous partial derivatives. A region $D$ (shaded gray) with three holes is shown below. The circulation, counter-clockwise around the region $D$ below has value 10. The value of the circulation along the three holes with orientation given by the arrows are shown written in the holes.

Finally, we are also given the fact that
\[
\int_D \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \, dx \, dy = 2.
\]
Find the value of $a$.

Hint: Use Greens theorem.

Answer: $a = -10$.

Exercise 31. Compute the flux across the parabola $z = 4 - x^2 - y^2$, $z \geq 0$ with normal pointing outwards, in the vector field $F = (x, y, z)$.

Answer: $24\pi$.

Exercise 32. Compute $\int_S F \cdot n \, d\sigma$ where $S$ is parametrized by
\[
\mathbf{r}(u, v) = u \mathbf{i} + v \mathbf{j} + u(1 - u) \mathbf{k}, \quad 0 \leq u, v \leq 1,
\]
and $F$ is the vector field $x \mathbf{i} + y \mathbf{j}$.

Answer: $1/6$.

Exercise 33. Find the area of the surface parametrized as
\[
\mathbf{r}(u, v) = (u^2, \sqrt{2}uv, v^2), \quad 0 \leq u, v \leq 1.
\]

Answer: $\frac{4\sqrt{2}}{3}$.

Exercise 34. Compute
\[
\oint_{\gamma_1} F \cdot \mathbf{n} \, dr
\]
where $\gamma_1$ is the unit circle oriented counterclockwise.

Hint:
- Show that the vector field $F = (-y, x)/(x^2 + 2y^2)$ is conservative in any region that do not contain the origin.
- Let $\gamma_2$ be the curve parametrized by $x = \cos(t)$, $y = \sin(t)/\sqrt{2}$, $0 \leq t \leq 2\pi$, and use path-independence to show that $\oint_{\gamma_1} F \cdot \mathbf{n} \, dr = \oint_{\gamma_2} F \cdot \mathbf{n} \, dr$.
- Compute $\oint_{\gamma_2} F \cdot \mathbf{n} \, dr$.

Answer: $\sqrt{2}\pi$. 
Exercise 35. Let \( z = f(x,y) \) be a surface \( S \) over some region \( R \) in the \( xy \)-plane, with normal pointing upwards, and let \( F = (M,N,P) \) be a vector field.

Show the following identity for computing the flux:

\[
\int\int_S F \cdot n \, d\sigma = \int\int_R -M \frac{\partial f}{\partial x} - N \frac{\partial f}{\partial y} + P \, dxdy.
\]

Exercise 36. Let \( C \) be the closed curve given by the intersection of the cylinder \( x^2 + y^2 = 4 \) and the plane \( x + y + z = 5 \). The curve is oriented counter-clockwise around the \( z \)-axis. Let

\[
F = (e^{x^2}, \sin(y^2), 1 - x^3)
\]

Find the work \( \oint_C F \cdot T \, ds \).

Hint: Use Stokes theorem and show that the work is given by the surface integral

\[
\int\int_S (0, 3x^2, 0) \cdot n \, d\sigma
\]

where \( S \) is the surface parametrized as

\[
r(r, \theta) = (r \cos(\theta), r \sin(\theta), 5 - r \cos(\theta) - r \sin(\theta)).
\]

Answer: \( 12\pi \).

Exercise 37. Let \( C \) be the curve parametrized as

\[
r(t) = (t, t(1 - t), t(1 - t)), \quad 0 \leq t \leq 1
\]

and let \( F = (x^2 + e^{x^2} (z - y), \sin(y^2), \cos(z^2)) \). Compute the work along \( C \).

Hint: Add the line \( L \) from \((1,0,0)\) to \((0,0,0)\) such that \( C \cup L \) is a simple closed loop that encloses a region in the plane \( y = z \). Use Stokes theorem to show that

\[
\int_C F \, dr = -\int_L F \, dr
\]

and then finally compute \(-\int_L F \, dr\). Be careful with normals and orientations of curves.

Answer: \( \frac{1}{3} \).

Exercise 38. Let \( S \) be surface of the paraboloid \( z = 4 - x^2 - y^2 \), \( z \geq 0 \) with outward-pointing normal, and let \( F = (-x^2y, xy^2, ze^{x^2+y^2}) \). Compute the surface integral

\[
\int\int_S (\nabla \times F) \cdot n \, d\sigma
\]

by converting it to a line integral via Stokes theorem.

Hint: You need to close the surface by adding a disk before applying Stokes.

Answer: \( 8\pi \).

Exercise 39. Consider curve parametrized as

\[
r(t) = (\sin(t) \cos(2t), \sin(t) \sin(2t)), \quad 0 \leq t \leq \pi.
\]

This is a simple closed curve with a counter-clockwise orientation. Compute the area of the region.
Exercise 40. We are given two curves:
\[ r_1 = (t^2, (t-1)^2 + t), \quad r_2 = (2t, 1 - t^2 + 3t), \quad 0 \leq t \leq 2. \]
They enclose a simple closed region (see figure). Compute the area of the region.

Hint: Be careful with orientation of the curves.
Answer: \( \frac{\pi}{2} \).

Exercise 41. Let \( x^2 + y^2 = z \) define a parabola with normal pointing outwards. Let \( S \) be the part of the parabola which lies between the planes \( z = 9 \) and \( z = 2x \). Consider the field \( F = ((2x - z)xe^z, y, -y) \) and compute the integral
\[ \int_S (\nabla \times F) \cdot n \, d\sigma. \]

Hint: Use Stokes theorem. You can see the surface integral as a difference of two integrals: let \( S_1 \) be the points on the parabola below the plane \( z = 9 \), and let \( S_2 \) be everything on the parabola below \( z = 2x \). Then \( S = S_1 - S_2 \). See also Exercise 30.
Answer: \( 2\pi \).