Math 340
Fall 2015
Midterm 2
29/10/15
Time Limit: 80 Minutes

Name (Print): ______________________

This exam contains 7 pages (including this cover page) and 5 problems. Enter all requested information on the top of this page, and put your name on the top of every page, in case the pages become separated.

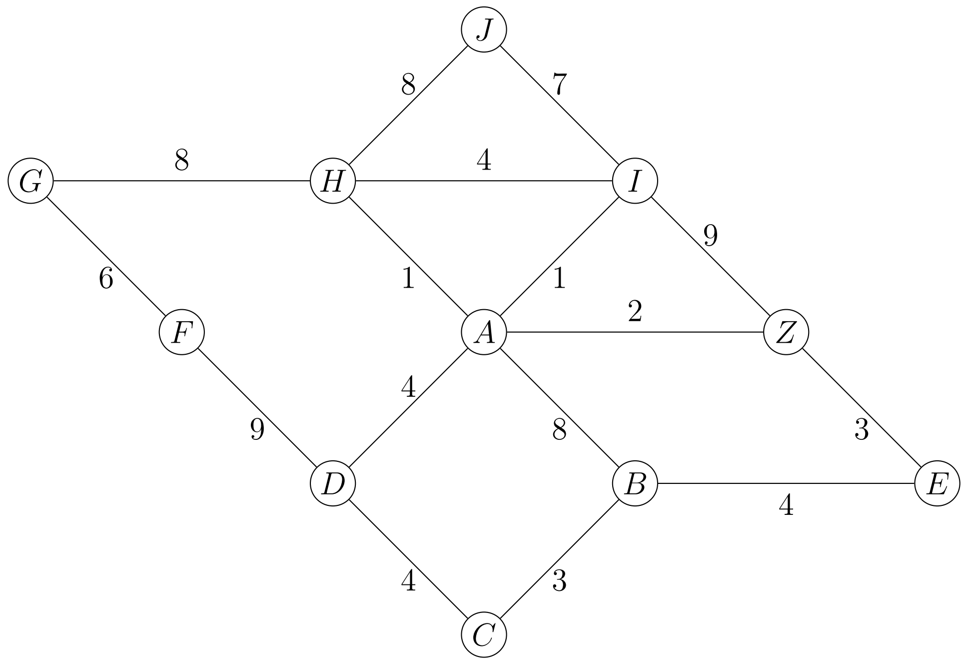
You are required to show your work on each problem on this exam.

- If you need more space, use the back of the pages; clearly indicate when you have done this.

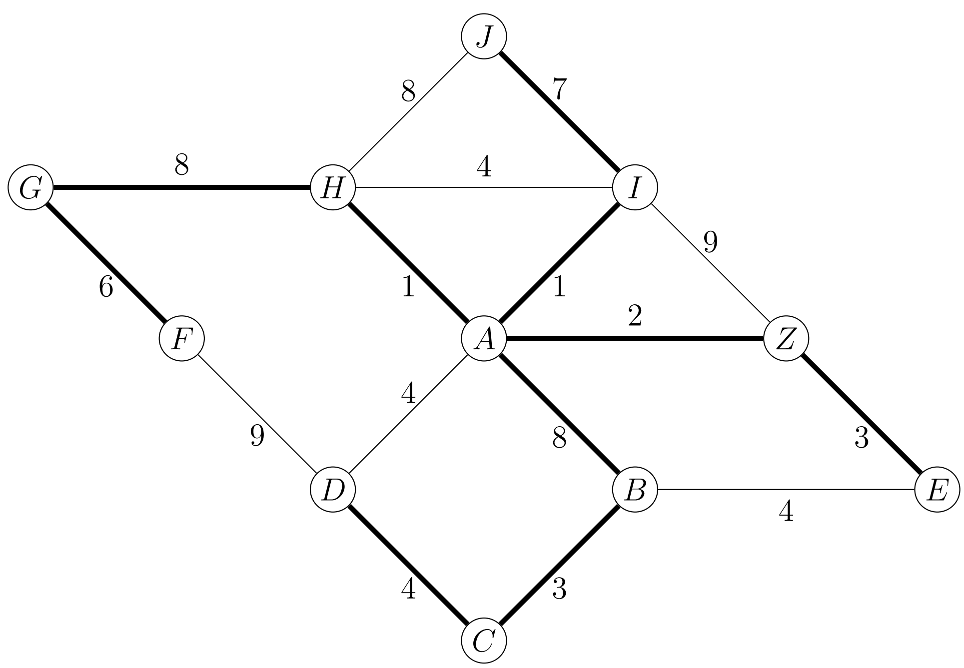
Do not write in the table to the right.

Problem	Points	Score
1	10	
2	20	
3	15	
4	15	
5	20	
Total:	80	

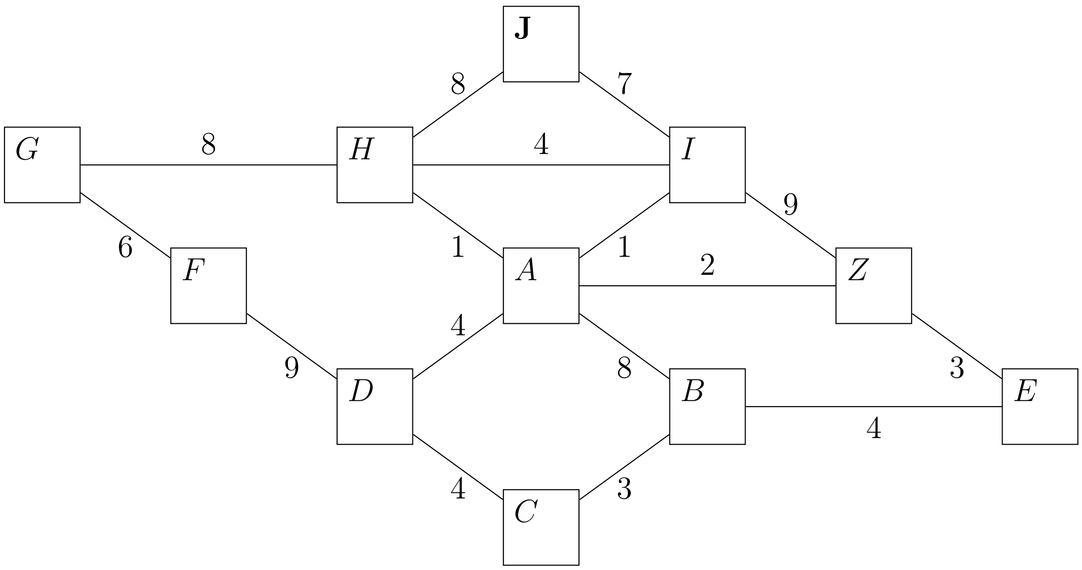
1. (10 points) Find a minimal spanning tree in the following graph, *that includes the edge* (A, B). It is enough to *clearly mark* the spanning tree you obtain.



Solution:



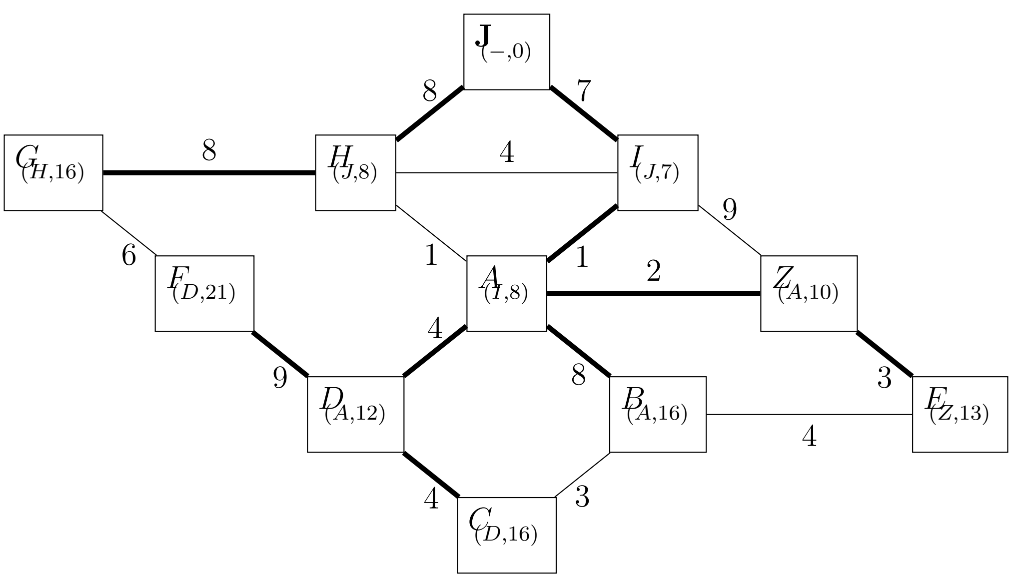
2. (a) (10 points) Perform *Djikstra's algorithm* on the weighted graph below to find the shortest distance from vertex J to any other vertex. In each box, write the shortest distance to J and the name of previous vertex in the shortest path from J.



(b) (10 points) A supervillain asks you for advice: She wants to travel from New York to San Francisco as quick as possible, but her *first priority* is to minimize the number of surveillance cameras along the road taken. You know the time needed to travel each road and the number of cameras on each road. Thus, you want to find the shortest path, among all paths with a minimal number of cameras.

Describe how you solve this problem using Djikstra's algorithm.

Solution: Part a) This is the end result. The thick edges are the ones used to to find a shortest path from a vertex to J.



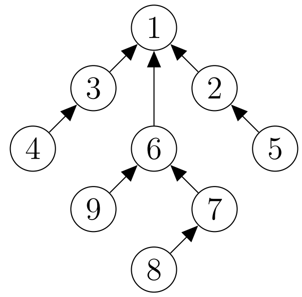
Part b)

Note that an unmodified Djikstra's algorithm only give you *one* optimal path. If there are several paths that optimize a certain total edge weight sum, the path it finds, might not be optimal w.r.t. a second condition. Thus, running Djikstra only on number of cameras, gives you *one* path, but this might not be quickest among all such camera-optimal paths.

We can modify the edge weights as follows: pick a really large number, M, and let the edge weight of an edge with c cameras and time t be $cM + t$. It is straightforward to show that only the number of cameras along a path matter if M is sufficiently large, and the time comes into play only if the number of cameras are equal.

This is the same as performing Djikstra, where whenever we compare two partial paths, we pick the one that minimize number of cameras, and in the case of a tie — pick the path that minimize total time traveled. Thus, we label nodes with total number of cameras so far *and* total time so far.

3. (15 points) Consider a labeled tree on n vertices, where 1 is considered the root vertex. We say that the tree is *decreasing* if the labels appear in a decreasing manner on every path from a vertex to the root. For example, the following labeled tree is such a tree:



Show that the number of such decreasing trees on n vertices is $(n-1)!$.

Hint: This is a perfect opportunity to use induction.

Solution: It is easy to verify the formula for $n = 1$.

Assume this formula holds for $n-1$ vertices. To construct a valid tree on n vertices, we first select a labeled tree on $n-1$ vertices. This can be done in $(n-2)!$ ways, according to the induction hypothesis.

Vertex n must now be attached to some other vertex with smaller label. There are $n-1$ vertices to attach it to, and all these choices give different results. In total, we get $(n-1)\cdot(n-2)! = (n-1)!$ possible trees on n vertices, which agrees with the formula.

Alternatively, we can construct a bijection with a set of objects that obviously has size $(n-1)!$. Consider all permutations of $\{1, 2, 3, \ldots, n\}$ that ends with a 1. This set of permutations has size $(n-1)!$.

Given such a permutation P we construct edges of a decreasing tree as follows: For each number $i \geq 2$ in P, find the first number $j < i$ to the right of i. Then $i \to j$ is an edge in T. Since 1 is the rightmost number in P, it is clear that such a j always exists. It is also clear that we cannot have two edges $i \to j_1$ and $i \to j_2$ where $i > j_1$ and $i > j_2$. This ensures that the resulting edges really determine a decreasing tree T.

In the other direction, given a decreasing tree T, we can invert the above construction as follows: First start with the permutation $P_1 = 1$. We read the vertices $i = 2, 3, \ldots$ in order, and if vertex i is a child of j, then insert i directly after j in the permutation, until we have processed all vertices.

For example, the tree in the question corresponds to the permutation 524387961.

4. (15 points) You are going to an amusement park. There are four attractions, (haunted house, roller coaster, a carousel, water ride). You buy 25 tokens. Each attraction cost 3 tokens each ride, except the roller coaster that costs 5. Obviously, you want to ride each ride at least once, but the order of the rides does not matter.

   In how many ways can you spend your tokens? You may have some remaining tokens in the end of the day.

Solution: We first ride all rides once. That leaves 11 tokens which can be spent as we please. We can ride the roller coaster 0, 1 or 2 times with the remaining tokens:

- 0 **times**. We need to count non-negative integer solutions to $3h+3c+3w \leq 11$. This is the same as solving $h+c+w+r=3$ where r represents the number of remaining tokens. Number of solutions: $C(3+4-1,3)$
- 1 **time**. Same strategy gives non-negative solutions to $3h+3c+3w \leq 6$, or $h+c+w+r=2$. This gives $C(2+4-1,2)$ number of solutions.
- 2 **times**. After riding the coaster 2 times, we have one token left and cannot ride anything else. Only 1 solution.

Total number of ways: $\binom{6}{3}+\binom{5}{2}+1=20+10+1=31$.

5. (20 points) Prove the identity

$$n \cdot 4^{n-1} = \sum_{k=0}^{n} \binom{n}{k} 3^k (n-k).$$

A combinatorial proof is encouraged, but you may use algebraic methods if you like.

Solution:

One possible solution is to expand $(3x + y)^n$ using the binomial theorem, take the derivative w.r.t. y on both sides, and then put $x = y = 1$.

We give a combinatorial interpretation of both sides. In the left hand side, there are n people who go on a wine tour: and since they are responsible adults, one person is selected to be the designated driver and he must stay sober. Each of the remaining $n - 1$ persons can pick one out of 4 menus, where the last menu is the non-alcoholic option.

In the right hand side, we pick a k-subset of people who prefers one of the first three options. They then pick which of these they like. The designated driver cannot, of course, be among these k people, and is selected among the remaining $(n-k)$ people (all who then pick the last option).