MIDTERM 2

Math 103
10/22/2014

Name: ____________________

ID: ____________________

"My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this"

Signature: ____________________

Read all of the following information before starting the exam:

- Check your exam to make sure all 8 pages are present.
- You may use writing implements and a single handwritten sheet of 8.5"x11" paper.
- NO CALCULATORS.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Good luck!

1	8		7	8	
2	8		8	8	
3	8		9	8	
4	8		10	8	
5	8		11	10	
6	8		12	10	
Total	100				

Name: ______________________

e ____ **1.** What is $\frac{d}{dx}e^{\tan^{-1}x}|_{x=\sqrt{3}}$?

a. $1/4$
b. $\frac{e^{\pi}}{2}$
c. $e^{\pi/3}$
d. $\frac{e^{\pi/3}}{2}$
e. $\frac{e^{\pi/3}}{4}$
f. 1
g. $\frac{e^{\pi/4}}{3}$
h. $\frac{e^{\pi/4}}{2}$

Using the chain rule,

$$\frac{d}{dx}e^{\tan^{-1}x} = e^{\tan^{-1}x}\frac{d}{dx}\tan^{-1}x = \frac{e^{\tan^{-1}x}}{1+x^2}.$$

Plugging in $\sqrt{3}$ and using the fact that $\tan^{-1}\sqrt{3} = \pi/3$,

$$\frac{e^{\tan^{-1}\sqrt{3}}}{1+\sqrt{3}^2} = \frac{e^{\pi/3}}{4}.$$

d ____ **2.** What values should a and b have so that

$$f(x) = \begin{cases} ax+b & \text{if } x < 1 \\ \ln x & \text{if } x \geq 1 \end{cases}$$

is differentiable everywhere.

a. $a = 0, b = -1$
b. $a = 0, b = 0$
c. $a = 0, b = 1$
d. $a = 1, b = -1$
e. $a = 1, b = 0$
f. $a = 1, b = 1$
g. $a = e, b = -1$
h. $a = e, b = 1$

In order for this to be diferentiable at 1, both the values and the functions must match at 1. $\ln 1 = 0$, so $a+b = 0$, and $\frac{d}{dx}\ln x = \frac{1}{x}$, so the derivative at 1 must be 1, so $\frac{d}{dx}(ax+b) = a = 1$. So $a = 1$ and $b = -1$.

g **3.** If $f(x) = \frac{x}{\sin x}$, what is $f'(\pi/6)$?

a. $\frac{2\sqrt{3}}{3} - \frac{\pi}{9}$
b. $\frac{2\sqrt{3}}{3} + \frac{\pi}{9}$
c. $\frac{\pi}{9} - \frac{2\sqrt{3}}{3}$
d. $\frac{\sqrt{3}}{16} - \frac{3\pi}{16}$
e. $\frac{\sqrt{3}}{16} + \frac{3\pi}{16}$
f. $\frac{3\pi}{16} - \frac{\sqrt{3}}{16}$
g. $2 - \frac{\pi\sqrt{3}}{3}$
h. $\frac{\pi\sqrt{3}}{3} - 2$

Using the quotient rule,

$$\frac{d}{dx}\frac{x}{\sin x} = \frac{\sin x - x\cos x}{\sin^2 x}.$$

Plugging in $\pi/6$,

$$\frac{\sin(\pi/6) - (\pi/6)\cos(\pi/6)}{\sin^2(\pi/6)} = \frac{1/2 - (\pi/6)\frac{\sqrt{3}}{2}}{1/4} = 2 - \frac{\pi\sqrt{3}}{3}.$$

e **4.** The chart below gives the functions f, f', g, and g' at several values. What is $\frac{d}{dx}f(xg(x))\mid_{x=1}$?

x	1	2	3	4
f(x)	1	2	3	4
f'(x)	2	1	3	1
g(x)	2	1	3	1
g'(x)	4	3	2	1

a. 1
b. 2
c. 3
d. 4
e. 6
f. 8
g. 9
h. 12

$$\frac{d}{dx}f(xg(x)) = f'(xg(x))\frac{d}{dx}xg(x) = f'(xg(x))[xg'(x) + g(x)].$$

When $x = 1$, this is

$$f'(g(1))[g'(1) + g(1)] = f'(2)[4 + 2] = 1 \cdot 6 = 6.$$

e 5. Find $\frac{d}{dx}(x\cos|x|)\,|_{x=0}$. (Remember this means the derivative of $x\cos|x|$ at the point $x=0$.)

a. $-\pi$
b. -2
c. -1
d. 0
e. 1
f. 2
g. π
h. DNE

Because of the absolute value, we can't use the chain rule, and fall back on the definition of the derivative:

$$\begin{aligned}\lim_{h\to 0}\frac{(0+h)\cos|0+h| - 0\cos 0}{h} &= \lim_{h\to 0}\frac{h\cos|h|}{h}\\ &= \lim_{h\to 0}\cos|h|\\ &= \cos|0|\\ &= 1.\end{aligned}$$

A popular approach was to use the equation $|x|=\sqrt{x^2}$. While this equation is correct, it isn't a good approach for this sort of question. This is because of a technicality in the definition of the product rule that we don't normally worry about: what it actually says is "If u and v are differentiable at x then so is their product uv and $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$". In this case, $u = x$ and $v = \cos|x|$. The problem is that v *is not* differentiable when $x = 0$, so the product rule doesn't apply. (This should remind you of the limit laws, where we do sometimes worry about this technicality: we can say $\lim_{x\to a} f(x)g(x) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x))$ *only when* both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.)

In this particular case, what happens if you try to use the product and chain rules is you get

$$\frac{d}{dx}(x\cos|x|) \stackrel{?}{=} \cos\sqrt{x^2} + x\left(-\sin\sqrt{x^2}\right)\frac{x}{\sqrt{x^2}}.$$

The requirement that $\cos|x|$ be differentiable is saying that you can't use the red x (which is u) to cancel the blue part (which is $\frac{dv}{dx}$). But the blue part is undefined at 0 (because there's a 0 in the denominator), so the product rule doesn't apply.

This is a subtle point; we don't delve into every subtlety of this kind in this course, and the ones you're responsible for knowing we talk about very explicitly (like we did with the limit laws). Since we hadn't covered this specific issue, we graded this problem as if that were a valid method. (Even so, very few students who tried this were apple to apply it correctly; no matter how you do it, you end up with a 0 in the denominator that you have to deal with carefully.)

f **6.** The graph of $y = ax^3 - 2ax^2 - x + 1$ is a curve such that the tangent line at $x = 2$ has slope 11. What is a?

a. -2
b. -1
c. 0
d. 1
e. 2
f. 3
g. 4
h. There is no such value a

The derivative is

$$y' = 3ax^2 - 4ax - 1.$$

We know that the derivative when $x = 2$ is 11, so

$$11 = 3a \cdot 2^2 - 4a \cdot 2 - 1,$$

so $12 = 12a - 8a = 4a$, so $a = 3$.

g **7.** What is the slope of the tangent line to the curve $x^2y^3 - y^2 = 4x - 4$ at $(1,1)$?

a. 0
b. 1/4
c. 1/3
d. 1/2
e. 2/3
f. 1
g. 2
h. 3

Taking the derivative of both sides gives

$$2xy^3 + 3x^2y^2y' - 2yy' = 4.$$

Plugging in $x = 1$ and $y = 1$ gives

$$2 + 3y' - 2y' = 4,$$

so $y' = 2$.

b **8.** What is the derivative of $\frac{\sqrt{e^x(x^2+4)}}{(x-1)^2(x+1)^3}$ at $x = 0$?

a. -2
b. -1
c. -1/2
d. -1/4
e. 2
f. 1
g. 1/2
h. 1/4

Let y be the function, and use logarithmic differentiation:

$$\ln y = \frac{1}{2}x + \frac{1}{2}\ln(x^2+4) - 2\ln(x-1) - 3\ln(x+1),$$

so

$$\frac{y'}{y} = \frac{1}{2} + \frac{x}{x^2+2} - \frac{2}{x-1} - \frac{3}{x+1},$$

so

$$y' = \left[\frac{1}{2} + \frac{x}{x^2+2} - \frac{2}{x-1} - \frac{3}{x+1}\right] y.$$

We know $y(0) = \frac{\sqrt{e^0(4)}}{(-1)^2(1)^3} = \frac{2}{1}$ so

$$y'(0) = [\frac{1}{2} + 0 - \frac{2}{-1} - \frac{3}{1}]2 = -1.$$

c **9.** Which of the following values is closest to $\sqrt[3]{124}$? (It may help to remember that $\sqrt[3]{125} = 5$.)

a. 5
b. $5 - 1/125$
c. $5 - 1/75$
d. $5 - 1/25$
e. $5 + 1/125$
f. $5 + 1/75$
g. $5 + 1/25$
h. 4

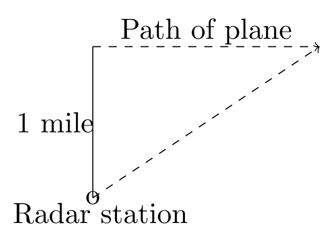
We use a linearization: the derivative of $\sqrt[3]{x} = x^{1/3}$ is $\frac{1}{3x^{2/3}}$; at 125 this is $\frac{1}{75}$.

$$L(x) = \sqrt[3]{125} + \frac{1}{75}(x - 125) = 5 + \frac{x - 125}{75}.$$

In particular, $L(126) = 5 + \frac{1}{75}$.

c **10.** A plane is flying horizontally at an altitude of 1 mile and a speed of 300 miles per hour. At one point in its trip, the plane passes directly over a radar station. Soon after, the distance from the plane to the radar station is 2 miles; at this moment, what is the rate at which the distance from the plane to the radar station is increasing?

a. $\frac{150}{\sqrt{3}}$mph
b. 150mph
c. $150\sqrt{3}$mph
d. $30\sqrt{3}$mph
e. $\frac{30}{\sqrt{3}}$mph
f. $\frac{300}{\sqrt{3}}$mph
g. $300\sqrt{3}$mph
h. 200mph



Let us call the horizontal distance between the plane and the radar station x, so $\frac{dx}{dt} = 300$. Let us call the total distance between the radar station and the path of the plane d, so we are looking for $\frac{dd}{dt}$.
By the Pythagorean theorem, $x^2 + 1^2 = d^2$, and taking the derivative,

$$2x\frac{dx}{dt} = 2d\frac{dd}{dt}.$$

We are looking at the moment where $d = 2$. At this moment $x = \sqrt{d^2 - 1^2} = \sqrt{4-1} = \sqrt{3}$, so

$$\frac{dd}{dt} = 300\sqrt{3}/2 = 150\sqrt{3}.$$

11. Consider the function $g(x) = x^5 + 2x^3 + 1$. What is $\frac{d}{dx} g^{-1}(x)|_{x=4}$? (That is, the derivative of $g^{-1}(x)$ at $x = 4$. It is helpful to notice that $g(1) = 4$.)

We know $g'(x) = 5x^4 + 6x^2$ and since $g(1) = 4$, $g^{-1}(g(1)) = g^{-1}(4)$, so $g^{-1}(4) = 1$. So using the formula for the derivative of the inverse,

$$\begin{aligned}\frac{d}{dx} g^{-1}(x)\,|_{x=4} &= \frac{1}{g'(g^{-1}(4))}\\ &= \frac{1}{g'(1)}\\ &= \frac{1}{5+6}\\ &= \frac{1}{11}.\end{aligned}$$

12. $L(x)$ is a new function with the property that $\frac{d}{dx} L(x) = \ln(\ln(x))$. What is $\frac{d}{dx} L(e^{e^x})$?

$$\begin{aligned}\frac{d}{dx} L(e^{e^x}) &= \ln(\ln(e^{e^x}))\frac{d}{dx} e^{e^x}\\ &= \ln(e^x) e^{e^x} e^x\\ &= x e^{e^x} e^x.\end{aligned}$$