MIDTERM 3

Math 103
11/18/2014

Name: ______________________

ID: ______________________

"My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this"

Signature: ______________________

Read all of the following information before starting the exam:

- Check your exam to make sure all 7 pages are present.
- You may use writing implements and a single handwritten sheet of 8.5"x11" paper.
- NO CALCULATORS.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Good luck!

1	8		7	8	
2	8		8	8	
3	8		9	8	
4	8		10	8	
5	8		11	10	
6	8		12	10	
Total	100				

Name: ______________________

e 1. Find $\lim_{x\to 0} \frac{\cos x - 1}{x+1-e^x}$

a. -1
b. $-1/e$
c. 0
d. $1/e$
e. 1
f. DNE because it approaches $-\infty$
g. DNE because it approaches $+\infty$
h. DNE for a different reason

$\lim_{x\to 0} \cos x - 1 = \cos 0 - 1 = 0$ and $\lim_{x\to 0} x + 1 - e^x = 0 + 1 - e^0 = 0$, so this is an indeterminate form of type $0/0$. We use L'Hospital's rule:

$$\lim_{x\to 0} \frac{\cos x - 1}{x + 1 - e^x} = \lim_{x\to 0} \frac{-\sin x}{1 - e^x}.$$

Again, $\lim_{x\to 0} -\sin x = 0$ and $\lim_{x\to 0} 1 - e^x = 0$, so we use L'Hospital's rule again:

$$\lim_{x\to 0} \frac{-\sin x}{1 - e^x} = \lim_{x\to 0} \frac{-\cos x}{-e^x} = \frac{-\cos 0}{-e^x} = \frac{-1}{-1} = 1$$

c 2. What intervals is the function $(x^2 - 1)^{3/5}$ increasing on?

a. $(-\infty, \infty)$
b. $(-\infty, 0)$
c. $(0, \infty)$
d. $(-\infty, -1) \cup (1, \infty)$
e. $(-1, 1)$
f. $(-\infty, -1) \cup (0, 1)$
g. $(-1, 0) \cup (1, \infty)$
h. $(-\infty, 0) \cup (1, \infty)$

Let $f(x) = (x^2 - 1)^{3/5}$. We find the derivative, $f'(x) = \frac{3}{5}(x^2 - 1)^{-2/5} 2x = \frac{6x}{5(x^2-1)^{2/5}}$. This function is 0 when $x = 0$ and is undefined when $x^2 - 1 = 0$, which is when $x^2 = 1$, so $x = \pm 1$. So the critical points are $-1, 0, 1$.
The denominator is always positive (it is raised to an even number). The numerator is positive exactly when $6x$ is positive, so this function is increasing on $(0, \infty)$.

d **3.** Find $\lim_{x\to\pi/2^-}(x+1-\pi/2)^{\sec(x)}$. (Don't forget that $\lim_{x\to\pi/2^-}\sec x = +\infty$.)

a. -1
b. $-1/e$
c. 0
d. $1/e$
e. 1
f. e
g. DNE because it approaches $-\infty$
h. DNE because it approaches $+\infty$

$\lim_{x\to\pi/2^-}(x+1-\pi/2) = 1$ and $\lim_{x\to\pi/2^-}\sec x = +\infty$, so this is an indeterminate form of type 1^∞. We write

$$\begin{aligned}
\lim_{x\to\pi/2^-}(x+(1-\pi/2))^{\sec(x)} &= e^{\ln \lim_{x\to\pi/2^-}(x+(1-\pi/2))^{\sec(x)}} &&= e^{\lim_{x\to\pi/2^-}\ln(x+(1-\pi/2))^{\sec(x)}}\\
&= e^{\lim_{x\to\pi/2^-}\sec(x)\ln(x+(1-\pi/2))} &&= e^{\lim_{x\to\pi/2^-}\frac{\ln(x+(1-\pi/2))}{\cos x}}\\
&=^{LH} e^{\lim_{x\to\pi/2^-}\frac{\frac{1}{x+1-\pi/2}}{-\sin x}} &&= e^{\frac{\frac{1}{1}}{-1}}\\
&= e^{-1}.
\end{aligned}$$

a **4.** Consider the function $h(x) = 16x^{-2}$ and the two intervals $[-2,2]$ and $[2,4]$. Which of the following can we conclude using the Mean Value Theorem? (Note: each answer below consists of two statements, one about each interval; pick the one which is correct about both intervals. It may help to check your answer: both statements should also be true.)

a. Nothing about the interval $[-2,2]$; there is a c in $[2,4]$ with $h'(c) = -\frac{3}{2}$

b. Nothing about the interval $[-2,2]$; there is a c in $[2,4]$ with $h'(c) = \frac{7}{4}$

c. There is a c in $[-2,2]$ with $h'(c) = 0$; nothing about the interval $[2,4]$

d. There is a c in $[-2,2]$ with $h'(c) = 0$; there is a c in $[2,4]$ with $h'(c) = -\frac{3}{2}$

e. There is a c in $[-2,2]$ with $h'(c) = 0$; there is a c in $[2,4]$ with $h'(c) = \frac{7}{4}$

f. There is a c in $[-2,2]$ with $h'(c) = -2$; nothing about the interval $[2,4]$

g. There is a c in $[-2,2]$ with $h'(c) = -2$; there is a c in $[2,4]$ with $h'(c) = -\frac{3}{2}$

h. There is a c in $[-2,2]$ with $h'(c) = -2$; there is a c in $[2,4]$ with $h'(c) = \frac{7}{4}$

We can't tell anything about $[-2,2]$ because $h(x)$ is not continuous on this interval. On the interval $[2,4]$, $h(x)$ is continuous and differentiable, so we can conclude that there is a c with

$$h'(c) = \frac{h(4)-h(2)}{4-2} = \frac{16/16-16/4}{4-2} = \frac{1-4}{2} = -\frac{3}{2}.$$

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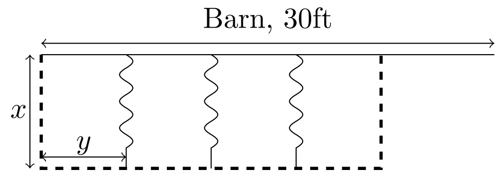
____e____ **5.** A farmer will build four equally sized rectangular stalls alongside a barn; each stall will have total size 20 square feet. The exterior fencing (dashes in the picture) costs \$5 per foot, and the interior fencing (wavy line in the picture) separating the stalls from each other costs \$2 per foot. (The side facing the barn needs no fencing.) The barn is 30 feet long, and the stalls must fit next to it (though they do not need to fill the whole side of the barn). What dimensions for the stalls make them as cheap as possible?



a. $x = 2$, $y = 10$
b. $x = 3$, $y = \frac{20}{3}$
c. $x = 4$, $y = 5$
d. $x = \sqrt{20}$, $y = \sqrt{20}$
e. $x = 5$, $y = 4$
f. $x = \frac{20}{3}$, $y = 3$
g. $x = 8$, $y = \frac{5}{2}$
h. $x = 10$, $y = 2$

The total cost of fencing is $P = 5 \cdot (2x + 4y) + 2(3x) = 10x + 6x + 20y$. We have the constraint $xy = 20$, so $x = 20/y$, so $P = \frac{16 \cdot 20}{y} + 20y$. We have the bounds $0 < y \leq 30/4$. Differentiating, $P' = 20 - \frac{320}{y^2}$. Setting equal to 0, $y^2 = \frac{320}{20} = 16$, so $y = \pm 4$. Since $y = -4$ makes no sense, the only candidate is $y = 4$. We can check that $P'' = \frac{320}{y^3} > 0$ when $y = 4$, so $y = 4$ is a local minimum.

____g____ **6.** At which x value does $g(x) = x^3 - 3x - 2$ achieve its absolute maximum on the interval $[-4, 4]$?

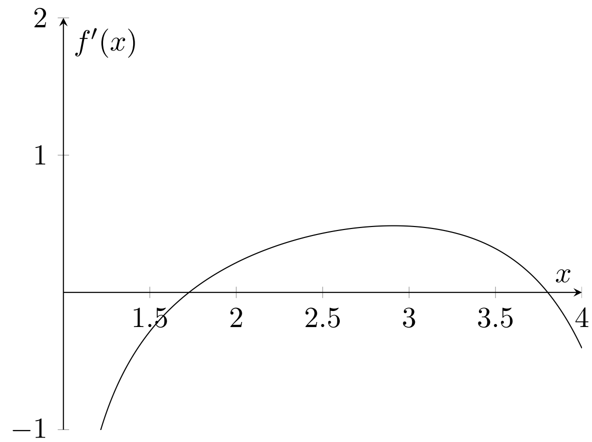
a. -4
b. -2
c. -1
d. 0
e. 1
f. 2
g. 4
h. $g(x)$ does not have an absolute maximum on $[-4, 4]$

$g'(x) = 3x^2 - 3$, so when the critical points are when $x = \pm 1$. We also consider the endpoints -4 and 4:

-4	$(-4)^3 - 3(-4) - 2 < 0$
-1	$1 + 3 - 1 = 0$
1	$1 - 3 - 2 = -4$
4	$4^3 - 14$

Even without calculating exactly, we can see that the last value is much larger than any of the others.

__c__ **7.** This is the graph of $f'(x)$ (note—this is the graph of the *derivative*, not the original function). At (approximately) which x-values in the interval $[2,4]$ does $f(x)$ (the *original function*) have an inflection point?



a. f has no inflection points in this interval
b. Only 1.7
c. Only 2.9
d. Only 3.8
e. 1.7 and 2.9
f. 1.7 and 3.8
g. 2.9 and 3.8
h. 1.7, 2.9, and 3.8

An inflection point happens when the second derivative changes sign, so when the first derivative changes from increasing to decreasing or vice versa. This happens a bit under 3, around 2.9.

__f__ **8.** Find $\frac{1}{121}\sum_{k=1}^{60} 2(k^2-121)$. (Note: don't multiply out big numbers until you need to; the numbers here are rigged to make the calculations possible without a calculator.)

a. 60
b. 550
c. 670
d. 940
e. 1001
f. 1100
g. 1340
h. 73870

$$\frac{1}{121}\sum_{k=1}^{60} 2(k^2-121) = \frac{2}{121}\sum_{k=1}^{60}(k^2-121) \qquad = \frac{2}{121}\left(\sum_{k=1}^{60} k^2 - \sum_{k=1}^{60} 121\right)$$

$$= \frac{2}{121}\left(\frac{60(61)(121)}{6} - 60\cdot 121\right) \qquad = 2(10(61)-60)$$

$$= 2(610-60) \qquad = 2(550)$$

$$= 1100$$

a **9.** Where does $f(x) = (e^x - 1)^{10/3}$ have inflection points (i.e. where does $f(x)$ change concavity)? It may help to know that $f'(x) = \frac{10}{3}e^x(e^x - 1)^{7/3}$ and

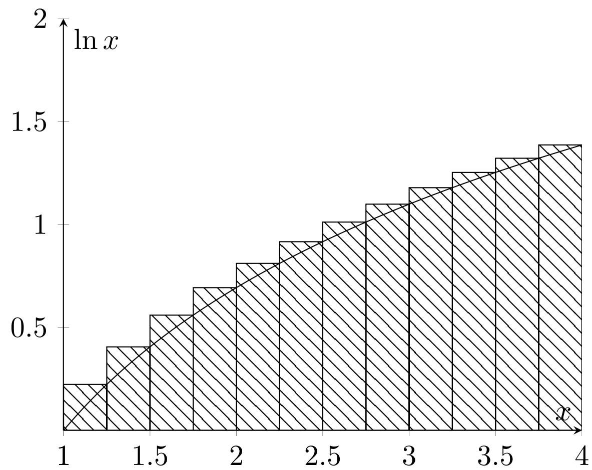
$$f''(x) = \frac{10}{9}e^x(e^x - 1)^{4/3}(10e^x - 3).$$

(Hint: $e^x(e^x - 1)^{4/3}$ is always positive.)

a. $\ln(3/10)$
b. 0
c. 1
d. $\ln(3/10), 0$
e. $\ln(3/10), 1$
f. $0, 1$
g. $\ln(3/10), 0, 1$
h. There are none

To have an inflection point, we must have $f''(x)$ be 0 or undefined. This happens when $e^x - 1 = 0$ or $10e^x - 3 = 0$, so when $e^x = 1$ or $e^x = \frac{3}{10}$, so $x = 0$ or $x = \ln(3/10)$. $e^x(e^x - 1)^{4/3}$ is always positive, so the only question is when the sign of $10e^x - 3$ is positive or negative. This happens exactly when $x > \ln(3/10)$; in particular, $\ln(3/10)$ is the only inflection point.

c **10.** This picture shows an estimation of the area under graph of $f(x) = \ln x$ from $[1, 4]$ using 12 rectangles. When k is an integer between 1 and 12, what is the area of the k-th rectangle?



a. $\frac{1}{12}\ln(\frac{k}{12})$
b. $\frac{1}{4}\ln(\frac{k}{4})$
c. $\frac{1}{4}\ln(1 + \frac{k}{4})$
d. $\frac{1}{4}\ln(\frac{k-1}{12})$
e. $\frac{1}{4}\ln(\frac{k-1}{4})$
f. $\frac{1}{4}\ln(1 + \frac{k-1}{4})$
g. $\frac{1}{12}\ln(1 + \frac{k}{12})$
h. $\frac{1}{4}\ln(1 + \frac{k}{12})$

The width of a rectangle is $\frac{4-1}{12} = \frac{1}{4}$. The k-th rectangle has height $\ln(1 + \frac{k}{4})$, so the area of the k-th rectangle is

$$\frac{1}{4}\ln(1 + \frac{k}{4}).$$

11. A pendulum clock uses the swinging of a pendulum to keep time. By cutting a metal rod to a precise length, we know that the pendulum will complete one swing in time

$$T = \sqrt{\frac{\ell}{g}}$$

where ℓ is the length of the rod and g is the gravitational constant. Any error in the length of the rod will make the clock less accurate, so we want to cut the rod very carefully. If the relative error in ℓ is 0.05, how big is the relative error in T?

$$\begin{aligned}\frac{\Delta T}{T} &\approx \frac{dT}{T} \\ &= \frac{\frac{1}{g}(1/2)(\ell/g)^{-1/2}d\ell}{(\ell/g)^{1/2}} \\ &= \frac{\frac{1}{g}(1/2)d\ell}{(\ell/g)} \\ &= \frac{d\ell}{2\ell} \\ &= 0.025\end{aligned}$$

12. Show that $\ln(1+e^{-x})-1$ does not have two distinct zeroes (in other words, that it has at most one zero). (You *do not* need to show that it has a zero.)

Let $f(x)=\ln(1+e^{-x})$. Suppose there were two zeroes, $a<b$, so $f(a)=f(b)$. Then by Rolle's Theorem, there would be a c with $f'(c)=0$.
But $f'(x)=\frac{1}{1+e^{-x}}(-e^{-x})$. This can never be 0, so there is no such c, so there could not be two distinct zeroes a and b.
(We can solve for the unique zero: $\ln(1+e^{-x})-1=0$ means $1+e^{-x}=e$, which means $e^{-x}=e-1$, so $x=-\ln(e-1)$.)