We name variables:

- x is the tons of CO_2e abated by the first policy.
- y is the total of CO_2e abated by the second policy.
- $t = x + y$ is the total tons of CO_2e abated.
- $C_1(x) = -30x + \frac{x^2}{4000}$ is the cost of the first policy
- $C_2(y) = 10y + \frac{y^2}{20000}$ is the cost of the second policy.
- $10000000 = C_1(x) + C_2(y)$ is the total cost of the combined policy.

We have the equation

$$10,000,000 = -30x + \frac{x^2}{4,000} + 10y + \frac{y^2}{20,000}$$

and we are trying to maximize,

$$t = x + y.$$

It would be hard to solve the the first equation for one variable, so instead we use implicit differentiation with respect to x:

$$0 = -30 + \frac{x}{2,000} + 10\frac{dy}{dx} + \frac{y\frac{dy}{dx}}{10,000}$$

and

$$\frac{dt}{dx} = 1 + \frac{dy}{dx}.$$

At a critical point, $\frac{dt}{dx} = 0$, so $\frac{dy}{dx} = -1$. Therefore

$$0 = -30 + \frac{x}{2,000} - 10 - \frac{y}{10,000}.$$

Now it's much easier to solve for one variable:

$$0 = -400,000 + 5x - y$$

so $y = 5x - 400,000$. Substituting back into the first equation,

$$10,000,000 = -30x + \frac{x^2}{4,000} + 10(5x - 400,000) + \frac{(5x - 400,000)^2}{20,000}$$

Multiplying everything out we have

$$200,000,000,000 = -600,000x + 5x^2 + 200,000(5x - 400,000) + (5x - 400,000)^2$$

which expands to

$$200,000,000,000 = -600,000x + 5x^2 + 1,000,000x - 80,000,000,000 + 25x^2 - 4,000,000x + 160,000,000,0$$

Combing terms,

$$0 = 30x^2 - 3,600,000x - 120,000,000,000,$$

or

$$0 = x^2 - 120,000x - 4,000,000,000,$$

After all that, it's just a quadratic:

$$\begin{aligned} x &= \frac{120,000 \pm \sqrt{(120,000)^2 + 4 \cdot 4,000,000,000}}{2} \\ &= \frac{120,000 \pm \sqrt{14,400,000,000 + 16,000,000,000}}{2} \\ &= 60,000 \pm \frac{1}{2}\sqrt{30,400,000,000} \\ &= 60,000 \pm 5,000\sqrt{304}. \end{aligned}$$

We can check (maybe with a calculator) that $\sqrt{304}$ is a bit over 17. In particular, $60,000 - 5,000\sqrt{304} < 0$, so the only critical point is when

$$x = 60,000 + 5,000\sqrt{304}$$

and

$$y = 25,000\sqrt{304} - 100,000.$$

No, I didn't really expect anyone to get through all that algebra during class. The really important part is setting up the equations and being able to use implicit differentiation to start the process—in other words, being able to get to the equation $y = 5x - 400000$. After that it's just algebra. The algebra is long and tedious, but it isn't really hard, *so it is something you should be able to do given time and a whole lot of paper. It's not something I'd ever ask you to do while timed.*