# Lecture 1

# Fourier-Laplace integrals in probability, combinatorics and statistics

The Fourier-Laplace integral is an integral over a neighborhood $\mathcal{N}$ in $\mathbb{R}^d$ of the form

$$\int_{\mathcal{N}} e^{-\lambda\phi(\mathbf{z})}\, A(\mathbf{z})\, d\mathbf{z}\,.$$

The function $A$ is called the **amplitude** and the function $\phi$ is called the **phase**. Both functions are assumed to have some degree of smoothness. These functions may be real or complex. When $\phi$ is real, we call it a **Laplace**-type integral and when $\phi$ is purely imaginary, we call it a **Fourier**-type integral, though we will see that mathematically these are both the same and are both subcases of the general Fourier-Laplace integral. Various conditions serve to localize the integral to a point $\mathbf{z}_0$ in the interior of $\mathcal{N}$; for example, $\mathrm{Re}\,\{\phi\}$ may assumed to have a strict minimum minimum at $\mathbf{z}_0$; or the minimum may not be strict but then $A$ must be compactly supported; a less elementary omnibus condition may be found in [PW09], along with complete proofs of everything whose proof is outlined in this lecture.

Fourier-Laplace integrals arise frequently in probability theory, as illustrated by the following examples.

(i) Thermodynamical ensembles: $\phi$ is energy, $\lambda$ is inverse temperature, and $A$ is a counting or weighting function.

(ii) Large deviations: $A$ is a density and $e^{-\lambda\phi}A$ is an associated exponential family.

(iii) Characteristic functions: take $\lambda = |\mathbf{r}|$ and $\lambda\phi(\mathbf{z}) = \mathbf{r} \cdot \mathbf{z}$ to be linear, with $A(\mathbf{z})\, d\mathbf{z}$ being any measure. The integral is the characteristic function (Fourier transform).

(iv) Analytic combinatorics: $F(\mathbf{z}) = \sum_{\mathbf{r}} p(\mathbf{r})\mathbf{z}^{\mathbf{r}}$ is a probaiblity generating function. If $A = F \circ \exp$ and $\phi$ is linear, then the integral inverts the series via Cauchy's integral formula, yielding formulae for $p_{\mathbf{r}}$:

$$(2\pi i)\, p_{\mathbf{r}} = \int \exp(-\mathbf{r} \cdot \mathbf{z}) A(\mathbf{z})\, d\mathbf{z}\,.$$

The reason for writing the phase as $\lambda \cdot \phi$ is that the asymptotics for such an integral as $\lambda \to \infty$ with $\phi$ fixed are well understood and easy to derive and remember. These asymptotics are also usually what we desire from such an integral. The goal of this first lecture is to describe how we evaluate these integrals, often at a glance, and to illustrate with a number of examples along the lines of the four classes above.

## 1.1 Laplace-type integrals in one variable

### A simple example: Stirling's formula

To motivate what follows, consider the derivation of Sterling's formula for factorials

$$n! \sim n^n e^{-n} \sqrt{2\pi n}\,. \tag{1.1}$$

This is surely one of the most useful asymptotic approximations you will have come across. To derive it, we express the factorial as a Gamma function $n! = \Gamma(n+1) = \int_0^\infty t^n e^{-t}\, dt$. Writing the integrand as the exponential $\exp(n \log t - t)$, the main contribution comes where the integrand is maximized, which calculus shows to occur at $t = n$. We therefore change variables to $t = n + u$ to obtain

$$\begin{aligned} n! &= \int_{-n}^{\infty} \exp(n \log(n+u) - (n+u))\, du \\ &= e^{n \log n - n} \int_{-n}^{\infty} \exp\left[n \log\left(1 + \frac{u}{n}\right) - u\right] du\,. \end{aligned}$$

Plugging in the two-term Taylor series $\log(1 + u/n) = u/n - u^2/(2n^2) + O(u^3/n^2)$ gives

$$n! = e^{n \log n - n} \int_{-n}^{\infty} \exp\left[-\frac{u^2}{2n} + O\left(\frac{u^3}{n^2}\right)\right] du\,. \tag{1.2}$$

The integral of $e^{-u^2/(2V)}$ is $\sqrt{2\pi V}$, so to prove (1.1) we need only bound the error created by the $O(u^3/n^2)$ term in the exponent. Changing variables to $v = u/\sqrt{n}$ gives

$$\begin{aligned} \exp\left[-\frac{u^2}{2n} + O\left(\frac{u^3}{n^2}\right)\right] &= \exp\left[-\frac{v^2}{2} + O\left(\frac{v^3}{\sqrt{n}}\right)\right] \\ &= \exp\left(-\frac{v^2}{2}\right)\left(1 + O\left(\frac{v}{\sqrt{n}}\right)\right) \end{aligned}$$

for $|v| \le n^{1/6}$, which we may assume because the integral outside of $|v| < n^\epsilon$ is rapidly decreasing for any $\epsilon > 0$. Integrability of $ve^{-v^2/2}$ implies that

$$\int \exp\left(-\frac{v^2}{2}\right)\left(1 + O\left(\frac{v}{\sqrt{n}}\right)\right) = \sqrt{2\pi} + O\left(\frac{1}{\sqrt{n}}\right)$$

and plugging this into (1.2) yields the Stirling approximation (1.1).

### The general one-dimensional case

Going back to the evaluation of $\int A(x)e^{-\lambda\phi(x)}\,dx$, the lesson we draw is that we should find the argument $x_0$ where the minimum of $\phi$ occurs and approximate $\phi(x_0 + t)$ by its leading Taylor term. If $\phi$ is smooth, then $\phi(x) - \phi(x_0) \sim C_k(x - x_0)^k$ for some $k \ge 2$. Generically, $k = 2$, though it is no more difficult to handle the general case. Necessarily, at the minimum of a real function, $k$ must be even.

Going from $\exp(-v^2/2 + O(v^3/n^2))$ to $\exp(-v^2/2)(1 + O(v/\sqrt{n}))$ above was somewhat *ad hoc*. The more general way to do this is with a change of variables. The function $y = (\phi(x) - \phi(x_0))^{1/k}$ is smooth and may therefore be inverted, so that $x$ becomes a function $x(y)$ and $\phi$ is naturally parametrized as

$$\phi(x(y)) = \phi(x_0) + y^k$$

in a neighborhood of $y = 0$. At zero, the derivative of the parametrization is $x'(0) = (\phi^{(k)}(0)/k!)^{-1/k}$. The Laplace integral then becomes

$$\int A(x)e^{-\lambda\phi(x)}\,dx = e^{-\lambda\phi(x_0)} \int A(x(y))x'(y)e^{-\lambda y^k}\,dy\,.$$

Here, the first integral is localized to $x_0$ and the second to zero, where it is easily verified that the main contributions occur. Writing $\psi = (A \circ x) \cdot x'$ and expanding $\psi$ in a Taylor series around zero as $\psi(y) = \sum_{j=0}^\infty b_j y^j$ gives

$$\int A(x)e^{-\lambda\phi(x)}\,dx = e^{-\lambda\phi(x_0)} \int \sum_{j=0}^{\infty} b_j y^j e^{-\lambda y^k}\,dy \tag{1.3}$$

where $b_0 = A(x_0)(\phi^{(k)}(0)/k!)^{-1/k}$ and the remaining coefficients $b_j$ may be computed via manipulation of formal power series. The series (1.3) may be integrated term by term. Plugging in

$$\int x^j e^{-x^k}\, dx = c(k,j) := \Gamma\left(\frac{1+j}{k}\right)$$

if $j$ is even and zero if $j$ is odd proves the following result.

**Definition 1.1** (asymptotic expansion)**.** *The asymptotic expansion* $f \sim \sum_{j=0}^{\infty} g_j$ *denotes the relation that* $f - \sum_{j=0}^{N-1} g_j = O(g_N)$ *for every* $N$*.*

**Theorem 1.2** (Laplace-type asymptotics in one variable)**.** *Let* $\phi$ *be a real analytic function with a strict minimum at* $x_0$*, let* $\phi(x) \sim C(x-x_0)^k$ *and let* $x(y)$ *satisfy* $\phi(x(y)) = y^k$*. Let* $A$ *be a real analytic function. Then*

$$\int A(x) e^{-\lambda\phi(x)}\, dx \sim \sum_{j=0}^{\infty} b_{2j} c(2j,k) \lambda^{(-1-2j)/k}$$

*where* $b_0 = A(x_0)C^{-1/k}$ *and the quantities* $b_j$ *for* $j > 0$ *are the power series coefficients for* $(A \circ x(y))x'(y)$*.*

The most common special case is when $k = 2$ and $A(0) \neq 0$. Then $C = \phi''(x_0)/2$ and $\Gamma(1/2) = \sqrt{\pi}$ so $b_0 = \sqrt{2\pi/\phi''(x_0)}$ and we have the following result.

**Corollary 1.3.** *If* $A(x_0) \neq 0$ *and* $k = 2$ *(quadratic nondegeneracy) then*

$$\int A(x) e^{-\lambda\phi(x)}\, dx \sim \sqrt{\frac{2\pi}{\lambda\phi''(x_0)}} e^{-\lambda\phi(x_0)} .$$

□

**Exercise 1.1.** Compute the second nonvanishing term of $\psi$ for the case of Stirling's formula and use it to derive the $O(1/n)$ correction term. Show that this more accurate formula may be written as

$$n^n e^{-n} \sqrt{2\pi\left(n + \frac{1}{6} + O(n^{-1})\right)} .$$

## 1.2 Laplace-type integrals in several variables

### Quadratically nondegenerate phase

Suppose now that $\phi$ and $A$ are smooth, real functions on $\mathbb{R}^d$. Recentering if necessary, we assume $\phi$ has a strict minimum of zero at the origin. We examine first the most common case, which is also the simplest, where $\phi$ is quadratically nondegenerate. This means that $\phi(\mathbf{x}) = Q(\mathbf{x}) + O(|\mathbf{x}|^3)$ where $Q(\mathbf{x}) = \sum q_{ij} x_i x_j$ is a positive definite quadratic. Observing that the matrix $(q_{ij})$ is just one half the Hessian matrix $\mathcal{H} := (\partial^2 \phi / \partial x_i \partial x_j)$, we write $|Q|$ to denote the determinant of the representing matrix, $(1/2)\mathcal{H}$. In the nondegenerate case $|Q| \neq 0$ and the Morse lemma (see [Ste93, VIII.3.2] or [PW09, Lemma 4.2]) shows that there is a change of variables $\mathbf{x}(\mathbf{y})$ such that $Q(\mathbf{x}(\mathbf{y})) = S(\mathbf{y}) := \sum_{i=1}^d y_i^2$, the standard quadratic form. This leads to

$$\int A(\mathbf{x}) e^{-\lambda \phi(\mathbf{x})} \, d\mathbf{x} = \int \psi(\mathbf{y}) e^{-\lambda S(\mathbf{y})} \, d\mathbf{y} \tag{1.4}$$

where $\psi = (A \circ \mathbf{x}) \cdot J$ and $J$ is the determinant of the Jacobian matrix of the map $\mathbf{y} \mapsto \mathbf{x}(\mathbf{y})$. Expansion of $\psi$ into a power series in $\mathbf{y}$ is theoretically trivial, although efficient computation of this requires some care. The leading coefficient is easy enough: when $A(\mathbf{x}_0) \neq 0$ then $\psi(\mathbf{0}) = A(\mathbf{x}_0)|Q|^{-1}$.

When $\psi$ is a monomial, the integral factors into $d$ one-dimensional integrals of the type in Theorem 1.2 with $k = 2$. Expanding $\psi$ in a multivariate Taylor series, and integrating term by term then gives the following theorem.

**Theorem 1.4** (quadratically nonegenerate phases in several variables)**.** *If $A$ is analytic and $\phi$ has a quadratically nondegenerate minimum at $\mathbf{x}_0$ then*

$$\int A(\mathbf{x}) e^{-\lambda \phi(\mathbf{x})} \, d\mathbf{x} \sim \left(\frac{2\pi}{\lambda}\right)^{d/2} |Q|^{-1/2} A(\mathbf{x}_0) \tag{1.5}$$

*with a full asymptotic expansion in decreasing powers of $\lambda$ computable term by term and taking the form*

$$\int A(\mathbf{x}) e^{-\lambda \phi(\mathbf{x})} \, d\mathbf{x} \sim \sum_{j=0}^{\infty} b_j \lambda^{-j-d/2} \, .$$

We will treat more general phases later because complex variable methods are needed even for the real case.

## A statistical example

The following example is taken from [DSS09, Example 2.2.3]. A data set consists of $N$ trials, each of which is a set of three coin flips. Either every trial is independent flips of the same coin, with parameter $p$, resulting in $N$ samples from a $\mathrm{BIN}(3, p)$ distribution, or there are two coins with parameters $\beta$ and $\gamma$ and each trial first selects between these with probability

$\alpha$, then flips the selected coin three times. Distinguishing between these possibilities is a problem of model selection. The usual procedure for this is to integrate the likelihood function over all parameter values with respect to uniformative reference measure. In this case, we need the integral of the likelihood function $P(\alpha,\beta,\gamma)$ over $\alpha,\beta,\gamma \in [0,1]$, where

$$P(\alpha,\beta,\gamma) := \mathbb{P}(\text{data} \,|\, \alpha,\beta,\gamma)\,.$$

Suppose the data contains $n_i$ instances of $i$ HEADS, for $0 \le i \le 3$ and $\sum_{i=0}^{3} n_i = N$. The likelihood function is given by

$$\binom{N}{n_0,n_1,n_2,n_3} \prod_{i=0}^{3} \left[ \binom{3}{i} \left( \alpha\gamma^i (1-\gamma)^{3-i} + (1-\alpha)\beta^i (1-\beta)^{3-i} \right) \right]^{n_i}.$$

Factoring out the multinomial coefficient and denoting $q_i := n_i/N$, this becomes

$$\frac{1}{N} \log P = C_N + \sum_{i=0}^{3} q_i \log f_i \tag{1.6}$$

where

$$f_i := \binom{3}{i} \left[ \alpha\gamma^i (1-\gamma)^{3-i} + (1-\alpha)\beta^i (1-\beta)^{3-i} \right] \tag{1.7}$$

is the probability of $i$ HEADS on any single trial.

The likelihood integral is $\int \exp(-\lambda\phi)$ where $\lambda = N$ and $-\phi = C_n + \sum_i q_i f_i(\alpha,\beta,\gamma)$. The first task in computing the integral is to find where $\phi$ is minimized. To maximize $\sum q_i \log f_i$ for fixed $\{q_i\}$ subject to $\sum_i f_i = 1$, one needs to take $f_i$ proportional to $q_i$, or in this case equal because $\sum_i q_1 = 1$ as well. This gives three equations (one for each $i$ except the last, which follows because $\sum_i f_i = 1 = \sum_i q_i$).

Symbolic algebra techniques may be used to show that for generic $\mathbf{q} := (q_0,q_1,q_2,q_3)$ there are precisely two triples $(\alpha,\beta,\gamma) \in [0,1]^3$ solving $f_i = q_i$. In case you don't know how to do this: you compute a Gröbner basis for

$$\{f_0(\alpha,\beta,\gamma) - q_0, f_1(\alpha,\beta,\gamma) - q_1, f_2(\alpha,\beta,\gamma) - q_2, f_2(\alpha,\beta,\gamma) - (1 - q_0 - q_1 - q_2)\}$$

with respect to the term order $\texttt{plex}(\gamma,\beta,\alpha)$; the first polynomial is quadratic in $\gamma$ (with coefficients that are polynomials in $q_i$) and the remaining polynomials express $\alpha$ and $\beta$ in terms of $\gamma$.

These algebraic points are then substituted into formulae for the second partial derivatives of $\phi(\alpha,\beta,\gamma) := \sum_i q_i \log f_i$ and the determinant evaluated. For generic $\mathbf{q}$, the Hessian determinant is nonzero, meaning that $\phi$ is quadratically degenerate. The Hessian determinant and the value of $\phi$ at the maximizing points are exact algebraic numbers, so we have found an exact formula for the coefficients of the asymptotic expansion of $\int \exp(N \sum_i q_i \log f_i)$ in decreasing powers of $N$.

## 1.3 Complex phases

### Back to one variable

Suppose we have a generating function $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If $f$ is analytic in some domain, we may recover $a_n$ from Cauchy's integral formula:

$$a_n = \frac{1}{2\pi i} \int z^{-n-1} f(z)\, dz\,.$$

The integral is taken initially over a small circle about the origin. Evaluating the integral usually requires moving the contour, so that it passes through a **stationary phase** point, where the derivative of $n \log z = \log f$ vanishes. When $f$ is entire (or in general, when the stationary phase point is interior to the comain of convergence of $f$), this process is known as **Hayman's method** after [Hay56]. An example of this is as follows.

**Example 1.5** (involutions)**.** Let $g(n)$ denote the number of involutions in the symmetric group $S_n$. When written in cycle product form, an involution is characterized by having cycles only of length 1 and 2. In the next lecture, we will examine the generating function for permutations with restrictions on cycle lengths via the **exponential formula** (see, e.g., [Wil94]). For involutions the e.g.f. is easily seen to be

$$f(z) = \exp\left(z + \frac{z^2}{2}\right)\,. \tag{1.8}$$

Cauchy's integral formula yields

$$a_n = \int_\gamma \exp\left[-(n+1)\log z + z + \frac{z^2}{2}\right] dz$$

where $\gamma$ is a small circle about the origin with a couterclockwise orientation.

If the integrand were real, we would look for the maximum of the argument of the exponential, setting the derivative equal to zero and solving. The method of stationary phase says to do the same thing: let $\phi_n := z + z^2/2 - (n+1)\log z$ and solve for $\phi_n'(z) = z + 1 - (n+1)/z = 0$. Sovling the quadratic yields

$$z_\pm = -\frac{1}{2} \pm \sqrt{n + \frac{5}{4}}\,.$$

The stationary phase method tells us to deform the contour so as to pass through one of these points in a direction such that the magnitude of the integrand is maximized at the point. If we expand $\gamma$ keeping it a circle centered at zero, we reach first the root of lesser magnitude, which we denote by $z_+ := \sqrt{(n + 5/4)} - 1/2$. It is easy to verify that on this circle, the real part of $\phi$ is maximized at $z_+$ and that the maximum is strict. If we now forget about the fact that the integrand is not real, we would have the Laplace-type estimate

$$\frac{1}{2\pi i} \int e^{-\phi_n(z)}\, dz \sim \frac{1}{2\pi i}\, e^{-\phi_n(z_+)} \left( \frac{2\pi}{n\phi_n''(z_+)} \right)^{1/2} . \tag{1.9}$$

The following result shows that we can indeed ignore the leap from real to complex integrands and contours.

**Theorem 1.6** (complex phases in one variable)**.** *Suppose that $\phi$ and $A$ are complex analytic functions. Suppose that $\gamma$ is a countour on which there is a unique critical point $z_0$ for $\phi$, and that the real part of $\phi$ on $\gamma$ is minimized (not necessarily strictly) at $z_0$. Then the conclusion of Theorem 1.2 holds. In particular, if $A$ and $\phi$ are both nonvanishing at $z_0$ then the conclusion of Corollary 1.3 holds:*

$$\int_\gamma e^{-\lambda\phi(z)}\, A(z)\, dz \sim \sqrt{\frac{2\pi}{\phi''(z_0)}}\, e^{-\lambda\phi(z_0)} .$$

Sketch of Proof: (1) Deforming the contour a small amount along the gradient flow, we may assume that the real part of $\phi$ is strictly minimized at $z_0$. (2) We may now localize, considering the integral only in a small neighborhood of $z_0$ in $\gamma$. (3) Using the same analytic change of variables as in (1.3), we reduce to the case where the phase is $z^k$ but the contour is now a small complex arc through the origin. (4) Deforming the contour by projecting out the inaginary part changes the integral by an exponentially small quantity, at the same time reducing it to the computation in Theorem 1.2. All of this is carried out in [PW09]. □

**Example 1.5 continued**. according to Theorem 1.6, we can now continue from (1.9). As it happens,

$$\begin{aligned} \phi_n(z_+) &= -\frac{1}{2} n \log n + \frac{1}{2} n + n^{1/2} - \frac{1}{2} \log n = \frac{1}{4} + o(1) ; \\ \phi_n''(z_+) &= 2 + o(1) \end{aligned}$$

which would then lead to

$$\begin{aligned} n!\, a_n &= n! e^{\phi_n(z_+)} \sqrt{\frac{1}{2\pi \phi_n''(z_+)}} \\ &= \exp\left(\frac{1}{2} n \log n - \frac{1}{2} n + n^{1/2} - \frac{1}{2}\log 2 - \frac{1}{4} + o(1)\right) \\ &\sim \sqrt{n!}\, e^{\sqrt{n}} \, (8\pi e n)^{1/4} \, . \end{aligned}$$

**Example 1.7** (weighted paths)**.** As an example, let us count weighted paths of length $n$ from 0 to 0, where the set of possible steps is some finite $E \subseteq \mathbb{Z}$ and an increment of $j$ gets weight $w_j$. Let $g(z) := \sum_{j \in E} w_j z^j$ be the Laurent polynomial generating function for the weighted steps. Then the generating function counting weighted paths of length $n$ by endpoint is $g^n$. The total weight $W_n(0,0)$ of paths returning to the orign at time zero is the constant coefficient which is given by

$$\frac{1}{2\pi i} \int_\gamma \frac{1}{z} g(z)^n \, dz = \frac{1}{2\pi i} \int_\gamma \frac{1}{z} e^{n \log g(z)} \, dz \, .$$

The integrand is analytic everwhere except the origin, so we may proceed as with the involutions, moving the contour to where $g'(z) = 0$. As long as $E$ contains at least one positive and at least one negative number, as $z$ varies from zero to infinity, the derivative of $\log g(z)$ will increase from $-\infty$ to $\infty$, so $g'$ has a unique zero, $z_0$. Setting $A(z) = z^{-1}$, we obtain the asymptotic expression

$$W_n(0,0) \sim \frac{1}{z_0} \sqrt{\frac{2\pi}{n (\log g)''(z_0)}} \, g(z_0)^n \, . \tag{1.10}$$

Suppose that $0 \le w_j \le 1$ for all $j$ with $\sum_{j \in E} w_j = 1$; then this derivation is formally quite similar to the large deviation computation of $\mathbb{P}(S_n = 0)$ where $S_n$ is the $n^{th}$ partial sum of IID random variables with probability generating function $g(z)$.

**Exercise 1.2.** Verify (1.10) in this using a local large deviation theorem. We will prove a general such theorem in a subsequent lecture.

## Many variables, quadratically nondegenerate phase

As in the univariate case, everything we know for real phases works for complex phases. For quadratically nondegenerate phase functions we have the following result.

**Theorem 1.8** (quadratically nongenerate complex phases in $d$ variables [PW09, Theorem 2.3])**.** *Suppose that $\phi$ and $A$ are complex analytic functions of several variables and suppose that $\gamma$ is a smooth chain of integration on which there is a unique critical point $z_0$ for $\phi$, and that the real part of $\phi$ on $\gamma$ is minimized (not necessarily strictly) at $z_0$. If $\phi$ is quadratically nondegenerate and $A(z_0) \neq 0$ then the conclusion of Theorem 1.4 holds.*

As an example, we compute the Fourier transform of a measure supported on a smooth hypersurface in $\mathbb{R}^d$; this will be useful in the lecture on quantum random walks.

**Example 1.9** (Fourier transform of hypersurface measure)**.** Let $\mathcal{M}$ be a compact $(d-1)$-manifold in $\mathbb{R}^d$ and let $\mu$ be a finite measure on $\mathcal{M}$ equivalent to $(d-1)$-dimensional Lebesgue measure (in any, hence every chart). For $\mathbf{r} \in \mathbb{R}^d$, denote the Fourier transform by

$$\hat{\mu}(\mathbf{r}) := \int_{\mathcal{M}} e^{i\mathbf{r}\cdot\mathbf{x}} \, d\mu(\mathbf{x}) \, .$$

Let $\hat{\mathbf{r}}$ denote the unit vector $\mathbf{r}/|\mathbf{r}|$ in the direction of $\mathbf{r}$. The critical points for the function $\mathbf{r} \cdot \mathbf{x}$ on $\mathcal{M}$ are precisely those points where the normal to $\mathcal{M}$ is parallel to $\hat{\mathbf{r}}$. Generically, this will be a finite set of points, depending only on $\hat{\mathbf{r}}$, which we denote by $E(\hat{\mathbf{r}})$. Letting $\lambda$ denote $|\mathbf{r}|$, the asymptotics for the Fourier transform are given by choosing a $(d-1)$-dimensional parametrization of $\mathcal{M}$ near each $\mathbf{x} \in E$ and using (1.4). Letting $A$ be the density of $\mu$ in this parametrization, the contribution near $\mathbf{x}$ is given asymptotically by

$$\Phi_{\mathbf{x}}(\mathbf{r}) = \left(\frac{2\pi}{|\mathbf{r}|}\right)^{d/2} A(\mathbf{x}) \left|\frac{1}{2}\mathcal{H}(\phi, \mathbf{x}_0)\right|^{-1/2}$$

where $\phi$ is the function $\mathbf{x} \mapsto -\hat{\mathbf{r}} \cdot \mathbf{x}$. This leads to

$$\hat{\mu}(\mathbf{r}) = \sum_{\mathbf{x}\in E} e^{i\mathbf{r}\cdot\mathbf{x}} \Phi_{\mathbf{x}}(\mathbf{r}) + O(|\mathbf{r}|^{-d/2-1})$$

as $\lambda \to \infty$ with $\mathbf{r} = \lambda\hat{\mathbf{r}}$. This gives the leading term asymptotic as long as the order $|\mathbf{r}|^{-d/2}$ terms do not cancel, this cancellation occurring for at most a discrete set of values of $\lambda$.

## Many variables, with degeneracies allowed

For more general phase functions, we see some of the possible intricacies by examining the real case. Suppose the phase $\phi$ is real with a minimum of zero zt the origin and let $B(\epsilon) :=:= \{\mathbf{x} : \phi(\mathbf{x}) < \phi(\mathbf{x}_0) + \epsilon\}$ denote the region where the phase is at most $\epsilon$. For quadratically nondegenerate phases, as $\epsilon \to 0$, $B$ becomes an ellipsoid with asymptotic volume $C\epsilon^{-d/2}$. When the quadratic term of $\phi$ is not positive definite, the volume of the set $B(\epsilon)$ is of larger order as $\epsilon \to 0$. This leads to asymptotics of greater order than $\lambda^{-d/2}$. The relevance of these volumes is intuitively obvious and may be made precise if one integrates by parts. Let $V(\epsilon)$ denote the volume of the set $B(\epsilon)$ above. Then

$$\int e^{-\lambda\phi(\mathbf{x})}\, d\mathbf{x} = \int e^{-\lambda t}\, V'(t)\, dt$$

and is a one-dimensional Laplace transform of a type that is well understood, provided one knows the asymptotics of $V$.

The first place I know where this is done effectively is Varchenko's 1976 paper [Var77]. A complete explanation is well beyond what I can cover here, but a summary is as follows. Treat the real and complex cases together (i.e., even if $\phi$ is real, treat is as a complex function on $\mathbb{C}^d$). Hironaka's celebrated theorem on resolution of singularities says that we my find a change of variables under which both $\phi$ and $A$ become monomials. Our canonical form is now $\phi(\mathbf{x}(\mathbf{y})) = \prod y_i^{n_i}$ rather than $\sum y_i^2$. This canonical form for the exponent is not quite as easy to handle as a sum of pure powers because it does not separate into a product. The monomial integrals

$$I(\mathbf{m}, \mathbf{r}) := \int e^{-\lambda \mathbf{y}^{\mathbf{m}}} \prod y_i^{n_i}$$

have however, been worked out. Thus, expanding $\psi$ in monomials,

$$\begin{aligned} \int A(\mathbf{x}) e^{-\lambda\phi}(\mathbf{x}) &= \int \sum_{\mathbf{r}} b_{\mathbf{r}} \mathbf{y}^{\mathbf{r}} e^{-\lambda \mathbf{y}^{\mathbf{m}}} \\ &= \sum_{\mathbf{r}} b_{\mathbf{r}}\, I(\mathbf{m}, \mathbf{r})\,. \end{aligned}$$

The integrals $I(\mathbf{m}, \mathbf{r})$ are known all to be of the form $C\lambda^{-\alpha}(\log \lambda)^{\beta}$, with $C, \alpha$ and $\beta$ determined by $\mathbf{m}$ and $\mathbf{r}$. The determination of $C, \alpha$ and $\beta$ is effective but not elementary. It is was first described in terms of the **Newton diagram** for $\phi(\mathbf{x} - \mathbf{x}_0)$ by Varchenko [Var77]. The (lower boundary of the) Newton diagram consists of all monomials $c_{\mathbf{r}}\mathbf{z}^{\mathbf{r}}$ occurring in the Taylor series for $\phi$ such that some choices of the coordinate moduli $|z_1|, \ldots, |z_d| << 1$ makes $c_{\mathbf{r}}\mathbf{z}^{\mathbf{r}}$ the dominant term of the series. Varchenko handled the case where where $\lambda$ is purely imaginary, however the real case is similar; see, e.g. [AGZV88, Section 7.2.4]. It is an interesting exercise to work out the result for quadratically nondegenerate phases from the more general result!