Lecture 5

Proofs

It will be instructive to see two proofs of Theorem 3.1. The first is elementary; it requires stronger hypotheses but not a lot of machinery. The second is more canonical and more general, but requires a bit of analytic technology.

5.1 First proof

To fix the notation, let $F = P/Q$ be a meromorhpic function in d variables and let $\mathbf{Z}_* = \exp(\mathbf{z}_*) = \exp(\mathbf{x}_* + i\mathbf{y}_*)$ be a point on the boundary of the domain of convergence of the series $F(\mathbf{Z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{Z}^{\mathbf{r}}$. Assume the hypotheses of Theorem 3.1:

(i) Smoothness: $\mathcal{V}$ is smooth in a neighborhood of $\mathbf{Z}_*$;

(ii) Strict minimality: $\mathcal{V} \cap \exp(\mathbf{x}_* + i\mathbb{R}^d) = \{\mathbf{Z}_*\}$;

(iii) Parametrization: $\partial Q/\partial Z_d(\mathbf{Z}_*) \neq 0$;

(iv) Nonsingularity of $\mathcal{H}(g, \mathbf{z}_*)$ where g parametrizes $\log \mathcal{V}$ near $\mathbf{z}_*$ as in (3.3).

Step 1: Localization

The multivariate Cauchy integral formula tells us that

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_T \mathbf{Z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{Z})\, d\mathbf{Z}$$

where T is any torus inside the domain of convergence. A convenient choice for T is $U \times V$ where U is the projection of the torus containing $\mathbf{Z}_*$ onto the first $d-1$ coordinates and V is a circle of any radius small enough so that Z_d is outside it. Using this decomposition, we write the Cauchy integral as an iterated integral:

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_U \left[\int_V \mathbf{Z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{Z})\, dZ_d\right] d(Z_1, \ldots, Z_{d-1}) .$$

We now show the outer integral to be localized at the point $(Z_1, \ldots Z_{d-1})_*$, meaning it is negligible outside of any neighborhood of this point. Here, negligible means exponentially smaller than the whole integral, which has magnitude $|\mathbf{Z}_*|^{-\mathbf{r}+o(\mathbf{r})}$; in fact we will see it has magnitude bounded above by $|\mathbf{Z}_*|^{-(1+\epsilon)\mathbf{r}}$. The reason is simple. Let $R := |(Z_*)_d|$ be the magnitude of the last coordinate of $\mathbf{Z}_*$. The inner integrand, $F(Z_1, \ldots, Z_{d-1}, \cdot)$, is analytic out to the least modulus solution W to $F(Z_1, \ldots, Z_{d-1}, W) = 0$. This has magnitude at least $R + \delta$ as long as we stay away from the strictly minimal point $\mathbf{Z}_*$. Localization now follows from the univariate theory (or directly from the univariate Cauchy formula).

Step 2: Residue

We now restrict to a small neighborhood $\mathcal{N} \subseteq U$. Let V' denote a circle whose radius is greater than R but less than the magnitude of any other solution W to $F((Z_*)_1, \ldots, (Z_*)_{d-1}, W) = 0$. Then, taking $\mathcal{N}$ sufficiently small, the set $\mathcal{N} \times V'$ will not intersect $\mathcal{V}$. We may therefore use univariate residue theory to compare the inner integrals over the contours V and V'. The integral over V' is negligible because $\mathbf{Z}^{-\mathbf{r}}$ is small there, so the integral over V is well approximated by the difference.

The function $F(Z_1, \ldots, Z_{d-1}, W)$ has a single simple pole bewteen V and V', so the difference between the inner integrals becomes $\mathbf{Z}^{-\mathbf{r}-\mathbf{1}}$ times the residue of F at $(Z_1, \ldots, Z_{d-1})$. The residue is equal to $(2\pi i)P(\mathbf{Z})/(\partial Q/\partial Z_d)(\mathbf{Z})$. Putting this together, letting E denote a term that is exponentially smaller than $\mathbf{Z}^{-\mathbf{r}}$, we have

$$a_{\mathbf{r}} = E + \left(\frac{1}{2\pi i}\right)^{d-1} \int_{\mathcal{N}} \mathbf{Z}^{-\mathbf{r}} \frac{P}{\partial Q/\partial Z_d}(\mathbf{Z})\, d(Z_1, \ldots, Z_{d-1}) , \tag{5.1}$$

where $\mathbf{Z}$ denotes the unique point $(Z_1, \ldots, Z_{d-1}, W)$ on $\mathcal{V}$ with $R < |W| < R + \delta$.

Step 3: Saddle integral

Typically, there are many strictly minimal smooth points. The localization and residue computation above work for all of them. Up to this point, we have not made use of the relation between $\mathbf{r}$ and $\mathbf{Z}_*(\hat{\mathbf{r}})$ that defines a particular point $\mathbf{Z}_*$. In order for the "large" part of the integrand of (5.1), namely $\mathbf{Z}^{-\mathbf{r}} = \exp(-\mathbf{r} \cdot \log \mathbf{Z})$, to have maximum magnitude, we need the gradient of $-\mathbf{r} \cdot \log \mathbf{Z}$ to be parallel to ∇Q; we have chosen $\mathbf{Z}_*(\hat{\mathbf{r}})$ to be where this happens. Changing to logarithmic coordinates (lower case variable names), we let

$$\begin{aligned} \phi(z_1, \ldots, z_{d-1}) &:= -\hat{\mathbf{r}} \log \mathbf{z} \\ A(\mathbf{z}) &= \frac{P(\mathbf{z})}{z_d \partial Q / \partial z_d}(\mathbf{z}) \end{aligned}$$

where $\mathbf{z}$ is the point on $\log \mathcal{V}$ near $\mathbf{z}_*$ whose first $d-1$ coordinates project to $\mathbf{z}_*$. We may now use Theorem 1.8 to evaluate the integral. This results in the desired conclusion.

Proof of cheap generalizations

1. While it is freshly in mind, examine the proof to see that we require meromorphicity of F only in an open set containing the torus $\exp(\mathbf{x} + i\mathbb{R}^d)$ through all minimal points $\mathbf{Z}(\hat{\mathbf{r}})$ corresponding to directions $\hat{\mathbf{r}}$ in which we wish our asymptotics to be uniform.

2. If, instead of one point in $\text{ReLog}^{-1}(\mathbf{x}_*) \cap \mathcal{V}$, there are several, then we may localize the integral into several neighborhoods, one for each point, argue as above, then sum the contributions near each point and disregard the negiligible remainder. This proves Corollary 3.2.

5.2 Second proof and canonical representations

The residue form

There is nothing magical about parametrizing by the first $(d-1)$ coordinates. In applications such as spacetime generating functions, this symmetry breaking may be natural, but hypotheses are invariant under permuting coordinates, so the conclusion must be and the proof should probably be as well.

Considering first the proof, we took a residue with respect to one eoordinate and integrated over a parametrization by the other coordinates. This cries out for a description in terms of differential forms. Indeed, the form

$$\frac{P}{\partial Q / \partial Z_j}(\mathbf{Z})\, dZ_1 \wedge \cdots \wedge \widehat{dZ_j} \wedge \cdots \wedge dZ_d \tag{5.2}$$

is independent of j when restricted to $\mathcal{V}$. It is well known that there is a unique form ω on $\mathcal{V}$, called the **Gelfand-Leray form**, satisfiying

$$\omega \wedge dQ = P \, d\mathbf{Z} .$$

The formula (5.2) holds for any j (see for example [DeV09, Proposition 2.6] or [AGZV88]). Alternatively, using natural coordinates that decompose into the tangent plane and a normal direction, one obtains the formula

$$\mathrm{Res}\left(\frac{P}{Q}\, d\mathbf{Z}\right) = \frac{P}{|\nabla Q|}\, dA \tag{5.3}$$

where $|\nabla Q| := \left[\sum_{j=1}^{d} \partial Q/\partial x_j)^2\right]^{1/2}$ is well defined up to sign and dA is the form $du_1 \wedge \cdots du_{d-1}$ on $\mathcal{V}$ for any unitary coordinates $\{u_1, \ldots, u_{d-1}, u_d\}$ such that the first $d-1$ coordinatize $\mathcal{V}$.

The chain of integration in the outer integral in the first proof is a $(d-1)$-dimensional chain parametrized by its projection to the first $(d-1)$ coordinates. In general, one can integrate over any intersection cycle. To define this, let $H : T \times [0,1]$ be a homotopy from the torus T in the original integral to a chain T' on which $-\hat{\mathbf{r}} \cdot \log \mathbf{Z}$ is strictly less that $-\hat{\mathbf{r}} \cdot \log \mathbf{Z}_*$. Perturbing H slightly if necessary so it intersects $\mathcal{V}$ transversely, let $\mathcal{C}$ be the intersection of H with $\mathcal{V}$. This cycle depends on the particular homotopy H, but its homology class in $H_{d-1}(\mathcal{V})$ does not. Let η be any holomorphic $(d-1)$-form on $\mathcal{V}$. This is the highest homology dimension (even though $\mathcal{V}$ has $2d-2$ real dimensions) so $d\eta = 0$ and Stokes' Theorem implies that $\int_{\partial \mathcal{D}} \eta = 0$ for any d-chain $\mathcal{D}$ on $\mathcal{V}$. Hence $\int_{\mathcal{C}} \eta$ depends only on the homology class of $\mathcal{C}$ in $H_{d-1}(\mathcal{V})$. The Cauchy-Leray residue theorem, proved in [DeV09, Theorem 2.8], says that

Theorem 5.1. *If ω has only simple poles on $\mathcal{V}$ then*

$$\int_T \omega = \int_{\mathcal{C}} \mathrm{Res}\,(\omega)$$

where $\mathrm{Res}\,(\omega)$ is the Gelfand-Leray residue form and $\mathcal{C}$ is any chain in the homology class of the intersection cycle of T with $\mathcal{V}$. □

As a consequence, we have the more canonical residue representation

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^{d-1} \int_{\mathcal{C}} \mathbf{Z}^{-\mathbf{r}-\mathbf{1}}\, \mathrm{Res}\,(F \, d\mathbf{Z}) . \tag{5.4}$$

Curvature

Let $\mathcal{M}$ be a real $(d-1)$-manifold in $\mathbb{R}^d$. For hypersurfaces, the Gaussian curvature has several equivalent definitions, all of them quite natural. The first involves the Gauss map $\mathcal{G} : \mathcal{M} \to S^{d-1}$, defined locally by a continuous choice of unit normal vector. The Gaussian curvature at the point $\mathbf{x} \in \mathcal{M}$ is defined by to be the determinant of the Jacobian matrix of the Gauss map at $\mathbf{x}$:

$$\mathcal{K}(\mathbf{x}) := |J(\mathcal{G})(\mathbf{x})| .$$

Because of the sign choice of the unit normal, in odd dimensions the curvature is defined only up to sign. An alternative definition begins with a description of $\mathcal{V}$ as a graph over its tangent plane at a point $\mathbf{x}$. If there is a smooth real function h on $T_{\mathbf{x}}(\mathcal{M})$ such that locally $\mathcal{V} = \{\mathbf{x}+\mathbf{u}+h(\mathbf{u})\mathbf{v} : \mathbf{u} \in T_{\mathbf{x}}(\mathcal{M})\}$, where $\mathbf{v}$ is a unit normal to $\mathcal{M}$ at $\mathbf{x}$, then the curvature is equal to the Hessian determinant of this map:

$$\mathcal{K}(\mathbf{x}) = \left| \frac{1}{2} \mathcal{H}(h, \mathbf{0}) \right| . \tag{5.5}$$

Notions of curvature are less natural for surfaces of co-dimension great than one, however, for complex manifolds of co-dimension one, either of these two definitions may be formally generalized. The second is the most straightforward because the representation h of $\mathcal{M}$ as a graph of a complex analytic function over its complex tangent plane is well defined and may be plugged directly into (5.5). The first definition also generalizes if some care is taken with the choice of unit normal. We now rewrite the coordinate-dependent conclusion of Theorem 3.1 and the even more blatantly coordinate-dependent (3.17) in a more natural form:

Theorem 5.2. *Assume the hypotheses of Theorem 3.1. Then*

$$a_{\mathbf{r}} \sim (2\pi)^{-d/2} e^{-\mathbf{r}\cdot\mathbf{x}_*} \frac{P}{|\nabla_{\log} Q|}(\mathbf{Z}_*) \mathcal{K}(\mathbf{z}_*)^{-1/2}$$

where the curvature refers to the complex hypersurface $\log \mathcal{V}/i$.

Proof: To put the integrand in (5.4) into a familiar saddle integral form, change coordinates to $\mathbf{Z} = \exp(\mathbf{z})$. This turns the integrand into

$$\exp(-\mathbf{r}\cdot\mathbf{z}) \operatorname{Res}(\tilde{\mathbf{F}}\, d\mathbf{z}) ,$$

where $\tilde{\mathbf{F}} := F \circ \exp = \tilde{\mathbf{P}}/\tilde{\mathbf{Q}}$. Applying (5.3) transforms this further into

$$\exp(-\mathbf{r}\cdot\mathbf{z}) \frac{\tilde{\mathbf{P}}}{|\nabla \tilde{\mathbf{Q}}|}(\mathbf{z})\, da$$

where da is the $(d-1)$-dimensional area form on $\log \mathcal{V}$. Letting $\phi(\mathbf{z}) := -\mathbf{r} \cdot \mathbf{z}$ and $A(\mathbf{z}) := \tilde{\mathbf{P}}/|\nabla Q_t|$, we apply Theorem 3.1 to find that

$$a_{\mathbf{r}} \sim \left(\frac{1}{2\pi i}\right)^{d-1} \frac{P}{|\nabla_{\log} Q|}(\mathbf{Z}_*) \int_{\log \mathcal{C}} \exp(-\mathbf{r} \cdot \mathbf{z}) \, da$$

where $\nabla_{\log} Q := (Z_1 \partial Q/\partial Z_1, \ldots, Z_d \partial Q/\partial Z_d)$ is the result of evaluating $\nabla \tilde{\mathbf{Q}}$ at $\log \mathbf{Z}$. It remains to see that the integral is equal to

$$i^d \sqrt{(2\pi)^{d-1}/\mathcal{K}(\mathbf{z}_*)} \, .$$

Taking a hint from the case where $\mathcal{C}$ is a torus and the contour $\log \mathcal{C}$ is parallel to the purely imaginary subspace, we change variables, trivially this time, to $\mathbf{z} = \mathbf{x}_* + i\mathbf{y}$. This removes the $d-1$ factors of $1/i$ and changes the chain of integration to $\log \mathcal{V}/i$. A factor of $\exp(\mathbf{x}_*)$ may be pulled out as well, leaving an integrand of $\exp(-i\mathbf{r} \cdot \mathbf{y}) \, da$. Recognizing the Fourier transform of the surface measure on $\log \mathcal{V}/i$ from Example 1.9, we observe this time that $-\hat{\mathbf{r}} \cdot \mathbf{y}$ is the height in the $\hat{\mathbf{r}}$ direction above the tangent plane, so the integral, which we already know to evaluate to $(2\pi)^{d/2} \mathcal{H}(-\mathbf{r} \cdot \mathbf{y})^{-1/2}$, must evaluate to $\mathcal{K}(\log \mathcal{V}/i)^{-1/2}$, proving Theorem 5.2.

5.3 Non-smooth points - a sketch of further results

Tilings of the Aztec diamond

Beyond the Gaussian regime, the possible limit behaviors of sequences with nice, even rational, generating functions are quite diverse. To illustrate this, the I will discuss a somewhat complicated example involving random tilings. I am particularly fond of this example, probably because it is one of two motivating applications that got me started on the project of mutlivariate asymptotics in the first place. An Aztec diamond is a union of lattice squares; the order four Aztec diamond is shown in figure 5.1. The region can be tiled by dominos (horizontally or vertically adjacent pairs of squares) in many ways, one of which is shown in the figure.

Color the lattice squares in alternating checkerboard fashion and classify the dominoes according to whether the white square is North, East, South or West of the black square. Let $p(n; i, j)$ denote the probability that the square (i, j) in a diamond of orer n is covered by a “northgoing” domino. The generating function $F(X, Y, Z) := \sum_{n,i,j} p(n; i, j) x^i y^j z^n$ is

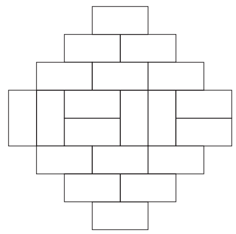


Figure 5.1: the Aztec diamond of order 4, tiled by dominoes

rational and was shown in [JPS98] to be given by

$$F(X,Y,Z) = \frac{P(X,Y,Z)}{Q(X,Y,Z)} = \frac{Z/2}{(1-YZ)(1-(X+X^{-1}+Y+Y^{-1})Z/2+Z^2)} . \tag{5.6}$$

The analysis of this generating function is treated in [BP08]; asymptotics for this were previously worked out by more combinatorial means in [CEP96]. The variety where the denominator Q vanishes is not smooth. The gradient of Q vanishes simultaneously with Q at the point $(1,1,1)$. Denote $q := A \circ \exp$ where A is the factor of the denominator other than $1-YZ$. The singularity of $Q \circ \exp$ at the origin has a power series expansion beginning with

$$q(x,y,z) \sim z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2 .$$

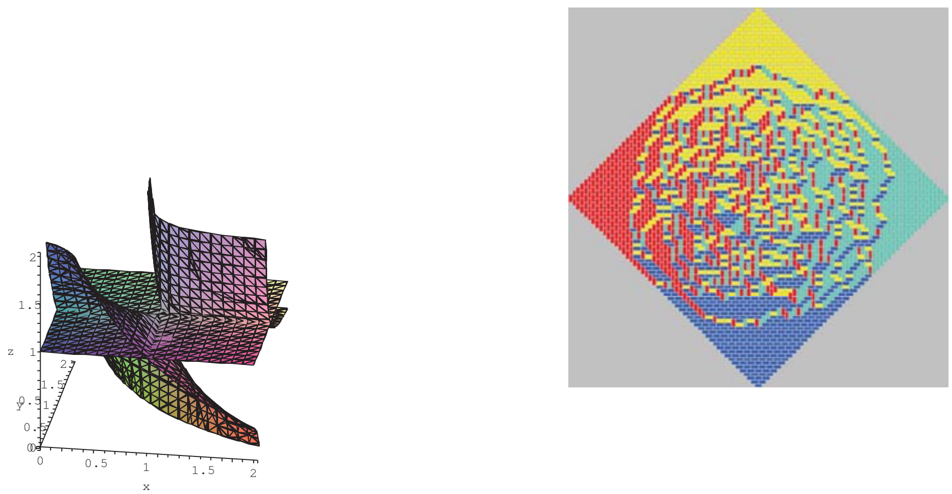
The first of the figures below shows the intersection of $\mathcal{V}$ with the real subspace of $\mathbb{C}^3$.

The limit theory in such a case is quite different, as may be gleaned from the second of the figures, which shows a uniformly chosen tiling of an order-47 Aztec diamond. The probabilities appear to be close to zero (or one) outside a circular region inscribed in the feasible square, and to vary smoothly across the region.

It is shown in [BP08, Theorem 3.9] that for any generating function of this type (a singularity of conic type at $(1,1,1)$ and another smooth factor vanishing at the origin), the coefficients are given asymptotically by

$$a_{\mathbf{r}} \sim C \arctan \left(\frac{\sqrt{b^*(\mathbf{r},\mathbf{r})}\sqrt{b^*(\ell^*,\ell^*)}}{b^*(\mathbf{r},\ell^*)} \right) \tag{5.7}$$

where b^* is the symplective bilinear form dual to the tangent cone to q at the origin and ℓ^* is the linear form dual to the tangent plane of the smooth factor. I close this lecture with a brief sketch of this and other similar results.



(a) Singular variety for the Aztec diamond (b) a uniformly sampled tiling

Figure 5.2: an algebraic hypersurface and a random tiling

Derivation of asymptotics

The Cauchy integral, in logarithmic coordinates, is of the form

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_{\mathbf{x}_* + i[-\pi,\pi]^d} e^{-\mathbf{r}\cdot\mathbf{z}} \frac{p(\mathbf{z})}{b(\mathbf{z})\ell(\mathbf{z})}\, d\mathbf{z}$$

where b vanishes to order two at the origin and ℓ vanishes to order one. As in the case of saddle point integrals, homogeneous functions are easier to handle; other functions may be expanded in series of homogeneous functions and integrated term by term. Proving this requires some work, but in the end we may assume that b is a homogeneous quadratic and ℓ is linear. The series approach also allows us to replace p by a monomial. We also change variables to $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ ($\mathbf{x}$ is fixed) and replace $[-\pi, \pi]^d$ by $\mathbb{R}^d$.

Ideally we would like to take $\mathbf{x}_* = \mathbf{0}$. In this case, p, b and ℓ evaluated at $\mathbf{x}_* + i\mathbf{y}$ are still repsectively monomial, quadratic and linear, and we have

$$a_{\mathbf{r}} \sim \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp(-i\mathbf{r}\cdot\mathbf{y}) \frac{\mathbf{y}^{\mathbf{m}}}{b(\mathbf{y})\ell(\mathbf{y})}\, d\mathbf{y}\,. \tag{5.8}$$

This integral is not convergent but is well studied as a **distribution** or generalized function: it is defined by its integrals against smooth, compactly suported test functions; see [GS64] for background on generalized functions. For functions of the form b^{-s}, the GFT was shown by M. Riesz [Rie49] to be a power of the dual quadratic, b^*. The GFT of a product is obtained in principle from a convolution; however, the GFT of $1/\ell$ is a delta function on a ray and the convolution is non-convergent, so one must go back to extend the proof and not simply use Riesz' result.

Finally, some work is needed to relate the generalized function back to the actual integral. Estimates on integrals require multivariate contour deformations; in particular a deformation is required that moves the original contour to a cone on which $\hat{\mathbf{r}} \cdot \mathbf{z}$ is postive and at least $\epsilon|\mathbf{z}|$. Following [ABG70], Baryshnikov and Pemantle [BP08] use the theory of **hyperbolic polynomials** to find such contours and finish the proof of (5.7).