Asymptotics of multivariate sequences IV: generating functions with poles on a hyperplane arrangement

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Abstract

Let $F(z_1, \ldots, z_d)$ be the quotient of an analytic function with a product of linear functions. Working in the framework of analytic combinatorics in several variables, we compute asymptotic formulae for the Taylor coefficients of F using multivariate residues and saddle-point approximations. Because the singular set of F is the union of hyperplanes, we are able to make explicit the topological decompositions which arise in the multivariate singularity analysis. In addition to effective and explicit asymptotic results, we study how asymptotics change between different regimes, and report on an accompanying computer algebra package implementing our results. It is also our hope that this paper will serve as an entry to the more advanced corners of analytic combinatorics in several variables for combinatorialists.

1 Introduction

In this paper we study the coefficients of meromorphic functions

$$F(\mathbf{z}) = F(z_1, \ldots, z_d) = \frac{G(\mathbf{z})}{\prod_{j=1}^{m} \ell_j(\mathbf{z})^{p_j}} \tag{1}$$

whose denominator is the product of positive integer powers of real linear functions ℓ_j. Such functions arise, among other places, in queuing theory.

Example 1.1. The so-called partition generating function for a closed multiclass queuing network with one infinite server has the form

$$F(\mathbf{z}) = \frac{e^{z_1+z_2+\cdots+z_d}}{\prod_{j=1}^{m}\left(1 - \sum_{i=1}^{d} \rho_{ij} z_j\right)} \tag{2}$$

for real constants $\rho_{ij} > 0$ depending on model parameters [8, Eq. (2.26)]. ◁

Bertozzi and McKenna [8] approached the asymptotic analysis of the queuing system described in (2) by noting that the multivariate Cauchy integral which evaluates these coefficients can be represented as a sum of integrals over basic homology cycles in the domain of holomorphy of the complex d-form $\mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) d\mathbf{z}$.

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In certain low-dimensional cases, they were able to exploit linear relations among these cycles to determine dominant asymptotics via multivariate residues. Since that time, the theory of multivariate coefficient extraction via the field of *analytic combinatorics in several variables (ACSV)* [34, 24] has grown substantially.

The techniques of ACSV aim to characterize the asymptotic behaviour of the series coefficients $\{a_{\mathbf{r}}\}$ of a convergent series expansion $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ as $\mathbf{r} \to \infty$ with the normalized vector $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$ staying in a bounded set, where $|\mathbf{r}|$ denotes the 1-norm $|\mathbf{r}| = |r_1| + \cdots + |r_d|$. Suppose that $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$, where $H(\mathbf{z}) = \prod_{j=1}^{m} h_j(\mathbf{z})^{p_j}$ for polynomials h_j each vanishing on a smooth variety (generalizing the case when H is a product of linear factors). The series coefficients $\{a_{\mathbf{r}}\}$ exhibit uniform asymptotic behaviour in $\mathbf{r}$ aside from certain degenerate cases having to do with nontransverse intersections of these varieties or boundary directions $\hat{\mathbf{r}}$ where there are transitions in asymptotic behaviour. An asymptotic expansion for $a_{\mathbf{r}}$ can usually be obtained by taking an integer sum of saddle-point integrals $I(\boldsymbol{\sigma})$ localized near certain *contributing points* $\boldsymbol{\sigma}$ where the denominator H vanishes. Pemantle and Wilson [33] characterize the asymptotic behaviour of the types of local integrals that come up in this paper (so-called *transverse multiple points*); see also [36] for explicit formulae.

Computing the set of contributing points $\boldsymbol{\sigma}$ over which to sum local integrals to determine asymptotics can be easy in some applications, but is difficult (perhaps even undecidable) in general. When the denominator under consideration is a product of linear factors, Bertozzi and McKenna [8] determine this set for a few examples and speculated on the existence of a theory to compute the set in general. Outside of the linear denominator case, which we cover extensively in this paper, when the singular set of F is a manifold then it is known how to compute the contributing points both in the bivariate case [12] and when the contributing points lie on the boundary of the domain of convergence of the power series [25]. There is currently no known algorithm to determine contributing singularities for general meromorphic (or even rational) functions, and such an algorithm would need to decide certain deep topological questions. However, the current state of knowledge is enough for many problems of combinatorial origin.

In addition to queuing theory, other areas where generating functions having a form similar to (1) arise include Markov modeling [15], lattice point enumeration [11], and discrete probability theory (see Example 3.15 below). Gaussian and other limit theorems also follow from the asymptotic extraction of coefficients of multivariate generating functions.

When the denominator of F has only linear factors, the singular set of F forms a hyperplane arrangement, which allows us to describe which critical points $\boldsymbol{\sigma}$ are contributing points that affect asymptotic behaviour of a coefficient sequence. In this paper we give an algorithm to determine dominant asymptotics for any such meromorphic function under two assumptions: (1) $\mathbf{r}$ is **generic**, meaning that $\mathbf{r}/|\mathbf{r}|$ does not approach one of a codimension 1 set of bad boundary directions, and (2) the numerator $G(\mathbf{z})$ is polynomial. Algorithm 1 below summarizes the procedure under the additional assumption that the functions ℓ_j are linearly independent, and a Maple implementation is detailed in Section 4.4. Later, Algorithm 2 handles the more general case where the $\{\ell_j\}$ can be linearly dependent. When G is allowed to be a more general entire function, the same two algorithms give valid asymptotic formulae, with the proviso that the determination of which among finitely many terms of the formula asymptotically dominate may require further investigation. In non-generic directions, while we have no complete algorithm, we show how the desired estimates can be written as integral transforms and compute these in a variety of cases. We also give the first results discussing transitions in asymptotics around non-generic directions.

The remainder of the paper is structured as follows. General background related to ACSV is given in Section 2. Results and motivating examples are given in Section 3. Section 4 gives the heart of the analysis, under the assumption of linear independence of the $\{\ell_j\}$. Here, the topological decomposition of the cycle of integration in Cauchy's integral formula is decomposed into certain Morse-theoretically determined cycles for which the integral has an easily computed asymptotic form. Section 5 extends the results to allow for linear dependencies among the $\{\ell_j\}$. Finally, Section 6 studies a number of cases where $\mathbf{r}/|\mathbf{r}|$ approaches a boundary direction, giving complete results in a scaling window of width $|\mathbf{r}|^{1/2}$ in the case of an ordinary boundary direction in terms of negative moments of Gaussian random variables.

Remark 1.2. Some of our exposition follows the textbook [24] of the second author, which was written at the same time as much of this paper. In addition to a self-contained presentation that makes clearer the relationship between our arguments and more general results in Morse theory, the new contributions of this paper include a Maple implementation of the main algorithms, the first results on transitions of behaviour around non-generic directions, a proof that all critical points must be real (giving a complete classification of when critical points can occur, and simplifying other arguments), and further clarification on when drops in the exponential growth of a sequence can occur.

ACSV requires a number of techniques not always familiar to combinatorialists: Morse theory, computational algebraic geometry and the theory of singular integral transforms. Nevertheless, our aim is to provide an introductory exposition. In order to help motivate the homological arguments taken in modern approaches to this topic, we illustrate our techniques on several examples and provide a Maple implementation of our work at

https://github.com/ACSVMath/ACSVHyperplane

2 ACSV background

The use of analytic techniques to derive asymptotic information about a sequence (a_n) from properties of its generating function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is the domain of *analytic combinatorics*. When the generating function $f(z)$ represents an analytic function at the origin, Cauchy's integral formula implies

$$a_n = \frac{1}{2\pi i}\int_C f(z)\frac{dz}{z^{n+1}},$$

where C is any positively oriented circle sufficiently close to the origin. By deforming the domain of integration C, one can typically use classical integral methods to obtain asymptotic results. Standard references include Flajolet and Sedgewick [13], Odlyzko [28], and Henrici [17].

More recently, ACSV has developed tools for the multivariate asymptotic analysis of generating functions. Fix a dimension $d \in \mathbb{N}$. Given a multi-dimensional vector $\mathbf{z} = (z_1, \dots, z_d) \in \mathbb{C}^d$ and index $\mathbf{i} \in \mathbb{Z}^d$, let $\mathbf{z}^{\mathbf{i}} := z_1^{i_1} \cdots z_d^{i_d} \in \mathbb{C}$. Generalizing from the univariate case, if the multivariate generating function

$$F(\mathbf{z}) = \sum_{\mathbf{r}\in\mathbb{N}^d} a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}}$$

represents a power series at the origin then the Cauchy integral formula gives an analytic representation

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_{\mathcal{T}} F(\mathbf{z})\frac{d\mathbf{z}}{\mathbf{z}^{\mathbf{r}+\mathbf{1}}}, \tag{3}$$

where $\mathcal{T}$ is the product of sufficiently small positively oriented circles. The aim is often to determine asymptotics of the sequence $[\mathbf{z}^{\mathbf{r}}]F(\mathbf{z}) = a_{\mathbf{r}}$ as $\mathbf{r} = n\hat{\mathbf{r}}$ and $n \to \infty$, for some fixed *direction* $\hat{\mathbf{r}} \in \mathbb{R}^d_{>0}$. Although $a_{\mathbf{r}}$ is only non-zero when $\mathbf{r}$ has non-negative integer coordinates, the theory as detailed in this paper shows that asymptotics typically vary smoothly with $\mathbf{r}$, allowing one to make asymptotic statements about $a_{\mathbf{r}}$ for generic directions $\hat{\mathbf{r}}$ or when a normalization of $\mathbf{r}$ approaches $\hat{\mathbf{r}}$ sufficiently quickly (this will be made precise below).

The earliest results in this area [6, 14, 19, 7] were based on showing that the *sections* $\sum_{\mathbf{r}:r_1=k} a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}}$ are well approximated by a quasi-power $C_k z_1^k g(z_2, \ldots, z_d)^k$, leading to a central limit theorem and Gaussian behaviour of the generating function coefficients. Since the early 2000s, a systematic program to determine asymptotics of multivariate generating function coefficients has been developed by the *analytic combinatorics in several variables* project[1] [32, 33, 5, 34, 3, 24]. In the earliest of these works [32, 33] the deformations were accomplished by *ad hoc* surgeries. In [5], existence of the relevant deformations was shown to follow from the theory of hyperbolic functions [1] and a Morse-theoretic structure was provided to compute and interpret them; this Morse-theoretic approach was recently made rigorous in [3] by working around certain standard assumptions of Morse theory that don't hold in ACSV contexts.

Let $F(\mathbf{z})$ be a meromorphic function with poles along an algebraic variety $\mathcal{V}$, and let $\mathcal{M} = \mathbb{C}_*^d \setminus \mathcal{V}$ denote the domain on which the integrand of the Cauchy integral (3) is holomorphic. The value of the Cauchy integral depends only on the homology class of $\mathcal{T}$ in $H_d(\mathcal{M})$ so, for instance, we can deform $\mathcal{T}$ in $\mathcal{M}$ without changing the value of the integral.

As $\mathbf{r} \to \infty$ the magnitude of the Cauchy integrand of (3) is controlled by the points $\mathbf{z}$ for which $|\mathbf{z}|^{-\mathbf{r}}$ is maximized. Integrals are usually easiest to approximate when the contribution from integration away from the maximum value of the integrand is negligible, leading us to try and deform $\mathcal{T}$ to a *minimax contour*. The key to computing integral transforms is that minimizing $\max_{\mathcal{T}} |\mathbf{z}|^{-\mathbf{r}}$ will produce a contour $\mathcal{T}$ where, near the point where $|\mathbf{z}|^{-\mathbf{r}}$ is maximized, the integrand is in *stationary phase*, and may thus (typically) be approximated using standard analytic techniques. This minimax problem is not affected by rescaling $\mathbf{r}$, so the analysis depends only on the vector $\hat{\mathbf{r}} := \mathbf{r}/|\mathbf{r}|$ where $|\mathbf{r}| := r_1 + \cdots + r_d$ is L^1-norm of $\mathbf{r}$ (the L^1-norm is natural in many contexts, but can be replaced by another norm if desired).

Taking logarithms, we need to look near points $\mathbf{z}$ such that the **height function**

$$h_{\hat{\mathbf{r}}}(\mathbf{z}) := -\sum_{j=1}^{d} \hat{r}_j \log |z_j| \,. \tag{4}$$

is maximized. The decomposition of cycles into canonical height-minimizing cycles is the province of Morse theory (when $\mathcal{V}$ is smooth) and stratified Morse theory (when $\mathcal{V}$ is a stratified space such as an algebraic variety).

The result of a Morse-theoretic analysis is as follows. Any algebraic set $\mathcal{V}$ can be decomposed into a finite set of smooth manifolds known as strata, each of which (under the assumptions of this paper) contains a finite set of critical points for the height function (points where the differential of the height function restricted to the strata is zero). Any cycle C may be written as the sum of certain 'attachment cycles' for which height is maximized uniquely at one of these critical points[2]. These attachment cycles are known as **linking tori**, the one near a critical point $\boldsymbol{\sigma}$ being denoted by $\boldsymbol{\tau}_{\boldsymbol{\sigma}}$. The integral over one of these linking tori is a standard integral transform whose asymptotics are sometimes easy to compute (e.g., in the generic directions referenced above) or are in the province of singular integral theory and are explicit to various degrees depending on the geometry near the critical point. Restricting to the case where $\mathcal{V}$ is the union of hyperplanes results in a bound on the complexity of the singular integral, and therefore a reasonably complete answer.

The case where the denominator of F is the product of linear functions with real coefficients simplifies for a second reason, beyond the bounded complexity of the geometry of $\mathcal{V}$. The analysis from above may be summarized by a homological computation

$$[\mathcal{T}] = \sum_{\boldsymbol{\sigma}\in\Omega} k_{\boldsymbol{\sigma}} \boldsymbol{\tau}_{\boldsymbol{\sigma}} \,, \tag{5}$$

[1] See acsvproject.com for a listing of papers in this project.

[2] The fact that the original cycle is in $\mathcal{M} \subseteq \mathcal{V}^c$ while the Morse theory is done on $\mathcal{V}$ will be reconciled in Section 4.

where Ω is the set of stratified critical points for h on $\mathcal{V}$ and the $k_{\boldsymbol{\sigma}}$ are integer invariants that typically hard to compute. When $\boldsymbol{\sigma}$ lies on the boundary of the domain of convergence of $F(\mathbf{z})$ and the singular geometry at $\boldsymbol{\sigma}$ is sufficiently nice, it is known that $k_{\boldsymbol{\sigma}} \in \{\pm 1, 0\}$. When this (rather strong and computationally expensive) property cannot be established, the most promising method to date for determining the coefficients $k_{\boldsymbol{\sigma}}$ relies on the fact that, when $\hat{\mathbf{r}}$ is a rational direction, the coefficient sequences $a_{n\mathbf{r}}$ satisfy linear differential equations with polynomial coefficients [10, 22] which can be computed using so-called Creative Telescoping methods [20]. Numeric analytic continuation [26] can then be used to rigorously determine the constants appearing in asymptotics up to any fixed accuracy, recovering the integers $k_{\boldsymbol{\sigma}}$. Representation of the denominator of F as a product of real linear functions allows explicit computation of $\{k_{\boldsymbol{\sigma}}\}$, going through the intermediate homology basis of **imaginary fibers**, which relies in turn on convexity properties following from the form of the denominator of F.

3 Notation and results

In this section we state our main results, after some basic definitions.

3.1 Definitions

We begin by introducing the quantities appearing in our analysis. We make use of terminology from ACSV, the study of hyperplane arrangements, and topological constructions.

Hyperplane arrangements

Fix a ratio

$$F(\mathbf{z}) = F(z_1, \ldots, z_d) = \frac{G(\mathbf{z})}{H(\mathbf{z})},$$

where $G(\mathbf{z})$ is an entire function and

$$H(\mathbf{z}) = \prod_{j=1}^{m} \ell_j(\mathbf{z})^{p_j}$$

for integers $p_j \geq 1$ and real linear functions

$$\ell_j(\mathbf{z}) = 1 - (\mathbf{b}^{(j)}, \mathbf{z}) = 1 - b_1^{(j)} z_1 - \cdots - b_d^{(j)} z_d\,.$$

We always assume that G and H are coprime, which in this case simply means that G does not identically vanish on the zero set of one of the ℓ_j (when G is a polynomial this means G and H are coprime polynomials). The singular set

$$\mathcal{V} = \{\mathbf{z} \in \mathbb{C}^d : H(\mathbf{z}) = 0\}$$

is composed of a union of hyperplanes and thus defines a *hyperplane arrangement* [31], a well-studied combinatorial and topological object from which we borrow some of our terminology. The notation $\mathcal{V}_{\mathbb{R}}$ denotes the real arrangement, a union of corresponding real hyperplanes in $\mathbb{R}^d$ that is the intersection of $\mathcal{V}$ with $\mathbb{R}^d$.

Given complex-valued functions $g_1, \ldots, g_s$, we let $\mathbb{V}(g_1, \ldots, g_s)$ denote their set of common complex zeroes. The hyperplane arrangement $\mathcal{V} = \mathbb{V}(\ell_1 \cdots \ell_m)$ can be decomposed into closed ***flats***: for each subset $\{k_1, \ldots, k_s\} \subset \{1, \ldots, m\}$ the flat $\mathcal{V}_{k_1,\ldots,k_s} := \mathbb{V}(\ell_{k_1}, \ldots, \ell_{k_s})$ is the intersection of the appropriate hyperplanes. Because two flats with distinct index sets could be equal, we let $\mathcal{A}$ be the collection of maximal

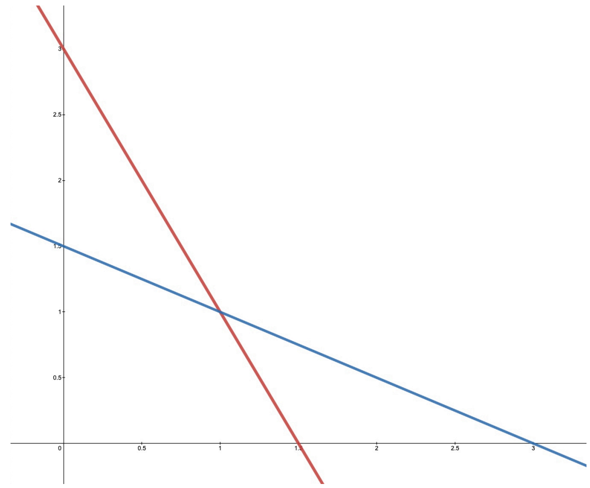

Figure 1: The real part of a simple arrangement of two hyperplanes in two dimensions

subsets of $\{1, \ldots, m\}$ corresponding to non-empty distinct flats: this is the ***hyperplane arrangement*** defined by $\mathcal{V}$. Thus, by definition, $\mathcal{V}_{S \cup \{k\}} \neq \mathcal{V}_S$ for any $S \in \mathcal{A}$ and $k \notin S$.

Definition 3.1 (simple arrangements)**.** The arrangement $\mathcal{A}$ (or singular set $\mathcal{V}$, or rational function $F(\mathbf{z})$) is said to be ***simple*** if for any subset $\{k_1, \ldots, k_s\} \subseteq [m]$ of indices such that the flat $\mathcal{V}_{k_1,\ldots,k_s}$ is nonempty, the coefficient vectors $\mathbf{b}^{(k_1)}, \ldots, \mathbf{b}^{(k_1)}$ are linearly independent. In other words, $\mathcal{A}$ is simple if hyperplanes with common points have linearly independent normals.

The ***stratum*** $\mathcal{S}_S$ corresponding to any $S \in \mathcal{A}$ is defined as the flat $\mathcal{V}_S$ with all subflats removed,

$$\mathcal{S}_S := \mathcal{V}_S \setminus \bigcup_{\mathcal{V}_T \subsetneq \mathcal{V}_S} \mathcal{V}_T \, .$$

The collection of strata form a ***stratification*** of $\mathcal{V}$, i.e., a partition of $\mathcal{V}$ into disjoint smooth sets, and the flat $\mathcal{V}_S$ is the closure of the stratum $\mathcal{S}_S$. The ***dimension*** of $\mathcal{S}_S$ is the dimension of $\mathcal{V}_S$ as an algebraic set. It is also the dimension of S as a complex manifold and half the dimension of S as a real manifold. When $\mathcal{V}$ is simple then $\mathcal{A}$ is the collection of all subsets of $\{1, \ldots, m\}$ such that the corresponding hyperplanes have non-empty intersection, and the dimension of a stratum $\mathcal{S}_S$ is simply $d - |S|$.

Example 3.2. Figure 1 shows an arrangement with two hyperplanes and three non-empty flats.

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Critical points

Let $\mathbf{r} \in \mathbb{R}^d_{>0}$. By definition, a ***critical point*** of $h = h_{\mathbf{r}}$ on a stratum $\mathcal{S}$ is a point where the differential of the restricted map $h|_{\mathcal{S}}$ is zero. The critical points on a stratum satisfy an explicit set of algebraic equations, which we now describe. Let $k_1, \ldots, k_s \in [m]$ be any indices such that $\{\mathbf{b}^{(k_1)}, \ldots, \mathbf{b}^{(k_s)}\}$ are linearly independent, defining a flat $\mathcal{V}_S$. Because the differential of the unrestricted map $h_{\mathbf{r}}$ is given by its gradient, a critical point on $\mathcal{V}_S$ is characterized by a rank deficiency in the matrix

$$M(\mathbf{z}) := \begin{pmatrix} -\nabla \ell_{k_1}(\mathbf{z}) \\ \vdots \\ -\nabla \ell_{k_s}(\mathbf{z}) \\ -\nabla h \end{pmatrix} = \begin{pmatrix} & \mathbf{b}^{(k_1)} & \\ & \vdots & \\ & \mathbf{b}^{(k_s)} & \\ r_1/z_1 & \cdots & r_d/z_d \end{pmatrix}. \tag{6}$$

Up to a reordering of variables and the ℓ_{k_j}, we may assume without loss of generality that M contains pivots in its first s diagonal entries. For $j = s+1, \ldots, d$, let M_j denote the $(s+1) \times (s+1)$ matrix constructed from the first s columns of M together with its $(s+j)$th column. Clearing denominators, the critical points on $\mathcal{V}_S$ are the real solutions of a polynomial system defined by

$$\ell_{k_1} = \cdots = \ell_{k_s} = \det M_1 = \cdots = \det M_{d-s} = 0\,. \tag{7}$$

As expected, the critical points are unchanged by scaling $\mathbf{r}$, and thus only depend on the normalized vector $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$.

Lemma 3.3. *If $\boldsymbol{\sigma}$ is a critical point of $h_{\mathbf{r}}$ then $\boldsymbol{\sigma} \in \mathbb{R}^d$.*

Proof. Assume, without loss of generality, that $\boldsymbol{\sigma}$ lies in the stratum $\mathcal{S}_{1,\ldots,s}$. Then there exist constants $\lambda_1, \ldots, \lambda_s \in \mathbb{C}$ such that

$$\lambda_1 \mathbf{b}^{(1)} + \cdots + \lambda_s \mathbf{b}^{(s)} = \left(\frac{r_1}{\sigma_1}, \ldots, \frac{r_d}{\sigma_d} \right).$$

Write $\boldsymbol{\sigma} = \mathbf{x} + i\mathbf{y}$ and $\lambda_j = a_j + ic_j$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ and each $a_j, c_j \in \mathbb{R}$. Taking the imaginary parts of both sides of the above equation gives

$$c_1 \mathbf{b}^{(1)} + \cdots + c_s \mathbf{b}^{(s)} = -\left(\frac{r_1 y_1}{x_1^2 + y_1^2}, \ldots, \frac{r_d y_d}{x_d^2 + y_d^2} \right),$$

and taking the dot product of both sides of this equation with $\mathbf{y}$ implies

$$\sum_{j=1}^{s} c_j \left(\mathbf{b}^{(j)} \cdot \mathbf{y} \right) = -\sum_{k=1}^{d} \frac{r_k y_k^2}{x_k^2 + y_k^2}.$$

Since $1 - (\mathbf{x} + i\mathbf{y}) \cdot \mathbf{b}^{(j)} = 0$ for all $1 \leq j \leq s$, we see that $\mathbf{y} \cdot \mathbf{b}^{(j)} = 0$ for all j, and thus

$$0 = \sum_{k=1}^{d} \frac{r_k y_k^2}{x_k^2 + y_k^2}.$$

Each term on the right-hand side of this equation is non-negative, and zero if and only if y_k vanishes, so we must have $y_k = 0$ for all $1 \leq k \leq d$, meaning $\boldsymbol{\sigma} = \mathbf{x} \in \mathbb{R}^d$ is real. $\square$

The function h is continuous and strictly convex on each orthant of $\mathbb{R}^d$. If $\mathcal{V}_{S,\mathbb{R}} = \mathcal{V}_S \cap \mathbb{R}^d$ is the real part of a flat and the intersection of $\mathcal{V}_{S,\mathbb{R}}$ with an open orthant $O \subset \mathbb{R}^d$ is bounded, then $\mathcal{V}_{S,\mathbb{R}} \cap O$ contains a unique critical point of h, which is necessarily a local minimum by convexity of h. If $\mathcal{V}_{S,\mathbb{R}} \cap O$ is unbounded, which happens in every orthant when $\mathcal{V}_{S,\mathbb{R}}$ is parallel to at least one coordinate plane, then the infimum of h on $\mathcal{V}_{S,\mathbb{R}} \cap O$ is $-\infty$ and h has no critical point on $\mathcal{V}_{S,\mathbb{R}} \cap O$. Thus, for each index set $S \in \mathcal{A}$ the critical points of the height function $h = h_{\hat{\mathbf{r}}}$ on the stratum $\mathcal{S}_S$ are real, have non-zero coordinates (h is not defined when one of its coordinates is zero), and there is at most one in each quadrant of $\mathbb{R}^d$.

We call a critical point of $h_{\mathbf{r}}$ restricted to the flat $\mathcal{V}_S$ a critical point of the stratum $\mathcal{S}_S$, although it may not lie in $\mathcal{S}_S$ (it could be in a proper subflat). Each critical point $\boldsymbol{\sigma}$ lies in a unique stratum of lowest dimension, which we denote $\mathcal{S}(\boldsymbol{\sigma})$.

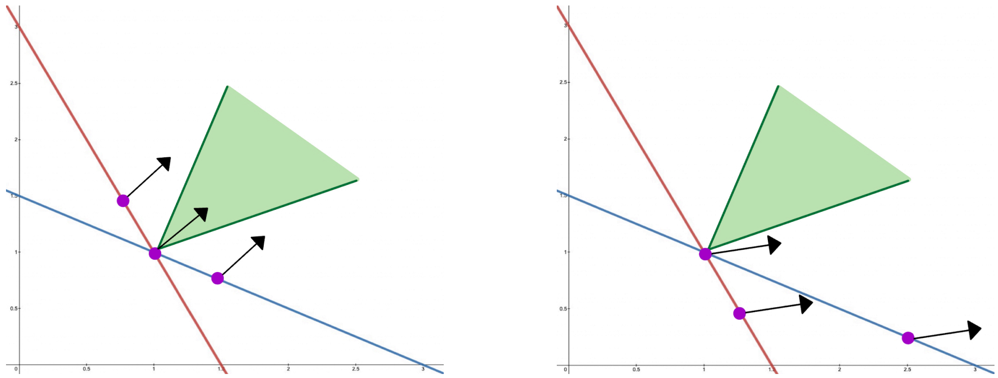

Figure 2: The real part of the hyperplane arrangement $\ell_1(x,y)\ell_2(x,y) = 0$ in Example 3.6, together with critical points, the lognormal cone $\tilde{N}(1,1)$ and $\hat{\mathbf{r}}$, when $\mathbf{r} = (1,1)$ (left) and $\hat{\mathbf{r}} = (5,1)$ (right).

Definition 3.4 (critical sets and generic directions)**.** The ***critical set*** Ω of F (in the direction $\hat{\mathbf{r}}$) is the union of all critical points of $h_{\hat{\mathbf{r}}}$ on the strata $\mathcal{S}_S$ as S ranges over all index sets $S \in \mathcal{A}$. The direction $\hat{\mathbf{r}}$ is said to be ***generic*** if no critical point of $h_{\hat{\mathbf{r}}}$ is a critical point of two distinct flats.

Definition 3.5 (normal and lognormal cones)**.** For any flat $\mathcal{V}_S$, the ***(positive) normal cone*** $N(S)$ is the cone

$$N(S) = \left\{ \sum_{j \in S} a_j \mathbf{b}^{(j)} : a_j > 0 \right\} \subset \mathbb{R}^d$$

and for any $\boldsymbol{\sigma} \in S \cap \mathbb{R}^d$ the ***(positive) lognormal cone*** $\tilde{N}_{\boldsymbol{\sigma}}(S)$ is the cone

$$\tilde{N}_{\boldsymbol{\sigma}}(S) = \left\{ \sum_{j \in S} a_j \tilde{\mathbf{b}}_{\boldsymbol{\sigma}}^{(j)} : a_j > 0 \right\} \subset \mathbb{R}^d$$

spanned by the ***lognormal vectors*** $\tilde{\mathbf{b}}_{\boldsymbol{\sigma}}^{(j)} := \left(b_1^{(j)}\sigma_1, \ldots, b_d^{(j)}\sigma_d\right)$. For convenience, we write

$$N(\boldsymbol{\sigma}) := N(S(\boldsymbol{\sigma})) \text{ and } \tilde{N}(\boldsymbol{\sigma}) := \tilde{N}_{\boldsymbol{\sigma}}(S(\boldsymbol{\sigma})).$$

As we will see, the ACSV analysis depends on identifying singularities $\boldsymbol{\sigma}$ with $-(\nabla h_{\hat{\mathbf{r}}})(\boldsymbol{\sigma}) \in N(\boldsymbol{\sigma})$ or, equivalently, with $\hat{\mathbf{r}} \in \tilde{N}(\boldsymbol{\sigma})$.

Example 3.6. We introduce a running example that clarifies some of the definitions, using x and y in place of z_1 and z_2 for clarity. Let $F(x,y) := 1/(\ell_1(x,y) \cdot \ell_2(x,y))$ where

$$\ell_1 = 1 - \frac{2x+y}{3} \qquad \text{and} \qquad \ell_2 = 1 - \frac{x+2y}{3}.$$

The real part of this hyperplane arrangement is illustrated in Figure 2. The flats are the two lines $\mathcal{V}_1$ and $\mathcal{V}_2$, and their intersection point $\mathcal{V}_{1,2} = \{(1,1)\}$.

For the flat $\mathcal{V}_1$ the critical point equations in the direction $\mathbf{r}$ are $\ell_1 = \det M_1 = 0$, with the latter expressing that the normal $\mathbf{b}^{(1)}$ of ℓ_1 is parallel to the gradient $(\nabla h_{\mathbf{r}})(x,y) = (r_1/x, r_2/y)$. This gives the system

$$\begin{aligned} 2x + y &= 3 \\ \frac{y}{r_2} &= \frac{2x}{r_1}, \end{aligned}$$

with solution $\boldsymbol{\sigma}_1 = \left(\frac{3\hat{r}_1}{2}, 3\hat{r}_2\right)$. A similar computation on the flat $\mathcal{V}_2$ gives the critical point $\boldsymbol{\sigma}_2 = \left(3\hat{r}_1, \frac{3\hat{r}_2}{2}\right)$, while the single point $\boldsymbol{\sigma}_{1,2} = (1,1)$ in $\mathcal{V}_{1,2}$ is trivially critical. We note that $\boldsymbol{\sigma}_1$ lies in the stratum $\mathcal{S}_1$ unless it coincides with $\boldsymbol{\sigma}_{1,2}$, which occurs when $\hat{r}_1 = 2/3$. Similarly, $\boldsymbol{\sigma}_2$ lies in the stratum $\mathcal{S}_2$ unless $\hat{r}_1 = 1/3$. Thus, $\mathbf{r}$ is generic unless it is a multiple of $(1,2)$ or $(2,1)$.

Figure 2 shows the lognormal cone $\tilde{N}(1,1)$ and the three critical points when $\mathbf{r} = (1,1)$ and $\mathbf{r} = (5,1)$. Note that $\hat{\mathbf{r}}$ lies inside $\tilde{N}(1,1)$ in the left sub-figure, but does not in the right sub-figure. The non-generic directions are precisely those where $\mathbf{r}$ lies on the boundary of $\tilde{N}(1,1)$. $\triangleleft$

Topological definitions

We now fix $\hat{\mathbf{r}}$, write $h(\mathbf{z}) := h_{\hat{\mathbf{r}}}(\mathbf{z})$, let $\mathcal{M}_{\le a}$ denote the sublevel set $\{\mathbf{z} \in \mathcal{M} : h(\mathbf{z}) \le a\}$, and set $h_{\min}$ to the least critical height,

$$h_{\min} := \min\{h(\boldsymbol{\sigma}) : \boldsymbol{\sigma} \in \Omega\}\,.$$

If $a < h_{\min}$ and C is a cycle in $\mathcal{M}_{\le a}$, then C can be continuously deformed in $\mathcal{M}$ so that $\max\{h(\mathbf{z}) : \mathbf{z} \in C\}$ is as small as desired. This follows from a non-proper extension of the fundamental Morse lemma described in [4] (because there are no so-called *critical points at infinity* in our case), or more directly from a concrete modification of the downward gradient flow on $\mathcal{M}$, using a partition of unity to keep the flow away from all strata of $\mathcal{V}$. It follows that the pairs $H_d(\mathcal{M}, \mathcal{M}_{\le a})$ with $a < h_{\min}$ are naturally isomorphic. We denote any of these naturally equivalent pairs by $(\mathcal{M}, -\infty)$ and use the symbol $\doteq$ to denote equality of classes in $H_d(\mathcal{M}, -\infty)$.

For those not topologically inclined, a class $[\mathcal{T}]$ in $H_d(\mathcal{M}, -\infty)$ can simply be viewed as a domain of integration $\mathcal{T}$ for the Cauchy integral (3) which is defined up to points in $\mathcal{M}$ of arbitrarily small height. The idea is that points at small height are asymptotically negligible, so only the class of $\mathcal{T}$ matters in our asymptotic calculations.

To that end, suppose that $C \doteq 0$, meaning C is a null-homologous chain in $H_d(\mathcal{M}, -\infty)$. For each $a < h_{\min}$, let C_a be a chain in $\mathcal{M}_{\le a}$ such that $C \doteq C_a$, and let $|C_a| = (2\pi)^{-d}\mu(C_a)$ where $\mu(C_a)$ is the area of C_a defined by the d-dimensional Lebesgue measure. Because C_a is supported on $\mathcal{M}_a$, the function F is continuous on C_a, hence $|F/\prod_{j=1}^d z_j|$ has some maximum value f_a on C_a. It follows that for all $\mathbf{r}$,

$$\begin{aligned} \left|(2\pi i)^{-d}\int_C F(\mathbf{z})\mathbf{z}^{-\mathbf{r}}\frac{d\mathbf{z}}{\mathbf{z}}\right| &= (2\pi)^{-d}\left|\int_C F(\mathbf{z})\mathbf{z}^{-\mathbf{r}}\frac{d\mathbf{z}}{\mathbf{z}}\right| \\ &\le f_a|\mathbf{z}|^{-\mathbf{r}}|C_a| \\ &\le f_a|C_a|e^{|\mathbf{r}|a} \end{aligned} \tag{8}$$

because $|\mathbf{z}|^{-\mathbf{r}} \le e^{|\mathbf{r}|a}$ on $\mathcal{M}_{\le a}$. This yields the following result.

Proposition 3.7. *Suppose $C \doteq 0$. Then the Cauchy integral integrated over C is smaller than any exponential. Specifically, for any $a < 0$ there exists $K > 0$ such that*

$$\left|\int_C F(\mathbf{z})\mathbf{z}^{-\mathbf{r}}\frac{d\mathbf{z}}{\mathbf{z}}\right| \le Ke^{a|\mathbf{r}|}\,.$$

Furthermore, if F grows at most polynomially then

$$\int_C F(\mathbf{z})\mathbf{z}^{-n\hat{\mathbf{r}}}\frac{d\mathbf{z}}{\mathbf{z}} = 0$$

for all n larger than some fixed natural number.

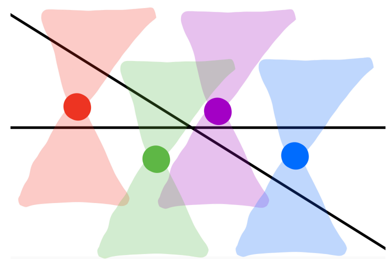

Figure 3: An oblique rendering of 3-space is used to depict $\mathbb{C}^2$, where the z-axis denotes imaginary directions in both the x and y coordinates. We take $d = 2$ and consider an arrangement of two intersecting hyperplanes. Each colored set represents an imaginary fiber over a real basepoint (large dot). The alternating sum of these is the linking torus for the point $\boldsymbol{\sigma}$ at the intersection of the two hyperplanes whose real parts are the lines in the figure.

Proof. The first part is an immediate consequence of (8). The second follows by fixing $\mathbf{r} = n\hat{\mathbf{r}}$ and letting $a \to -\infty$. Both f_a and C_a grow at most exponentially in $|a|$ (because f_a grows at most polynomially in $\mathbf{z}$), hence for $|\mathbf{r}|$ greater than the sum of these, the right-hand side of (8) goes to zero. As these are all bounds for the same fixed integral, its value must be zero. $\square$

We close this section with two more topological definitions that will appear in the statements of results. These are illustrated in Figure 3.

Definition 3.8 (imaginary fibers)**.** Let $\mathbf{x} \in \mathbb{R}^d_*$ be any point not in the real hyperplane arrangement $\mathcal{V}_\mathbb{R}$. The ***imaginary fiber*** $\mathcal{C}_\mathbf{x}$ is the chain

$$\mathcal{C}_\mathbf{x} = \mathbf{x} + i\mathbb{R}^d = \{\mathbf{x} + i\mathbf{y} : \mathbf{y} \in \mathbb{R}^d\},$$

oriented via the standard orientation on $\mathbb{R}^d$. Because the linear functions ℓ_j defining $\mathcal{A}$ have real coefficients, $\ell_j(\mathbf{x} + i\mathbf{y}) = 0$ only if $\ell_j(\mathbf{x}) = 0 = \ell_j(\mathbf{y})$, so the entire fiber $\mathcal{C}_\mathbf{x}$ is disjoint from $\mathcal{V}$. The point $\mathbf{x}$ is called the ***basepoint*** of the fiber $\mathcal{C}_\mathbf{x}$. If B is a connected component of the ***real domain of holomorphicity*** $\mathcal{M}_\mathbb{R} := \mathbb{R}^d_* \setminus \mathcal{V}_\mathbb{R}$ then the homotopy

$$t\mathbf{x} + (1-t)\mathbf{x}' + i\mathbb{R}^d \text{ for } t \in [0,1]$$

defines a homotopy between $\mathcal{C}_\mathbf{x}$ and $\mathcal{C}_{\mathbf{x}'}$ for any $\mathbf{x}, \mathbf{x}' \in B$. Thus, the homology class of $\mathcal{C}_\mathbf{x}$ in $H_d(\mathcal{M}, -\infty)$ depends only on the component B of $\mathcal{M}_\mathbb{R}$ containing $\mathbf{x}$, and we denote this homology class by $\mathcal{C}_B$.

Remark 3.9. The height function $h_{\hat{\mathbf{r}}}$ is unbounded from below on any unbounded component of $\mathcal{M}_\mathbb{R}$. Thus, $\mathcal{C}_\mathbf{x} \doteq 0$ whenever $\mathbf{x}$ lies in an unbounded component of $\mathcal{M}_\mathbb{R}$ since we can move $\mathbf{x}$ within B until its height is less than $h_{\min}$, at which point the whole fiber $\mathcal{C}_\mathbf{x}$ lies in $\mathcal{M}_{<h_{\min}}$. The fact that imaginary fibers of components on which $h_{\hat{\mathbf{r}}}$ is bounded from below form a basis of $H_d(\mathcal{M}, -\infty)$ was proved in [39, page 268].

Definition 3.10 (signs and linking tori)**.** Let $\mathcal{A}$ be a simple arrangement, $\hat{\mathbf{r}}$ be fixed, and $\boldsymbol{\sigma} \in \Omega$. Suppose $\boldsymbol{\sigma} \in \partial B$ for some component B of the real complement $\mathcal{M}_\mathbb{R} = \mathbb{R}^d_* \setminus \mathcal{V}_\mathbb{R}$ of the arrangement. Let s be the codimension of the stratum containing $\boldsymbol{\sigma}$ and let $k_1, \ldots, k_s$ be the indices such that the stratum $\mathcal{S}_{\boldsymbol{\sigma}}$ is defined by $\ell_{k_1} = \cdots = \ell_{k_s} = 0$. For each of the 2^k components B of $\mathcal{M}_\mathbb{R}$ with $\boldsymbol{\sigma} \in \overline{B}$, define the ***sign*** of B with respect to $\boldsymbol{\sigma}$ by

$$\operatorname{sgn}_{\boldsymbol{\sigma}}(B) := \operatorname{sgn}\left(\prod_{i=1}^{s} \ell_{k_i}(\mathbf{x})\right),$$

where $\mathbf{x}$ is any point of B. Thus, the sign is positive on the component not separated from the origin by any of the s hyperplanes, and alternates across each hyperplane. We also define the ***absolute sign*** of B by

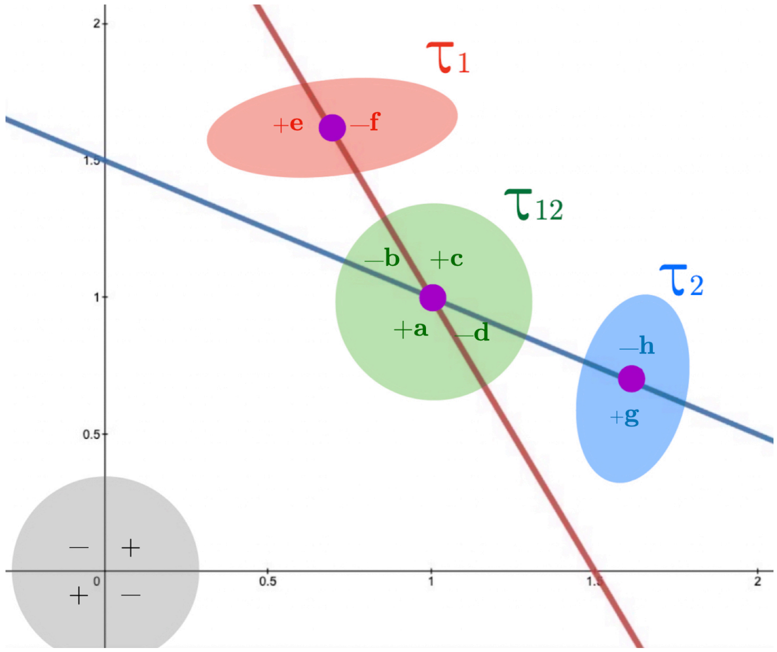

Figure 4: Linking tori for Example 3.6, with the signs associated to each basepoint pictured. A 'linking torus' around the origin with the shown signs will be used to decompose the Cauchy integral representing generating function coefficients.

$\text{sgn}(B) = \text{sgn}(\prod_{j=1}^{d} x_j)$ for any $\mathbf{x} \in B$. In other words, the sign is $+1$ if B is in an even orthant and -1 otherwise. The ***linking torus*** of $\boldsymbol{\sigma}$ is

$$\boldsymbol{\tau}_{\boldsymbol{\sigma}} := \sum_{B} \text{sgn}(B)\, \text{sgn}_{\boldsymbol{\sigma}}(B) \mathcal{C}_B,$$

where the sum is over all B with $\boldsymbol{\sigma} \in \overline{B}$.

Example 3.11 (linking tori for Example 3.6). Continuing Example 3.6, recall that the stratum $\mathcal{S}_{1,2}$ is the singleton point $\boldsymbol{\sigma}_{1,2} := (1,1)$. This is a stratum of codimension two, and the linking torus of $\boldsymbol{\sigma}_{1,2}$ is

$$\tau_{12} := \tau_{\boldsymbol{\sigma}_{1,2}} = \mathcal{C}_{\mathbf{a}} - \mathcal{C}_{\mathbf{b}} + \mathcal{C}_{\mathbf{c}} - \mathcal{C}_{\mathbf{d}},$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and $\mathbf{d}$ are points near $\boldsymbol{\sigma}$ in the four regions carved out by removing the two real lines ℓ_1 and ℓ_2 from $\mathbb{R}^2$, enumerated in counterclockwise order with $\mathbf{a}$ in the region whose closure contains the origin (see Figure 4).

Suppose $\mathbf{r} = (1,1)$, so we are in the situation pictured in Figure 4. The critical point $\boldsymbol{\sigma}_1$ lies on the codimension one stratum $\mathcal{S}_1$ and has linking torus

$$\tau_1 := \tau_{\boldsymbol{\sigma}_1} = \mathcal{C}_{\mathbf{e}} - \mathcal{C}_{\mathbf{f}},$$

where e lies in the halfspace defined by ℓ_1 whose closure contains the origin. Similarly, the critical point $\boldsymbol{\sigma}_2$ on $\mathcal{S}_2$ has linking torus

$$\tau_2 := \tau_{\boldsymbol{\sigma}_2} = \mathcal{C}_{\mathbf{g}} - \mathcal{C}_{\mathbf{h}},$$

where g lies in the halfspace defined by ℓ_2 whose closure contains the origin. Note that many of the basepoints in these linking tori lie in the same components of $\mathcal{M}_{\mathbb{R}}$. In particular,

$$\mathcal{C}_{\mathbf{b}} = \mathcal{C}_{\mathbf{e}}, \quad \mathcal{C}_{\mathbf{d}} = \mathcal{C}_{\mathbf{g}}, \quad \text{and} \quad \mathcal{C}_{\mathbf{c}} = \mathcal{C}_f = \mathcal{C}_g,$$

so the sum of the linking tori around the critical points is

$$\tau_1 + \tau_2 + \tau_{1,2} = \mathcal{C}_{\mathbf{a}} - \mathcal{C}_{\mathbf{c}}.$$

In fact, since $\mathbf{c}$ lies in an unbounded component of $\mathcal{M}_{\mathbb{R}}$ we have $\tau_1 + \tau_2 + \tau_{1,2} \doteq \mathcal{C}_{\mathbf{a}}$, which will come up again later in our analysis. ◁

Linking tori appear, in a more general form, in stratified Morse theory. The work of Goresky and MacPherson [16, Chapter 10] shows how the topology of $\mathcal{M}$ is generated by certain *attachment cycles* near $\boldsymbol{\sigma}$ as $\boldsymbol{\sigma}$ varies over the critical points of each stratum of $\mathcal{V}$. The cycle near a codimension s critical point $\boldsymbol{\sigma}$ has the structure of the product of the *normal link*, a local cycle which in this case is a small s-torus, with the *tangential cycle*, a $(d-s)$-disk modulo its boundary (specifically the $(d-s)$-dimensional downward gradient flow subspace of h on $\mathcal{V}$ near $\boldsymbol{\sigma}$). This toral tube intersects $\mathbb{R}^d$ in 2^s points, and is in fact equal to $\boldsymbol{\tau}_{\boldsymbol{\sigma}}$ in $H_d(\mathcal{M}, -\infty)$.

3.2 Results

We split our results into increasing complicated cases.

Simple arrangements in generic directions

All of our topological arguments can already be seen in the case of simple arrangements and generic directions. First, we identify which collection of critical points will be important in the analysis, which is, in the general (non-linear) case, a very hard and perhaps undecidable step.

Definition 3.12 (contributing points)**.** Assume $\mathcal{A}$ is simple and fix $\hat{\mathbf{r}} \in \mathbb{R}^d_{>0}$. We call $\boldsymbol{\sigma}$ a ***contributing point*** if $-\nabla h_{\hat{\mathbf{r}}}(\boldsymbol{\sigma}) \in N(\boldsymbol{\sigma})$ and let $\mathsf{contrib} \subseteq \Omega$ denote the set of contributing points.

Remarks. (*i*) The quantity $-\nabla h_{\hat{\mathbf{r}}}(\boldsymbol{\sigma})$ is given by $(\hat{r}_1/\sigma_1, \ldots, \hat{r}_d/\sigma_d)$. If this is in the positive cone $N(\boldsymbol{\sigma})$ then it is in linear space spanned by normals to hyperplanes containing $\boldsymbol{\sigma}$, so every contributing point is a critical point. (*ii*) Suppose $-\nabla h_{\hat{\mathbf{r}}}(\boldsymbol{\sigma})$ is on the relative boundary of $N(\boldsymbol{\sigma})$ (e.g., is in the span of some proper subset of the vectors $\{\mathbf{b}^{(k)} : k \in S(\boldsymbol{\sigma})\}$). Then $\boldsymbol{\sigma}$ is a critical point for the larger flat defined by this proper subset of hyperplanes and $\hat{\mathbf{r}}$ is not generic. In fact, non-genericity is equivalent to the existence of a critical point $\boldsymbol{\sigma} \in \Omega$ with $-\nabla h_{\hat{\mathbf{r}}}(\boldsymbol{\sigma}) \in \partial N(\boldsymbol{\sigma})$.

The heart of this paper, proven in Section 4, is the following result.

Theorem 3.13. *If $\mathcal{A}$ is simple and $\hat{\mathbf{r}}$ is generic then the domain of integration $\mathcal{T}$ in the Cauchy integral representation* (3) *satisfies*

$$\mathcal{T} \doteq \sum_{\boldsymbol{\sigma} \in \mathsf{contrib}} \boldsymbol{\tau}_{\boldsymbol{\sigma}}\,.$$

Theorem 3.13 gives a determination of the coefficients $\{k_{\boldsymbol{\sigma}}\}$ in Equation (5), namely $k_{\boldsymbol{\sigma}} = 1$ when $\boldsymbol{\sigma} \in \mathsf{contrib}$ and zero otherwise. Is this expected or surprising? If the torus $\mathcal{T}$ may be expanded up to the point $\boldsymbol{\sigma}$ while remaining in $\mathcal{T}$, then the condition $\boldsymbol{\sigma} \notin \mathsf{contrib}$ is precisely the condition needed to extend the deformation to bypass $\boldsymbol{\sigma}$ and show that $\mathcal{T} \doteq C$ for some C in $\mathcal{M}_{<h(\boldsymbol{\sigma})}$. Theorem 3.13 may be understood to say that the converse holds and, moreover, it holds even if one has to pass through $\mathcal{V}$ at various points en route to $\boldsymbol{\sigma}$.

One might conjecture that Theorem 3.13 should hold for general rational functions, but this is false. A counterexample is given in [2], where for certain critical points $\boldsymbol{\sigma}$ the coefficient $k_{\boldsymbol{\sigma}}$ may be seen not only to be nonzero but to take on the surprising value of 3. In other words, when $\mathcal{T}$ passes through $\mathcal{V}$ en route to $\boldsymbol{\sigma}$, it twists so as to hit $\mathcal{V}$ near $\boldsymbol{\sigma}$ in an unpredictable manner. Hyperplane arrangements are special in that they do not allow these kinds of shenanigans. We will give more intuition as to why when we discuss the relation between real hyperplane arrangements and their complexifications in Section 4.2.

To complete the analysis for simple arrangements in generic directions, saddle point integration results are quoted from [34], yielding in Theorem 4.16 from Section 4.3 an asymptotic expansion of $[\mathbf{z}^{n\mathbf{r}}]F(\mathbf{z})$ in terms of effective constants which can be determined an automatic manner. In many cases the leading asymptotic term is given explicitly. Algorithm 1 summarizes the steps needed to automatically produce these formulae. Details on our Maple implementation of Algorithm 1 and additional examples are given in Section 4.4. The main wrinkle in determining dominant asymptotics from the explicit formulae is the possibility that leading constants in Theorem 4.16 may vanish due to cancellation, which is discussed in Section 4.5.

Algorithm 1: SimpleHyperplaneAsymptotics

```
Input: Rational function F(z) = G(z)/∏_{j=1}^m ℓ_j(z)^{p_j} with ℓ_j(z) = 1 − (b^(j), z) linearly independent
       Generic direction r = (r_1, ..., r_d)
Output: Asymptotic expansion of [z^{nr}]F(z) as n → ∞, uniform in compact set around r

if matrix with rows b^(j) not full rank for a collection with common solution then
 | return FAIL, "not a simple arrangement"
end
get the flats defined by the ℓ_j(z)
find the (unique) critical point in each orthant for each flat by solving Equation (7)
sort the saddles by height function h_r and set σ equal to the highest critical point
while no contributing points have been found do
    compute the cone N(σ)
    if (−∇h)(σ) on boundary of N(σ) then
     | return FAIL, "r not a generic direction"
    end
    if (−∇h)(σ) ∉ N(σ) then
     | set σ to next lowest height critical point and repeat loop
    end
    Compute the ideal J(σ) in Equation (15)
    if G ∈ J(σ) then
     | set σ to next lowest height critical point and repeat loop
    else
        σ is a contributing singularity
        repeat above steps to identify contributing singularities of the same height as σ
    end
end
calculate asymptotic Φ_σ at each contributing singularity σ of highest height using Theorem 4.16
return sum of the Φ_σ from the last step
```

Non-simple arrangements

Section 5 deals with non-simple $F(\mathbf{z})$ in generic directions $\mathbf{r}$ using a partial fraction decomposition. When $F(\mathbf{z}) = G(\mathbf{z})/\prod_{j=1}^{m} \ell_j(\mathbf{z})^{p_j}$, with linear dependencies among the normals $\mathbf{b}^{(j)}$ to ℓ_j, it is possible to rewrite F as $\sum_{i=1}^{s} F_i$ where each F_i is of this same form except that the vectors $\mathbf{b}^{(j)}$ are linearly independent.

This is a known fact from the theory of matroids: the **broken circuit** complex defined in [9] leads to the canonical **no broken circuit** basis [31] for the matroid $\{\mathbf{b}^{(j)} : 1 \le j \le m\}$, which may be harnessed for easy computation of the partial fraction decomposition $F = \sum_{i=1}^s F_i$. The main result of Section 5 is the following algorithm.

Algorithm 2: SimpleDecomp

Input: Rational function $F(\mathbf{z}) = G(\mathbf{z}) / \prod_{j=1}^m \ell_j(\mathbf{z})^{p_j}$
Output: Collection $F_1(\mathbf{z}), \ldots, F_r(\mathbf{z})$ of rational functions with $\sup(F_j) \subset \sup(F)$ such that $F = F_1 + \cdots + F_r$ and each F_j has independent support

$S \leftarrow F(\mathbf{z})$
while *there exists a summand* $\tilde{F}$ *of* S *with broken circuit* $\{i_1, \ldots, i_s\}$ **do**
    apply Equation (17) with $i_j > i_s$ such that $\{i_1, \ldots, i_s, i_j\}$ is dependent
    add the result to $S - \tilde{F}$
end
collect and simplify summands in S with the same denominator
while *there exists summand* $\tilde{F}$ *of* S *with numerator in the ideal generated by* $\sup(\tilde{F})$ **do**
    pick among all such $\tilde{F}$ one with maximal total degree in its denominator
    write the numerator as a polynomial combination of the elements in its denominator
    expand this new fraction and simplify, removing terms in the denominator
    collect and simplify summands with the same denominator
end

Non-generic directions

Section 6 deals with non-generic directions $\mathbf{r}$ which, beyond some two-dimensional work of Lladser [23], has not received much previous attention in the literature. Due to the results of Section 5, when addressing non-generic directions one may assume without loss of generality that F is simple.

Nongeneric directions arise when the domains of dominant singularities from different strata collide. If $\mathcal{S}$ is a stratum of codimension k, then each $\boldsymbol{\sigma} \in \mathcal{S}$ lies in the set $\mathsf{crit}(\mathbf{r})$ for a set of vectors $\mathbf{r}$ of dimension k (projective dimension $k-1$). As $\boldsymbol{\sigma}$ varies over $\mathcal{S}$, these sets foliate an open cone K in $\mathbb{R}^d$. Typically, as $\mathbf{r}$ passes through ∂K there will be a point $\boldsymbol{\sigma}(\mathbf{r}) \in \mathsf{crit}(\mathbf{r})$ such that $\boldsymbol{\sigma}(\mathbf{r}) \in \mathsf{contrib}$ on one side of ∂K while $\boldsymbol{\sigma}(\mathbf{r}) \notin \mathsf{contrib}$ on the other side. Precisely when $\mathbf{r} \in \partial K$, it is not defined whether the critical point $\boldsymbol{\sigma}(\mathbf{r})$ is in $\mathsf{contrib}$ because this is not defined for non-generic directions (as mentioned above, $-\nabla h_{\hat{\mathbf{r}}}(\boldsymbol{\sigma})$ lies on the boundary of the positive cone $N(\boldsymbol{\sigma})$).

The word *typically* above refers to the fact that these cones themselves are curved versions of central hyperplane arrangements. Their boundaries are mostly smooth except when two or more bounding surfaces intersect, corresponding to the stratum of $\boldsymbol{\sigma}$ dropping in dimension. Our most explicit results are limited to the most common case, when there is a drop of 1 in dimension. In this sense there is still more work to be done to obtain a complete understanding of non-generic directions.

We give two types of result. The first is the behavior precisely along a non-generic direction when the critical points of two strata, one of which is precisely one dimension less than the other, overlap. In this setting, we consider $a_{n\hat{\mathbf{r}}}$ as $n \to \infty$ when $\hat{\mathbf{r}}$ is a fixed non-generic rational direction; our results may be extended to $\mathbf{r} = n\hat{\mathbf{r}} + O(1)$, though we do not pursue that here. Proposition 6.7 in Section 6 gives the leading asymptotic behavior in terms of certain negative moments of Gaussian distributions, computed in Proposition 6.2. The second type of result concerns the behavior of $a_{\mathbf{r}}$ as $\mathbf{r}$ crosses ∂K. Patching from one formula to the other occurs in a scaling window of width $\sqrt{|\mathbf{r}|}$, with our most explicit result occurs when all divisors have power 1.

3.3 Applications

Before going into details we list a few applications of our results.

Example 3.14 (Example 1.1 continued). The simplest case of (2) is the two-server case, where (2) becomes

$$F(x,y) = \frac{e^{x+y}}{(1-\rho_{11}x-\rho_{12}y)(1-\rho_{21}x-\rho_{22}y)} \tag{9}$$

By scaling the x and y variables, asymptotics in all cases of interest follow from a study of

$$H(x,y) = \frac{e^{x/\rho_{21}+y/\rho_{22}}}{(1-ax-by)(1-x-y)}$$

where $a = \rho_{11}/\rho_{21} > 0$ and $b = \rho_{12}/\rho_{22} > 0$. By symmetry we may assume $a \geq b$; if $a = b$ then H is not simple, so we postpone this case until the end of the example. The asymptotic expansion in Theorem 4.16 gives coefficient asymptotics for H, as follows.

Parametrize directions by taking $\mathbf{r} = (r, 1)$ with $r > 0$. The flat defined by $1-ax-by = 0$ has the critical point

$$c_1 = \left(\frac{r}{(r+1)a}, \frac{1}{(r+1)b}\right),$$

which is always contributing. Similarly, the flat defined by $1-x-y = 0$ has the critical point

$$c_2 = \left(\frac{r}{r+1}, \frac{1}{r+1}\right),$$

which is always contributing. The flat defined by $1-ax-by = 1-x-y = 0$ contains a single point

$$c_{12} = \left(\frac{1-b}{a-b}, \frac{a-1}{a-b}\right),$$

which is always a critical point. The point c_{12} is contributing, and $(r, 1)$ is a generic direction, if and only if there exist $A_1, A_2 > 0$ such that

$$(-\nabla h)(c_{12}) = \left(\frac{r(a-b)}{1-b}, \frac{a-b}{a-1}\right) = A_1(a,b) + A_2(1,1);$$

since we assume $a \geq b$, this occurs precisely when $0 < b < 1 < a$ and

$$\frac{1-b}{a-1} < r < \frac{(b-a)a}{(a-1)b}.$$

The points c_1, c_2, and c_{12} have corresponding heights

$$\begin{aligned}
h_1 &= -r\log\left(\frac{r}{r+1}\right) - \log\left(\frac{1}{r+1}\right) + \log(a^r b) \\
h_2 &= -r\log\left(\frac{r}{r+1}\right) - \log\left(\frac{1}{r+1}\right) \\
h_{12} &= -r\log\left|\frac{1-b}{a-b}\right| - \log\left|\frac{a-1}{a-b}\right|.
\end{aligned}$$

The theory developed below shows that h_{12} is the highest height contributing point whenever c_{12} is contributing. Otherwise, h_1 has the highest height among the contributing points if and only if $a^r b > 1$. Theorem 4.16 determines asymptotics for each of the three different asymptotic regimes: if $0 < b < 1 < a$ and $\frac{1-b}{a-1} < r < \frac{(b-a)a}{(a-1)b}$ then

$$[x^{rn}y^n]H(x,y) \sim \left(\frac{a-b}{1-b}\right)^{rn}\left(\frac{a-b}{a-1}\right)^n \frac{\exp\left[\frac{(1-b)\rho_{22}+(a-1)\rho_{21}}{(a-b)\rho_{21}\rho_{22}}\right](1-b)(a-1)}{a-b}.$$

If those conditions do not hold and $a^r b > 1$ then

$$[x^{rn}y^n]H(x,y) \sim \left(a+\frac{a}{r}\right)^{rn}(b+br)^n \frac{\exp\left[\frac{a\rho_{21}+rb\rho_{22}}{ab(r+1)\rho_{21}\rho_{22}}\right]ab(r+1)^{3/2}}{((a-1)rb+a(b-1))\sqrt{2n\pi r}},$$

while if those conditions do not hold and $a^r b < 1$ then

$$[x^{rn}y^n]H(x,y) \sim \left(1+\frac{1}{r}\right)^{rn}(1+r)^n \frac{\exp\left[\frac{\rho_{21}+r\rho_{22}}{(r+1)\rho_{21}\rho_{22}}\right](r+1)^{3/2}}{(1-b+r-ar)\sqrt{2n\pi r}}.$$

If $a = b \neq 1$ then we are in the non-simple case, and Algorithm 2 provides the decomposition

$$H(x,y) = \frac{e^{x/\rho_{21}+y/\rho_{22}}}{(1-ax-ay)(1-x-y)} = \frac{a}{a-1}\cdot\frac{e^{x/\rho_{21}+y/\rho_{22}}}{1-ax-ay} - \frac{1}{a-1}\cdot\frac{e^{x/\rho_{21}+y/\rho_{22}}}{1-x-y}.$$

Thus, as coefficient extraction distributes over a sum,

$$[x^{rn}y^n]H(x,y) \sim \left(a+\frac{a}{r}\right)^{rn}(a+ar)^n \frac{\exp\left[\frac{\rho_{21}+r\rho_{22}}{a(r+1)\rho_{21}\rho_{22}}\right]a\sqrt{r+1}}{(a-1)\sqrt{2n\pi r}} + \left(1+\frac{1}{r}\right)^{rn}(1+r)^n \frac{\exp\left[\frac{\rho_{21}+r\rho_{22}}{(r+1)\rho_{21}\rho_{22}}\right]\sqrt{r+1}}{(a-1)\sqrt{2n\pi r}}.$$

Finally, if $a = b = 1$ then $H(x,y) = e^{x/\rho_{21}+y/\rho_{22}}/(1-x-y)^2$ is simple but has a second order pole. Theorem 4.16 then implies

$$[x^{rn}y^n]H(x,y) \sim \left(1+\frac{1}{r}\right)^{rn}(1+r)^n\sqrt{n}\,\frac{\exp\left[\frac{\rho_{21}+r\rho_{22}}{(r+1)\rho_{21}\rho_{22}}\right](r+1)^{3/2}}{\sqrt{2\pi r}}.$$

◁

Example 3.15 (Pemantle and Wilson [34, Ex. 12.1.3]). Fix $r, s \in \mathbb{N}$. A sequence of $r+s$ independent biased coin flips occurs, with the first n flips coming up heads with probability $2/3$ and the remaining $r+s-n$ flips coming up heads with probability $1/3$. A player wants to see r heads and s tails, and may pick n. On average, how many choices of n will be winning choices?

By independence of the coin flips, the probability that n is a winning choice equals

$$p_{r,s}(n) = \sum_{0\le a\le n}\binom{n}{a}(2/3)^a(1/3)^{n-a}\binom{r+s-n}{r-a}(1/3)^{r-a}(2/3)^{s-(n-a)},$$

so $f_{r,s} = \sum_{n\ge 0} p_{r,s}(n)$ counts the expected number of winning values of n. It follows that the bivariate generating function $F(x,y) = \sum_{r,s\ge 0} f_{r,s}x^r y^s$ is a product of rational functions with linear denominators,

$$F(x,y) = \frac{1}{(1-2x/3-y/3)(1-x/3-2y/3)}.$$

Theorem 4.16 then immediately implies that, for any $\epsilon > 0$,

$$f_{r,s} \sim \begin{cases} \left(\frac{2(r+s)}{3s}\right)^{sn} \left(\frac{r+s}{3r}\right)^{rn} n^{-1/2} \frac{(r+s)^{3/2}\sqrt{2}}{\sqrt{rs\pi}(s-2r)} & : r < s/2 - \epsilon \\ 3 & : s/2 + \epsilon < r < 2s - \epsilon \\ \left(\frac{2(r+s)}{3r}\right)^{rn} \left(\frac{r+s}{3s}\right)^{sn} n^{-1/2} \frac{(r+s)^{3/2}\sqrt{2}}{\sqrt{rs\pi}(r-2s)} & : 2s + \epsilon < r \end{cases}$$

as $(r, s) \to \infty$. In particular, the expected number of winning values of n exponentially decreases in the first and last cases, but approaches the constant 3 when $s/2 + \epsilon < r < 2s - \epsilon$. Moreover, in this case the leading term is not an asymptotic series but the single term 3, as discussed in Remark 4.17 (see also [35]). It follows that when $(r, s) \to \infty$ with $s/2 + \epsilon < r < 2s - \epsilon$, the error term $|3 - f_{r,s}|$ decreases exponentially in s.

This work allows, for the first time, detailed information at slopes 1/2 and 2 and how behaviour transitions around such directions. Proposition 6.3 implies that both $f_{2s,s}$ and $f_{s,2s}$ approach 3/2 as $s \to \infty$. More precisely, Proposition 6.7 implies that the transition, in which dominant asymptotics go from the constant 3 through the constant 3/2 to end up exponentially small, occurs in a scaling window of width $s^{1/2}$:

$$f_{2s+t\sqrt{s},s} \sim \frac{3}{2} \left(\Psi(t) + 1\right),$$

where $\Psi(z) = \frac{2}{\pi} \int_0^z e^{-u^2/2}\, du$ is the Gaussian error function. $\triangleleft$

4 Simple Arrangements in Generic Directions

Unless otherwise stated, in this section we assume that

$$F(\mathbf{z}) = \frac{G(\mathbf{z})}{\prod_{j=1}^m \ell_j^{p_j}(\mathbf{z})} \text{ is simple and } \hat{\mathbf{r}} \text{ is generic.} \tag{$H1$}$$

We begin by describing an iterative *slide and replace* approach to finding critical points which affect dominant asymptotics. This iterative algorithm, while correctly determining asymptotics, becomes unnecessary once we prove Theorem 3.13. Our topological results allow one to shortcut the slide and replace algorithm, since we know a priori that the easily determined contributing points will be those critical points selected by the algorithm.

4.1 Slide and replace

Let $\mathcal{B}$ denote the collection of *bounded* components of $\mathcal{M}_{\mathbb{R}}$. For $\boldsymbol{\sigma} \in \mathsf{crit}$ denote by $\mathrm{Adj}\,(\boldsymbol{\sigma}) \subseteq \mathcal{B}$ the collection $\{B : \boldsymbol{\sigma} \in \overline{B}\}$ of bounded components of $\mathcal{M}_{\mathbb{R}}$ adjacent to $\boldsymbol{\sigma}$. For $B \in \mathcal{B}$, we recall that the imaginary fiber $\mathcal{C}_B$ is well defined in $H_d(\mathcal{M}, -\infty)$ and independent of the particular basepoint $\mathbf{x} \in B$. Let $\mathbb{U}$ denote the set unbounded components of $\mathcal{M}_{\mathbb{R}}$, so $\mathcal{C}_B \doteq 0$ for any $B \in \mathbb{U}$ by Remark 3.9. In any orthant the height function $h_{\hat{\mathbf{r}}}$ is strictly convex, so its minimizer on any bounded closed convex set is unique and occurs on the boundary of the set.

Proposition 4.1. *For $B \in \mathcal{B}$, let $\mathbf{x}_B$ denote the unique $\mathbf{x} \in \overline{B}$ on which h is minimized. Then $B \mapsto \mathbf{x}_B$ defines a one-to-one correspondence between $\mathcal{B}$ and* crit.

Proof. The minimizer $\mathbf{x}_B$ lies in $\partial\overline{B}$ and thus on some stratum $\mathcal{S} = \mathcal{S}_L$. The intersection $\mathcal{S} \cap \overline{B}$ is a neighborhood of $\mathbf{x}$ in $\mathcal{S}$, therefore $\mathbf{x}$ is a local minimum for h on $\mathcal{S}$ and thus $\mathbf{x}$ is the critical point for the stratum $\mathcal{S}$. We have shown that the range of the map $B \mapsto \mathbf{x}_B$ is contained in crit, and it remains to show that the map is a bijection.

Pick any $\boldsymbol{\sigma} \in \mathsf{crit}$ and let $\mathcal{S} = \mathcal{S}_L$ be the stratum containing $\boldsymbol{\sigma}$. Writing the flat $\mathcal{V}_L$ as the common zero set of $\ell_1, \ldots \ell_k$, the k hyperplanes defined by the ℓ_j divide $\mathbb{R}^d$ into 2^k regions each containing a unique (possibly unbounded) component B of $\mathcal{M}_\mathbb{R}$ whose closure contains $\boldsymbol{\sigma}$. If $k = 1$ then $\mathbf{b}^{(1)}$ is a non-zero multiple of $\nabla h(\boldsymbol{\sigma})$, and $\boldsymbol{\sigma}$ is a local minimum for h precisely on the closure of the component $B \in \mathrm{Adj}\,(\boldsymbol{\sigma})$ closer to the origin. When $k > 1$ none of these k hyperplanes are orthogonal to $\nabla h(\boldsymbol{\sigma})$, by our assumption of genericity of $\mathbf{r}$. Therefore there is a unique component $B(\boldsymbol{\sigma})$ of $\mathcal{M}_\mathbb{R}$ such that, in a neighborhood of $\boldsymbol{\sigma}$, if $\mathbf{x}$ is on the boundary of $B(\boldsymbol{\sigma})$ and $\mathbf{v}$ is a vector from $\mathbf{x}$ pointing into $B(\boldsymbol{\sigma})$ then h increases in the direction $\mathbf{v}$. Because the point $\boldsymbol{\sigma}$ is a minimum for h on $\mathcal{S}$, it follows that $\boldsymbol{\sigma}$ is a local minimum for h on $B(\boldsymbol{\sigma})$, hence by convexity the global minimum for h on $B(\boldsymbol{\sigma})$. Thus, we have $\mathbf{x}_{B(\boldsymbol{\sigma})} = \boldsymbol{\sigma}$.

On the other hand, for any other component $B' \neq B$ adjacent to $\boldsymbol{\sigma}$ there is a point $\mathbf{x}$ on the boundary of B', which can be taken arbitrarily close to $\boldsymbol{\sigma}$, and a vector $\mathbf{v}$ pointing into B' such that h decreases at a linear rate along $\mathbf{v}$. Because $\boldsymbol{\sigma}$ is a critical point $h(\mathbf{x}) = h(\boldsymbol{\sigma}) + O(|\mathbf{x} - \boldsymbol{\sigma}|^2)$ on $\mathcal{V}_L$, hence h cannot be minimized on $\overline{B'}$ at $\boldsymbol{\sigma}$. It follows that the choice $B(\boldsymbol{\sigma})$ is unique, finishing the proof. $\square$

Given $\mathbf{x} \in B$ for a bounded component $B \in \mathcal{B}$, the *slide and replace* operation represents the fiber $\mathcal{C}_\mathbf{x}$ in terms of linking tori and lower fibers. The slide consists of moving $\mathbf{x}$ near to the critical point $\boldsymbol{\sigma} := \mathbf{x}_B$ on the boundary of B at which h is minimized. The replacement consists of replacing $\mathcal{C}_\mathbf{x}$ by $\boldsymbol{\tau}_{\boldsymbol{\sigma}}$ plus a sum of fibers $\mathcal{C}_\mathbf{y}$ over points $\mathbf{y}$ in components B' with $h(\mathbf{x}_{B'}) < h(\mathbf{x}_B)$. We illustrate this in an example.

Example 4.2 (slide and replace). The left side of Figure 5 illustrates the slide and replace operation for our running Example 3.6 when $\hat{r}_1 > 2/3$. Starting with a basepoint $\mathbf{x}$ near the origin in the first quadrant, we slide $\mathbf{x}$ near the critical point $\boldsymbol{\sigma}_1$ in the flat $\mathcal{V}_1$. Ideally we would like to replace $\mathcal{C}_\mathbf{x}$ by $\boldsymbol{\tau}_{\boldsymbol{\sigma}_1}$ but these chains are not equal. Instead, the linking torus equals $\mathcal{C}_\mathbf{x} - \mathcal{C}_\mathbf{y}$, where $\mathbf{y}$ is in the component B' lying just to the right of $\boldsymbol{\sigma}_1$, and we replace $\mathcal{C}_\mathbf{x}$ by $\boldsymbol{\tau}_{\boldsymbol{\sigma}} + \mathcal{C}_\mathbf{y}$. Repeating the slide and replace operation on $\mathcal{C}_\mathbf{y}$ yields $\mathcal{C}_\mathbf{y} = \boldsymbol{\tau}_{\boldsymbol{\sigma}_2} + \mathcal{C}_\mathbf{z}$, where $\boldsymbol{\sigma}_2$ is the critical point on the next line segment to the right and $\mathbf{z}$ is in the unbounded component above $\boldsymbol{\sigma}_2$. Because $\mathcal{C}_\mathbf{z} \doteq 0$ all fibers have been replaced by tori, so the algorithm is complete and returns the identity

$$\mathcal{C}_\mathbf{x} \doteq \boldsymbol{\tau}_{\boldsymbol{\sigma}_1} + \boldsymbol{\tau}_{\boldsymbol{\sigma}_2}\,.$$

The right side of Figure 5 depicts the result of running the same algorithm $\hat{r}_1 \in (1/3, 2/3)$. This time $\mathbf{x}$ slides near $\boldsymbol{\sigma}_{1,2} = (1,1)$ and replacing $\mathcal{C}_\mathbf{x}$ by $\boldsymbol{\tau}_{\boldsymbol{\sigma}_{1,2}}$ generates three additional imaginary fibers,

$$\mathcal{C}_\mathbf{x} = \boldsymbol{\tau}_{\boldsymbol{\sigma}} + \mathcal{C}_\mathbf{b} - \mathcal{C}_\mathbf{c} + \mathcal{C}_\mathbf{d}$$

with $\mathbf{b}, \mathbf{c}$, and $\mathbf{d}$ as in Figure 4 above. The points $\mathbf{b}$ and $\mathbf{d}$ slide to the respective critical points $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, while the point $\mathbf{c}$ slides to infinity. Ultimately, we obtain the same relation

$$\mathcal{C}_\mathbf{x} \doteq \boldsymbol{\tau}_{\boldsymbol{\sigma}_1} + \boldsymbol{\tau}_{\boldsymbol{\sigma}_2} + \boldsymbol{\tau}_{\boldsymbol{\sigma}_{1,2}}$$

observed above. $\triangleleft$

Formalizing the slide and replace algorithm leads to the following two results. By the previous proposition we may index the cycles $\{\mathcal{C}_B : B \in \mathcal{B}\}$ by crit, denoting

$$\mathcal{C}_{\boldsymbol{\sigma}} := \mathcal{C}_{B(\boldsymbol{\sigma})}\,.$$

Proposition 4.3. *The spans of* $\{\mathcal{C}_{\boldsymbol{\sigma}} : \boldsymbol{\sigma} \in \mathsf{crit}\}$ *and* $\{\boldsymbol{\tau}_{\boldsymbol{\sigma}} : \boldsymbol{\sigma} \in \mathsf{crit}\}$ *as* $\mathbb{Z}$*-modules in* $H_d(\mathcal{M}, -\infty)$ *are equal.*

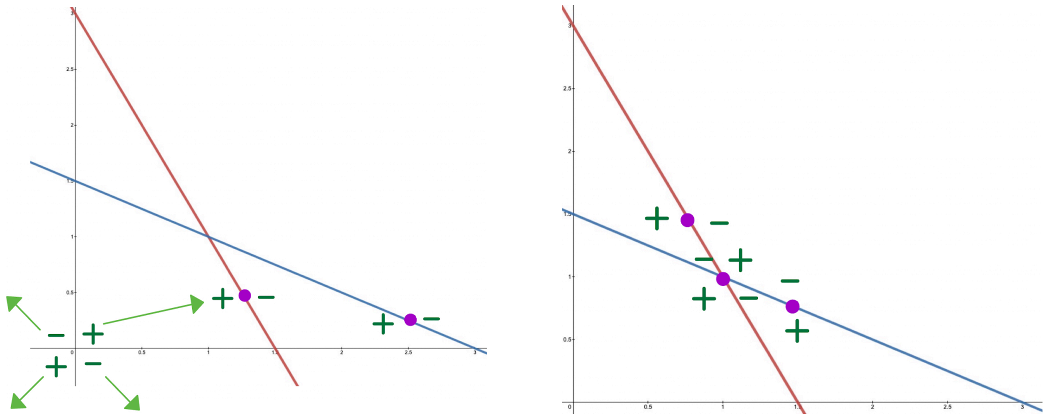

Figure 5: *Left:* When $\mathbf{r} = (5, 1)$ we start with the point $\mathbf{x}$ near the origin, bring $\mathbf{x}$ to the critical point $\boldsymbol{\sigma}_1$, then add and subtract imaginary fibers to ultimately obtain $\mathcal{C}_{\mathbf{x}}$ as a sum of linking tori. Note that $\mathcal{C}_{\mathbf{x}'} \doteq 0$ for $\mathbf{x}'$ in any component of $\mathcal{M}_{\mathbb{R}}$ whose closure contains the origin but not $\mathbf{x}$. *Right:* When $\mathbf{r} = (1, 1)$ we apply the slide and replace algorithm to represent $\mathcal{C}_{\mathbf{x}}$ as a sum of the linking tori seen in Figure 4. Note that the signs in all bounded components not containing $\mathcal{C}_{\mathbf{x}}$ cancel in both cases.

Proof. Given $\boldsymbol{\sigma} \in \mathsf{crit}$, we use the definition of $\boldsymbol{\tau}_{\boldsymbol{\sigma}}$ to write

$$\mathcal{C}_{\boldsymbol{\sigma}} = \boldsymbol{\tau}_{\boldsymbol{\sigma}} - \operatorname{sgn}(\boldsymbol{\sigma}) \sum_{B' \in \mathrm{Adj}\,(\boldsymbol{\sigma}) \setminus \{B(\boldsymbol{\sigma})\}} \operatorname{sgn}_{\boldsymbol{\sigma}}(B') \mathcal{C}_{B'} \,. \tag{10}$$

For each B' in the sum, the infimum value of h on B' is achieved somewhere other than $\boldsymbol{\sigma}$, hence $B' = B(\boldsymbol{\sigma}')$ for some $\boldsymbol{\sigma}'$ with $h(\boldsymbol{\sigma}') < h(\boldsymbol{\sigma})$. Therefore, (10) writes $\mathcal{C}_{\boldsymbol{\sigma}}$ as $\boldsymbol{\tau}_{\boldsymbol{\sigma}}$ plus a linear combination of classes $\mathcal{C}_{\boldsymbol{\sigma}'}$ with $h(\boldsymbol{\sigma}') < h(\boldsymbol{\sigma})$. Iterating, we express $\mathcal{C}_{\boldsymbol{\sigma}}$ as an integer sum of classes $\boldsymbol{\tau}_{\boldsymbol{\sigma}}$ (because the iteration ends at null classes $\mathcal{C}_{B'}$ where the component B' is unbounded). The converse holds since $\boldsymbol{\tau}_{\boldsymbol{\sigma}}$ is by definition an integer combination of classes $\mathcal{C}_B$ for $B \in \mathrm{Adj}\,(\boldsymbol{\sigma})$. $\square$

Proposition 4.4. *The collection $\{\boldsymbol{\tau}_{\boldsymbol{\sigma}} : \boldsymbol{\sigma} \in \mathsf{crit}\}$ forms a module basis of $H_d(\mathcal{M}, -\infty)$ with coefficients in $\mathbb{Z}$. Furthermore, the change of basis matrix from $\{\mathcal{C}_{\boldsymbol{\sigma}} : \boldsymbol{\sigma} \in \mathsf{crit}\}$ is upper-triangular, for any total order $\prec$ that extends the partial order $h(\boldsymbol{\sigma}) < h(\boldsymbol{\sigma}')$. This means*

$$\mathcal{C}_{\boldsymbol{\sigma}} = \sum_{\boldsymbol{\sigma}' \in \mathsf{crit}} \nu_{\boldsymbol{\sigma}, \boldsymbol{\sigma}'} \, \boldsymbol{\tau}_{\boldsymbol{\sigma}'}$$

for ***linking constants*** *$\nu_{\boldsymbol{\sigma}, \boldsymbol{\sigma}'}$, where $\nu_{\boldsymbol{\sigma}, \boldsymbol{\sigma}'} = 0$ unless $\boldsymbol{\sigma}' \preceq \boldsymbol{\sigma}$.*

Proof. This relies on the fact that $\{\mathcal{C}_B : B \in \mathcal{B}\}$ form a basis for $H_d(\mathcal{M}, -\infty)$ with coefficients in $\mathbb{Z}$. This is essentially Lemma 5.1 and Theorem 5.2 of [30], most of it having been proved in [29] and in [31, Chapters 3 and 5]. The result then follows from Proposition 4.3, with upper triangularity following from the fact that $\boldsymbol{\sigma}' \preceq \boldsymbol{\sigma}$ whenever $B(\boldsymbol{\sigma}') \in \mathrm{Adj}\,(\boldsymbol{\sigma})$. $\square$

4.2 Proof of Theorem 3.13

Proposition 4.4 implies that the (class of the) domain of integration $\mathcal{T}$ in the Cauchy integral (3) can be written as a finite integer linear combination of linking tori in $H_d(\mathcal{M}, -\infty)$, while Theorem 3.13 states that this linear combination is simply a sum of the linking tori corresponding to contributing points. We begin our proof of Theorem 3.13 by characterizing contributing points via a minimizing property of the height function. Informally, $\boldsymbol{\sigma} \in \mathsf{contrib}$ if and only if $\boldsymbol{\sigma} = \mathbf{x}_B$ when all hyperplanes other than those containing $\boldsymbol{\sigma}$ are dropped from $\mathcal{A}$ and B is the component whose closure contains both the origin and $\boldsymbol{\sigma}$.

Proposition 4.5. *The point $\boldsymbol{\sigma}$ in the flat $\mathcal{V}_{k_1,\ldots,k_s}$ is a contributing singularity if and only if $\boldsymbol{\sigma}$ is the unique minimizer of $h_{\hat{\mathbf{r}}}$ on the convex connected set*

$$\Gamma = \left\{\mathbf{z} : \ell_{k_j}(\mathbf{z}) \geq 0 \text{ for } j = 1, \ldots, s\right\} \cap \{\mathbf{z} : \sigma_j z_j > 0 \text{ for } j = 1, \ldots, d\}$$

consisting of the polyhedron in the orthant containing $\boldsymbol{\sigma}$ defined by the $\ell_j(\mathbf{z})$ and the coordinate axes.

Proof. In any fixed orthant, the strictly convex function $h(\mathbf{z})$ has a unique minimizer on the convex set Γ. Because $\boldsymbol{\sigma}$ is critical, and we are in a generic direction, we can write

$$(-\nabla h)(\boldsymbol{\sigma}) = \sum_{j=1}^{s} a_j \mathbf{b}^{(k_j)}$$

for some $a_j \in \mathbb{R}_*$. If $a_j > 0$ for all j then

$$(\nabla h)(\boldsymbol{\sigma}) \cdot (\mathbf{w} - \boldsymbol{\sigma}) = \sum_{j=1}^{s} a_j \left(1 - \mathbf{w} \cdot \mathbf{b}^{(k_j)}\right) \geq 0$$

for all $\mathbf{w} \in \Gamma$, so $h(\mathbf{z})$ decreases as $\mathbf{z}$ moves from $\mathbf{w}$ towards $\boldsymbol{\sigma}$, and $\boldsymbol{\sigma}$ minimizes $h(\mathbf{z})$. Conversely, if, say, $a_1 < 0$ and we pick $\mathbf{w}$ such that $\ell_{k_j}(\mathbf{w}) = 0$ for $j = 2, \ldots, s$ and $\ell_{k_1}(\mathbf{w}) > 0$, then $(\nabla h)(\boldsymbol{\sigma}) \cdot (\mathbf{w} - \boldsymbol{\sigma}) = a_1(1 - \mathbf{b}^{(k_1)} \cdot \mathbf{w}) < 0$ and for $\epsilon > 0$ sufficiently small we have $\mathbf{v} = \boldsymbol{\sigma} + \epsilon(\mathbf{w} - \boldsymbol{\sigma}) \in \Gamma$ with $h(\mathbf{v}) < h(\boldsymbol{\sigma})$. Thus, $\boldsymbol{\sigma}$ is contributing if and only if it is the unique minimizer of $h_{\hat{\mathbf{r}}}$ on Γ. ☐

Example 4.6. Continuing Example 3.6, let B be the bounded component of $\mathcal{M}_{\mathbb{R}}$ whose closure contains the origin. The cones $N(\boldsymbol{\sigma}_1)$ and $N(\boldsymbol{\sigma}_2)$ are rays which contain $-(\nabla h)(\boldsymbol{\sigma}_1)$, meaning they are always contributing points. The critical point $\boldsymbol{\sigma}_{1,2} = (1, 1)$ is more interesting. By definition $\boldsymbol{\sigma} \in \mathsf{contrib}$ if and only if $\hat{r}_1 \in [1/3, 2/3]$ and $\hat{\mathbf{r}}$ is generic (which removes the endpoints). For precisely these values of $\hat{r}_1$, the point $\boldsymbol{\sigma}_{1,2}$ is the minimizer of $h_{\hat{\mathbf{r}}}$ on $\overline{B}$. When $\hat{r}_1 < 1/3$ the minimizer of $h_{\hat{\mathbf{r}}}$ on $\overline{B}$ is $\boldsymbol{\sigma}_2$ on the upper edge of $\overline{B}$, while for $\hat{r} > 2/3$ the minimizer is $\boldsymbol{\sigma}_1$ on the right edge of $\overline{B}$. ◁

Because we want to compute linking constants around critical points far from the origin, we next examine how they can change when crossing over $\mathcal{V}$. If B is an unbounded component of $\mathcal{M}_{\mathbb{R}}$ then we write $h(\sigma(B)) := -\infty$ and $\nu_{\sigma(B),\boldsymbol{\sigma}_*} := 0$ for any $\boldsymbol{\sigma}_* \in \mathsf{crit}$.

Lemma 4.7. *Fix $\boldsymbol{\sigma}_* \in \mathsf{crit}$ and let B, B' be two components of $\mathcal{M}_{\mathbb{R}}$ in the same orthant as $\boldsymbol{\sigma}_*$. Suppose there B and B' are separated by a unique hyperplane $\mathcal{V}_j$ not containing $\boldsymbol{\sigma}_*$ (so that there is a unique index j such that ℓ_j is positive on B and negative on B', or vice versa, and that $\ell_j(\boldsymbol{\sigma}_*) \neq 0$). Then $\nu_{\sigma(B),\boldsymbol{\sigma}_*} = \nu_{\sigma(B'),\boldsymbol{\sigma}_*}$.*

Proof. Assume, without loss of generality, that B and B' lie in the positive orthant. If both components are unbounded then $\nu_{\sigma(B),\boldsymbol{\sigma}_*} = \nu_{\sigma(B'),\boldsymbol{\sigma}_*} = 0$, and we are done. Otherwise, one of $\sigma(B)$ and $\sigma(B')$ is finite and must lie on $\mathcal{V}_j$: if not then by strict convexity of h the line segment between them intersects $\mathcal{V}_j$ at a point of lower height than $\sigma(B)$ and $\sigma(B')$, contradicting Proposition 4.1.

We may therefore assume that $\ell_j(\sigma(B)) = 0$, and write

$$\mathcal{C}_B - \mathcal{C}_{B'} = \boldsymbol{\tau}_{\sigma(B)} + \sum_{B'' \in \mathrm{Adj}\,(\sigma(B)) \setminus \{B, B'\}} \mathrm{sgn}_{\sigma}(B'')\mathcal{C}_{B''}\,. \tag{11}$$

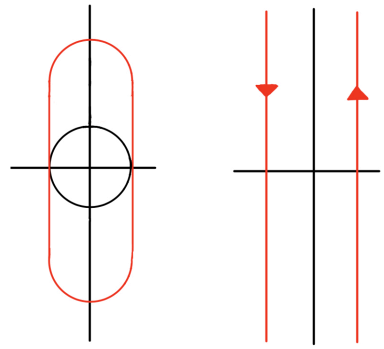

Figure 6: Deforming the domain of integration from circle to the difference of two imaginary fibers: stretching the grey circle in the imaginary direction, on any bounded region the chain converges locally to the red difference of fibers.

Grouping the elements of $\text{Adj}\,(\sigma(B))$ into pairs of components (β, β') separated only by $\mathcal{V}_j$, we note that $\mathcal{C}_\beta$ and $\mathcal{C}_{\beta'}$ appear with opposite signs on the right-hand side of (11), and that both $\sigma(\beta)$ and $\sigma(\beta')$ have height less than $\sigma(B)$. The result then follows by induction on the maximum heights of $\sigma(B)$ and $\sigma(B')$: if $\sigma(B)$ and $\sigma(B')$ have height $-\infty$ then $\mathcal{C}_B \doteq \mathcal{C}_B \doteq 0$, otherwise we have decomposed them as a sum of pairs (β, β') with smaller maximum height whose contributions to (11) cancel. $\square$

The last preliminary step is to we express the torus $\mathcal{T}$ from the original Cauchy integral in the fiber basis. Let $\text{Adj}\,(\mathbf{0})$ denote the components of $\mathcal{M}_\mathbb{R}$ adjacent to the origin and define

$$\boldsymbol{\tau}_\mathbf{0} := \sum_{B \in \text{Adj}\,(\mathbf{0})} \text{sgn}(\mathcal{O}) \mathcal{C}_B \,,$$

where $\text{sgn}(\mathcal{O})$ is 1 for the positive orthant and alternates across coordinate planes. This was depicted by the gray fibers in Figure 4.

Proposition 4.8. *If $\mathcal{T}$ is the chain of integration in (3) then $\mathcal{T} \doteq \boldsymbol{\tau}_\mathbf{0}$.*

Proof. The chain $\mathcal{T}$ is parametrized by a map η from the standard torus $(\mathbb{R}/(2\pi\mathbb{Z}))^d$ defined by

$$\eta(\boldsymbol{\theta}) := (\varepsilon \cos\theta_1 + i\delta \sin\theta_1, \ldots, \varepsilon \cos\theta_d + i\delta \sin\theta_d)$$

with $\delta = \varepsilon$ arbitrarily small. Increasing δ to infinity, we note that the chain remains in $\mathcal{M}$ and that the intersection with $\mathcal{M}_{\geq h_{\min}}$ converges to the alternating sum of imaginary fibers $\boldsymbol{\tau}_\mathbf{0}$. The one-dimensional story is illustrated in Figure 6. $\square$

The following result now immediately implies Theorem 3.13.

Theorem 4.9.

$$\boldsymbol{\tau}_\mathbf{0} = \sum_{\boldsymbol{\sigma} \in \mathsf{contrib}} \boldsymbol{\tau}_{\boldsymbol{\sigma}} .$$

Proof. Let $\boldsymbol{\sigma}^*$ be a critical point in the positive orthant $\mathcal{O}$ and B be the unique component of $\mathcal{M}_\mathbb{R}$ in $\mathcal{O}$ whose closure contains the origin. The hyperplanes through $\boldsymbol{\sigma}_*$ divide $\mathcal{O}$ into regions on which $\nu_{\cdot,\boldsymbol{\sigma}_*}$ is constant by Lemma 4.7, taking the common value 1 on the region containing $B(\boldsymbol{\sigma}_*)$ and vanishing on the other regions (as their critical points minimizing h have smaller height than $\boldsymbol{\sigma}_*$). Proposition 4.5 implies that the origin is in the closure of the region containing $B(\boldsymbol{\sigma}^*)$ if and only if $\boldsymbol{\sigma}_*$ is contributing, so $\nu_{\sigma(B),\boldsymbol{\sigma}_*} = 1$ if $\boldsymbol{\sigma}_* \in \mathsf{contrib}$ and zero otherwise. A similar argument proves the result for the other orthants of $\mathbb{R}^d$. $\square$

4.3 Integrals over linking tori

Throughout this section, the assumption that $F(\mathbf{z}) = G(\mathbf{z})/\prod \ell_j(\mathbf{z})^{p_j}$ is simple and $\mathbf{r}$ is generic remain in force. If G is a polynomial, Proposition 3.7 and Theorem 3.13 imply that when F is a rational function,

$$[\mathbf{z}^{n\mathbf{r}}]F(\mathbf{z}) = \frac{1}{(2\pi i)^d} \sum_{\boldsymbol{\sigma} \in \mathsf{contrib}} \int_{\boldsymbol{\tau}_{\boldsymbol{\sigma}}} \frac{F(\mathbf{z})}{\mathbf{z}^{\mathbf{r}+\mathbf{1}}} \, d\mathbf{z} \, . \tag{12}$$

More generally, when G is analytic, there is an extra term on the right decaying faster than any exponential $e^{-c|\mathbf{r}|}$. In any case, we see that the problem is reduced to evaluating Cauchy integrals over linking tori, which has been solved, for instance, in [33] and [34, Section 10.3].

In order to provide some intuition, before quoting the general result we look at a special case. Suppose that $\boldsymbol{\sigma}$ forms a zero-dimensional stratum, that is, a single point where d of the m hyperplanes intersect. The linking torus is then homeomorphic to an actual torus (cf. the end of Section 3.1, there is no tube). In coordinates given by the linear functions defining $\boldsymbol{\sigma}$, the linking torus is the product of circles winding once counterclockwise about the origin in each coordinate. Taking d iterated residues (or, equivalently, a d-fold residue) one arrives at Proposition 4.11 below.

Definition 4.10 (logarithmic gradients). The ***logarithmic gradient*** of any differentiable function f is

$$\nabla_{\log} f(\mathbf{z}) := \left(z_1 \frac{\partial f}{\partial z_1}, \cdots, z_d \frac{\partial f}{\partial z_d} \right) . \tag{13}$$

If $\boldsymbol{\sigma}$ is a critical point on the stratum defined by the intersection of $s \leq d$ hyperplanes $\mathcal{V}_{k_1}, \ldots, \ell_{k_s}$ then the ***log-gradient matrix*** $\Gamma = \Gamma(\boldsymbol{\sigma})$ is the $d \times d$ matrix whose first k rows are $\{\nabla_{\log} \ell_{k_j}(\boldsymbol{\sigma}) : 1 \leq j \leq s\}$ and whose last $d - k$ rows are any $d - k$ standard basis vectors that complete these logarithmic gradients to a positively oriented basis for $\mathbb{R}^d$.

Proposition 4.11 ([34, Theorem 10.3.3]). *Suppose that $\boldsymbol{\sigma}$ has codimension d and let Γ be the log-gradient matrix as in Definition 4.10. If G is a polynomial then*

$$\frac{1}{(2\pi i)^d} \int_{\boldsymbol{\tau}_{\boldsymbol{\sigma}}} \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z}) \, d\mathbf{z} = \frac{\boldsymbol{\sigma}^{-\mathbf{r}}}{(\mathbf{p}-1)!} \frac{G(\boldsymbol{\sigma})}{|\det \Gamma|} (\mathbf{r}\Gamma^{-1})^{\mathbf{p}-\mathbf{1}} \, .$$

When G is analytic, rather than a polynomial, then there is a remainder term decaying faster than an exponential function in $|\mathbf{r}|$. When all powers are 1, the right-hand side simplifies to $\boldsymbol{\sigma}^{-\mathbf{r}} G(\boldsymbol{\sigma})/|\det \Gamma|$. ☐

Example 4.12 (d-fold residue). Continuing Example 4.6, suppose $\hat{r}_1 \in (1/3, 2/3)$ so that $\boldsymbol{\sigma}_{1,2} = (1,1) \in \mathsf{contrib}$. Then $\Gamma(\boldsymbol{\sigma}_{1,2}) = -\begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$ and $\mathbf{p} = \mathbf{1}$, so

$$\frac{1}{(2\pi i)^d} \int_{\boldsymbol{\tau}_{\boldsymbol{\sigma}}} \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z}) \, d\mathbf{z} = \frac{1^n}{|\det \Gamma|} = 3 \, .$$

◁

In general, any linking torus can be expressed as the product of an s-torus with a $(d-s)$-dimensional imaginary fiber. Under our current assumptions, we can thus always compute s residues and be left with a $(d-s)$-dimensional integral in stationary phase. The following result shows that h is well-behaved around its critical points, so that we will be able to compute the stationary phase integral.

Proposition 4.13 (h is Morse). *Let $\mathcal{V}$ be the complexification of any real hyperplane arrangement and let $\boldsymbol{\sigma} \in \mathcal{V}_*$ be a critical point of $h|_{\mathcal{S}}$ for any stratum $\mathcal{S}$ of any dimension k. Then the Hessian matrix of $h|_{\mathcal{S}}$ at $\boldsymbol{\sigma}$ in any local k-dimensional complex coordinates is nonsingular.*

Proof. Because $\boldsymbol{\sigma}$ is a critical point the differential dh vanishes on $\mathcal{S}$ at $\boldsymbol{\sigma}$, so $h(\boldsymbol{\sigma}+\mathbf{z})$ is locally a quadratic form in $\mathbf{z}$. If $\mathbf{x}$ is any real vector then every coordinate of $\mathbf{x}+i\mathbf{y}$ has increasing modulus as any coordinate of the real vector $\mathbf{y}$ moves away from the origin. Thus, if $\mathcal{S}_{\mathbb{R}}$ denotes the real part of the stratum $\mathcal{S}$ then $h_{\mathbf{r}}(\mathbf{z}) = -\sum_j r_j \log|z_j|$ has a local maximum at $\mathbf{z}=\boldsymbol{\sigma}$ on the space $\boldsymbol{\sigma}+i\mathcal{S}_{\mathbb{R}}$ with real dimension k and a local minimum at $\mathbf{z}=\boldsymbol{\sigma}$ on the space $\boldsymbol{\sigma}+\mathcal{S}_{\mathbb{R}}$ with real dimension k. In other words, $h(\boldsymbol{\sigma}+\mathbf{z})$ is Morse at the origin of $\mathcal{S}\oplus i\,\mathcal{S}$ with middle index. ☐

Proposition 4.13 implies quadratic nondegeneracy of the phase function for the stationary phase integral, allowing it to be evaluated as a standard saddle point integral.

Proposition 4.14 ([34, Theorem 10.3.4])**.** *Let $S=\{k_1,\dots,k_s\}$ and suppose $\boldsymbol{\sigma}$ lies on the flat $\mathcal{V}_S$. Let Γ be the log-gradient matrix in Definition 4.10 and M be the $d\times d$ matrix whose first k rows are the coefficient vectors $\{\mathbf{b}^{(k_j)} : 1\le j\le k\}$ and whose last $d-k$ rows are any $d-k$ standard basis vectors that complete these gradients to a positively oriented basis for $\mathbb{R}^d$. Define the $(d-k)\times(d-k)$ matrix $\mathcal{H}$ to be the Hessian of*

$$\phi(\mathbf{y}) = \mathbf{r}\cdot\log\left(\boldsymbol{\sigma} - iM^{-1}\binom{\mathbf{0}}{\mathbf{y}}\right)$$

evaluated at $\mathbf{y}=(y_1,\dots,y_{d-k})=\mathbf{0}$, where the logarithm is taken coordinate-wise. Then there is an explicitly computable asymptotic series in $|\mathbf{r}|=|r_1|+\cdots+|r_d|$ beginning

$$\frac{1}{(2\pi i)^d}\int_{\boldsymbol{\tau}_{\boldsymbol{\sigma}}} \mathbf{z}^{-\mathbf{r}-\mathbf{1}}F(\mathbf{z})\,d\mathbf{z} = \left[\frac{\boldsymbol{\sigma}^{-\mathbf{r}}G(\boldsymbol{\sigma})}{\prod_{j\notin S}\ell_j(\boldsymbol{\sigma})^{p_j}\prod_{1\le j\le s}(p_{k_j}-1)!\sqrt{\det\mathcal{H}}\;|\det\Gamma|}\right](2\pi|\mathbf{r}|)^{-(d-k)/2}(\mathbf{r}\Gamma^{-1})^{\mathbf{p}-1}$$
$$\times\left(1+O\left(\frac{1}{|\mathbf{r}|}\right)\right).$$

All asymptotic terms in this series are uniform as $\boldsymbol{\sigma}$ varies over a compact subset of S. ☐

Remark 4.15. Raichev and Wilson [36] give explicit formulas for the coefficients in the asymptotic expansion in Proposition 4.14, building on expansions of Hörmander [18, Theorem 7.7.5]; see also Melczer [24, Proposition 5.3].

We summarize the results of this Section as follows.

Theorem 4.16. *Suppose $F(\mathbf{z})$ is simple and $\mathbf{r}$ is a generic direction, and let* $\mathsf{contrib}$ *denote the set of contributing singularities for $F(\mathbf{z})$. Then as $\mathbf{r}\to\infty$ with $\hat{\mathbf{r}}$ staying in a compact subset of $\mathbb{R}^d_{>0}$ consisting of only generic directions there exist asymptotic series $\Phi_{\boldsymbol{\sigma}}(\mathbf{r})$ such that*

$$[\mathbf{z}^{\mathbf{r}}]F(\mathbf{z}) = \sum_{\boldsymbol{\sigma}\in\mathsf{contrib}(\hat{\mathbf{r}})}\Phi_{\boldsymbol{\sigma}}(\mathbf{r}). \tag{14}$$

If $\boldsymbol{\sigma}$ lies on the flat $\mathcal{V}_S$ with $S=\{k_1,\dots,k_s\}$ then, for any positive integer K, there exist effective constants $C_j^{\boldsymbol{\sigma}}$ such that

$$\Phi_{\boldsymbol{\sigma}}(\mathbf{r}) = \boldsymbol{\sigma}^{-\mathbf{r}}\,|\mathbf{r}|^{p_{k_1}+\cdots+p_{k_s}-(s+d)/2}\left(\sum_{j=0}^{K}C_j^{\boldsymbol{\sigma}}|\mathbf{r}|^{-j} + O\left(|\mathbf{r}|^{-K-1}\right)\right).$$

If $G(\boldsymbol{\sigma})\neq 0$ then the leading asymptotic term of $\Phi_{\boldsymbol{\sigma}}$ is given by Proposition 4.14. The error term varies uniformly as $\hat{\mathbf{r}}$ varies without crossing nongeneric directions and $\boldsymbol{\sigma}$ varies over a compact subset of any stratum.

Remark 4.17. When $\boldsymbol{\sigma}$ has dimension zero and G is a polynomial then $\Phi_{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^{-n\mathbf{r}}P_{\boldsymbol{\sigma}}(n)$, where $P_{\boldsymbol{\sigma}}(n)$ is a polynomial in n of degree $p_{k_1}+\cdots+p_{k_d}-d$ which can be determined exactly. The asymptotic expansion in deceasing powers of $|\mathbf{r}|$ has no further terms and the remainder is exponentially decreasing in $|\mathbf{r}|$. This phenomenon, that asymptotic behaviour can be determined in this case up to an exponentially small remainder, was previously noted by Pemantle [35].

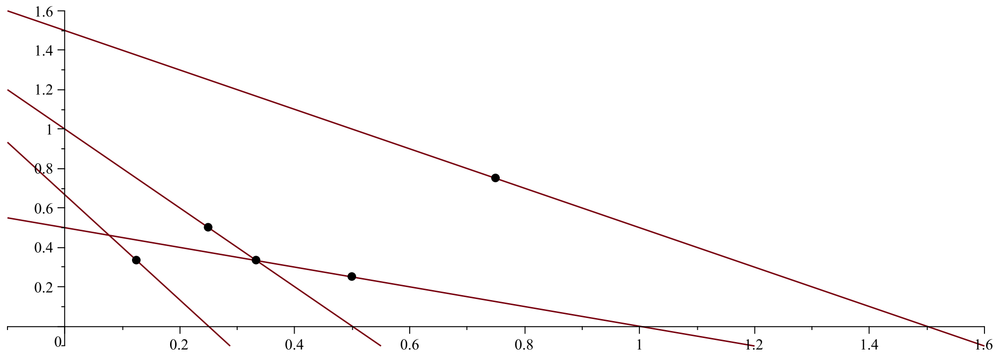

Figure 7: Real singularities and contributing points of the rational function in Example 4.18. No singularities outside the first quadrant are contributing.

4.4 Implementation and Additional Examples

A Maple implementation of Theorem 4.16 via Algorithm 1 is available at

https://github.com/ACSVMath/ACSVHyperplane

and allows the user to easily derive asymptotics for a generic direction in the non-simple case. We illustrate a few additional examples here, pointing out the computer algebra issues which arise.

Example 4.18. Consider asymptotics in the main diagonal direction $\mathbf{r} = (n, n)$ for the rational function

$$F(x, y) = \frac{1}{(1 - 2x - y)(1 - x - 2y)(1 - 4x - 3y/2)(1 - 2x/3 - 2y/3)},$$

whose real singularities and contributing points are illustrated in Figure 7. The algorithm detects 5 contributing points: note that one of the common intersections of two factors is contributing while the other intersections (one in the first quadrant and further intersections in the other quadrants) are not contributing. The algorithm then computes asymptotic contributions with leading terms

$$\frac{64 \cdot 8^n}{11\sqrt{n\pi}}, \quad \frac{9^n}{3}, \quad \frac{32 \cdot 8^n}{3\sqrt{n\pi}}, \quad \frac{10368 \cdot 24^n}{625\sqrt{n\pi}}, \quad \frac{-128 \cdot (16/9)^n}{625\sqrt{n\pi}},$$

giving dominant asymptotics

$$[x^n y^n]F(x, y) = \frac{10368 \cdot 24^n}{625\sqrt{n\pi}} \left(1 + O\left(\frac{1}{n}\right)\right).$$

◁

Example 4.19. Our previous examples have been restricted to dimension 2, and admitted contributing points in the first quadrant only, in order to guide intuition and allow for visualization. In order to illustrate our results in a more general setting we consider the rational function

$$F(x, y, z, w) = \frac{1}{(1 + 2x + y + z + w)(1 - x - 3y - z - w)(1 - x + y - 4z + w)(1 - x - y + z - 5w)}$$

in the direction $\mathbf{r} = (n, 2n, n, 2n)$. Here there are 20 contributing singularities in multiple orthants, many flats have contributing singularities in multiple quadrants, and several contributing points have irrational coordinates. We use the symbolic-numeric methods of Melczer and Salvy [25] to store the coordinates of the contributing points: for each contributing point $\boldsymbol{\sigma}$ the algorithm outputs an algebraic number α defined by a square-free integer polynomial $P(u)$ and isolating interval, together with integer polynomials $Q_j(u)$ for each coordinate, such that the jth coordinate $\sigma_j = Q_j(\alpha)/P'(\alpha)$.

For example, the flat defined by

$$0 = 1+2x+y+z+w = 1-x-3y-z-w = 1-x+y-4z+w = 1-x-y+z-5w$$

contains three critical points, two of which are contributing points with coordinates

$$\left\{\left(\frac{-3w-1}{2}, \frac{3-3w}{4}, \frac{11w-3}{4}\right) : 99w^3 - 61w^2 - 9w + 3 = 0,\ w \approx .1791\ldots \text{ or } .6843\ldots\right\}$$

containing algebraic numbers of degree three (the final algebraic conjugate is critical but not contributing). The algorithm encodes these numbers as

$$(x,y,z,w) = \left(\frac{Q_x(u)}{P'(u)}, \frac{Q_y(u)}{P'(u)}, \frac{Q_z(u)}{P'(u)}, \frac{Q_w(u)}{P'(u)}\right)$$

evaluated at the roots of $P(u) = 2970u^3 - 282927u^2 + 8961876u - 94409378$ of value approximately 29.731 and 34.86, where

$$Q_x(u) = -7200u^2 + 459750u - 7326354 \qquad Q_y(u) = 5310u^2 - 335979u + 5298699$$
$$Q_z(u) = -1650u^2 + 100215u - 1504811 \qquad Q_w(u) = 1830u^2 - 117882u + 1896944.$$

The roots of u are given to sufficient precision to separate them among those of P. The algorithm also computes the constants giving $-(\nabla h_{\mathbf{r}})(\mathbf{z})$ as a linear combination of the coefficient vectors $\mathbf{b}^{(j)}$, and determines u with enough accuracy to decide whether all these constants are positive (in which case the critical point is contributing) or whether any are zero (in which case $\mathbf{r}$ is a non- generic direction).

There is some randomness in selecting the polynomial $P(u)$; see Melczer and Salvy [25] for the advantages of this representation and how to compute it, and the Maple worksheet accompanying this paper for more details on this example. Computing the asymptotic contributions of each contributing point here gives dominant asymptotics of the form

$$[x^n y^n z^n w^n]F(x,y) = \frac{C}{\pi}\left(\frac{-11072781\sqrt{249} - 54475983}{3125}\right)^n n^{-1}\left(1 + O\left(\frac{1}{n}\right)\right),$$

where $C = -0.109\ldots$ is an explicit degree 4 algebraic number. ◁

4.5 Vanishing of Leading Coefficients

To find dominant asymptotics of $[\mathbf{z}^{n\hat{\mathbf{r}}}]F(\mathbf{z})$ using Theorem 4.16, one starts with a contributing singularity $\boldsymbol{\sigma}$ of largest height and tries to find the first constant $C_j^{\boldsymbol{\sigma}}$ which is non-zero, repeating the process until all maximum height contributing singularities have been examined. If the right-hand side of (14) is non-zero, such a non-zero constant $C_j^{\boldsymbol{\sigma}}$ exists and will eventually be found. When there is a unique contributing point $\boldsymbol{\sigma}$ of maximal height with $G(\boldsymbol{\sigma}) \neq 0$ then this gives dominant asymptotic behaviour. If, however, there is more than one such contributing point then it is possible that their sum will sometimes be of smaller order: for instance, if $F(z) = 1/(1-z^2)$ our algorithm will correctly return $[z^n]F(z) = 1 + (-1)^n$. For generic directions $\mathbf{r}$ the leading term in our asymptotic statement will be non-zero.

A more serious difficulty occurs when all $C_j^{\boldsymbol{\sigma}}$ vanish at all contributing points $\boldsymbol{\sigma}$ of largest height. This implies an exponentially smaller estimate $\Phi_{\boldsymbol{\rho}}$ holds, where $\boldsymbol{\rho}$ is a critical point of height less than $\boldsymbol{\sigma}$. We give additional details on this in Theorem 4.27 below, after some examples illustrating the possible outcomes.

Example 4.20. Consider the rational functions

$$A(x,y) = \frac{1-x-y}{1-x-y}, \quad B(x,y) = \frac{1}{1-x-y}, \quad C(x,y) = \frac{x-2y^2}{1-x-y}, \quad \text{and} \quad D(x,y) = \frac{x-y}{1-x-y}.$$

These functions share the same denominator, and all appear to admit the single contributing singularity $\boldsymbol{\sigma} = (1/2, 1/2)$. Of course, A only appears to admit this point because it is not simplified in lowest terms (simplified it is the constant 1) but $\boldsymbol{\sigma}$ is a contributing point of B, C, and D. Applying Theorem 4.16 to the main diagonal of B gives

$$[x^n y^n]B(x,y) = 4^n \left(\frac{1}{\sqrt{\pi} n^{1/2}} + O\left(n^{-3/2}\right) \right).$$

Because the numerator of $C(x,y)$ vanishes at $\boldsymbol{\sigma}$ we determine dominant asymptotic behaviour of its main diagonal by computing higher order constants, ultimately obtaining

$$[x^n y^n]C(x,y) = 4^n \left(\frac{1}{4\sqrt{\pi} n^{3/2}} + O\left(n^{-5/2}\right) \right).$$

Finally, direct inspection shows that the main diagonal of D is identically zero, but applying Theorem 4.16 in an automatic manner only allows us to show for any $K > 0$ that

$$[x^n y^n]D(x,y) = O\left(\frac{4^n}{n^K} \right).$$

Note that for any $a > 0$ we have

$$[x^{an} y^n]D(x,y) = \left(\frac{(a+1)^{a+1}}{a^a} \right)^n \left(\frac{a-1}{\sqrt{2\pi a(a+1)n}} + O\left(n^{-3/2}\right) \right),$$

so that the exponential growth of $[x^{an}y^n]C(x,y)$ approaches 4^n as $a \to 1$. ◁

Remark 4.21. Such unexpected exponential drops are related to the *connection problem* from enumerative combinatorics: given a sequence satisfying a linear recurrence relation with polynomial coefficients—including any diagonal sequence $[\mathbf{z}^{n\mathbf{r}}]F(\mathbf{z})$ with F rational—one can compute a finite asymptotic basis of functions such that dominant asymptotics of the sequence under consideration is a $\mathbb{C}$-linear combination of basis elements. The real coefficients in such a linear combination of basis elements can be determined rigorously to any desired numerical accuracy [38, 27], however it is still unknown whether it is decidable to determine which constants are exactly zero (see also Section VII.9 of Flajolet and Sedgewick [13]).

When $\boldsymbol{\sigma}$ forms a zero-dimensional stratum, things are better. The degree of the polynomial produced by Proposition 4.11 is bounded above by the sum of $(p_j - 1)$ over those j such that $\ell_j(\boldsymbol{\sigma}) = 0$. It follows that testing $\Phi_{\boldsymbol{\sigma}} = 0$ can be done rigorously by computing a sufficient number of effective coefficients. When the contribution of all maximal height contributing singularities is known to vanish, one can repeat this effective process on the contributing singularities of next highest height, and so on. We now give an example where the highest contributing critical point $\boldsymbol{\sigma}$ is of dimension zero and it is indeed necessary to go to the next lower critical point.

Example 4.22. Let

$$F(x,y) = \frac{x-y}{\left(1 - \frac{2x+y}{3}\right)\left(1 - \frac{x+2y}{3}\right)}$$

and consider asymptotics in the direction $(r,s) = n(\alpha, 1-\alpha)$ for $\alpha = \hat{r}_1 \in (2/3, 3/4)$. This rational function has the same denominator the one in Example 3.6, so all critical points, contributing singularities, etc. are the same. In particular, the point $\boldsymbol{\sigma}_{1,2} = (1,1)$ is a contributing singularity of maximal height. This time the contribution at $\Phi_{\boldsymbol{\sigma}_{1,2}}$ is identically zero: this is easy to discover because our degree bound implies the leading asymptotic term is a polynomial of degree zero, and one need only evaluate the first coefficient.

Alternatively, one may verify that the numerator $x - y$ is in a specific ideal described below. Dominant asymptotics of $a_{r,s}$ are thus determined by $\Phi_{\boldsymbol{\sigma}_1}$ and $\Phi_{\boldsymbol{\sigma}_2}$, where $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the one-dimensional critical points

$$\boldsymbol{\sigma}_1 = \left(\frac{3\alpha}{2}, 3(1-\alpha)\right) \qquad \text{and} \qquad \boldsymbol{\sigma}_2 = \left(3\alpha, \frac{3(1-\alpha)}{2}\right).$$

Exponentiating the heights of $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ gives the values

$$h_1 = \frac{1}{3(1-\alpha)}\left(\frac{2(1-\alpha)}{\alpha}\right)^{\alpha} \qquad\qquad h_2 = \frac{2}{3(1-\alpha)}\left(\frac{1-\alpha}{2\alpha}\right)^{\alpha},$$

and basic calculus shows

$$\begin{cases} h_1 < h_2 & : \alpha \in (1/3, 1/2) \\ h_1 = h_2 & : \alpha = 1/2 \\ h_1 > h_2 & : \alpha \in (1/2, 2/3) \end{cases}.$$

Thus, Theorem 4.16 implies

$$a_{r,s} = \left(\frac{1}{3\alpha}\right)^r \left(\frac{2}{3(1-\alpha)}\right)^s (r+s)^{-1/2} \left(\frac{-3\sqrt{2}}{2\sqrt{(1-\alpha)\alpha\pi}} + O\left((r+s)^{-1}\right)\right)$$

for $\alpha \in (1/3, 1/2)$, while

$$a_{r,s} = \left(\frac{2}{3\alpha}\right)^r \left(\frac{1}{3(1-\alpha)}\right)^s (r+s)^{-1/2} \left(\frac{3\sqrt{2}}{2\sqrt{(1-\alpha)\alpha\pi}} + O\left((r+s)^{-1}\right)\right)$$

for $\alpha \in (1/2, 1/3)$. When $\alpha = 1/2$ dominant asymptotic behaviour is given by the sum of these contributions, which cancel, reflecting the fact that the main diagonal of $F(x, y)$ is zero by symmetry. $\triangleleft$

Exponential growth rates and the annihilating ideal

As we have seen, asymptotic growth of a coefficient sequence can be lower than that predicted by the highest height contributing points for some fixed direction. For this reason, it is useful to introduce two different directional exponential rates.

Definition 4.23. The (limsup) ***exponential rate*** of a sequence $a_{\mathbf{s}}$ in the direction $\mathbf{r}$ is the quantity

$$\beta(\mathbf{r}) := \limsup_{n\to\infty} \frac{1}{n} \log |a_{n\hat{\mathbf{r}}}|,$$

while the (limsup) ***neighborhood exponential rate*** is

$$\overline{\beta}(\mathbf{r}) := \limsup_{\substack{\mathbf{s}\to\infty \\ \mathbf{s}/|\mathbf{s}|\to\hat{\mathbf{r}}}} \frac{1}{|\mathbf{s}|} \log |a_{\mathbf{s}}|.$$

The (neighbourhood) exponential growth rate of a generating function is the (neighbourhood) exponential growth rate of its coefficient sequence.

Using the limsup helps smooth behaviour because many combinatorial sequences have periodicity, either exact with period k (e.g., terms vanish for certain values of n mod k) or via a phase term such as a factor of $\cos(n\theta\pi)$ for some, possibly irrational, θ. The reason to look at neighborhood growth rates is explained by Example 4.20: for all the functions but the trivial case of $A(x, y) = 1$ we saw that

$$\overline{\beta}(1, 1) = h_{1/2,1/2}(\boldsymbol{\sigma}) = \log 4,$$

while the behavior precisely on the diagonal has a polynomial drop (for C) or is identically zero (for D).

Lemma 4.24. *The neighborhood exponential rate does not increase when multiplying $F(\mathbf{z})$ by any Laurent polynomial $\kappa(\mathbf{z})$.*

Proof. Multiplication by κ translates into the convolution of the series F with the Laurent polynomial κ. The claim follows as κ has finite support. $\square$

Remark 4.25. The (non-neighbourhood) exponential rate of a sequence can change both increase and decrease when multiplying its generating function by a polynomial. For instance,

$$F(x,y) = \frac{1}{1-x-y} \implies \beta(1,1) = \log 2$$

$$F(x,y) = \frac{x-y}{1-x-y} \implies \beta(1,1) = 0$$

$$F(x,y) = \frac{x(x-y)}{1-x-y} \implies \beta(1,1) = \log 2.$$

On a heuristic level, what is going on is that algebraic and analytic techniques can detect the neighborhood rate better than they can detect delicate behaviors such as cancellation among sums of powers of algebraic numbers. Algebraic cancellation[3] can be detected using the following definition.

Definition 4.26 (annihilating ideal)**.** The ***annihilating ideal*** of a flat $\mathcal{V}_{k_1,\dots,k_s}$ is the polynomial ideal

$$\mathcal{J}(L) := \langle \ell_{k_j}(\mathbf{z})^{p_{k_j}} : j = 1, \dots, s \rangle \tag{15}$$

and the ***annihilating ideal*** of a point $\boldsymbol{\sigma}$ on the stratum $\mathcal{S}_{k_1,\dots,k_s}$ is $\mathcal{J}(\boldsymbol{\sigma}) := \mathcal{J}(\mathcal{V}_{k_1,\dots,k_s})$.

The annihilating ideal contains precisely the functions G such that the singular set of $\frac{G(\mathbf{z})}{\prod_{j=1}^m \ell_j(\mathbf{z})^{p_j}}$ does not contain $\mathcal{V}_{k_1,\dots,k_s}$. Indeed, if $G(\mathbf{z}) \in \mathcal{J}(\boldsymbol{\sigma})$ then $G(\mathbf{z})$ can be rewritten as a polynomial linear combination of the $\ell_{k_j}(\mathbf{z})^{p_{k_j}}$, meaning that F can be decomposed as a sum of meromorphic functions whose poles do not contain L. For polynomials, the condition $G \in \mathcal{J}(\boldsymbol{\sigma})$ can be verified automatically by, for instance, a Gröbner basis computation.

Theorem 4.27. *Consider a simple arrangement in a generic direction* $\mathbf{r}$*, let* $\boldsymbol{\sigma}$ *be a point in a stratum* S *of codimension* s *defined by the vanishing of the linear functions* $\ell_{k_1}, \dots, \ell_{k_s}$*, and let* $\mathcal{J}(\boldsymbol{\sigma})$ *be the annihilating ideal.*

(i) If $G \in \mathcal{J}(\boldsymbol{\sigma})$ *then the asymptotic series* $\Phi_{\boldsymbol{\sigma}}(\mathbf{r})$ *defined in Theorem 4.16 has exponential growth strictly less than* $h_{\mathbf{r}}(\boldsymbol{\sigma})$*. Hence, if all other critical points* $\boldsymbol{\sigma}'$ *with* $h_{\hat{\mathbf{r}}}(\boldsymbol{\sigma}') \geq h_{\hat{\mathbf{r}}}(\boldsymbol{\sigma})$ *are ruled out by Algorithm 1 then*

$$\overline{\beta}(\hat{\mathbf{r}}) < h_{\hat{\mathbf{r}}}(\boldsymbol{\sigma}).$$

(ii) If $G \notin \mathcal{J}(\boldsymbol{\sigma})$ *then* $\Phi_{\boldsymbol{\sigma}}(\mathbf{r})$ *has neighbourhood exponential growth at least* $h_{\mathbf{r}}(\boldsymbol{\sigma})$*. In other words, if all critical points* $\boldsymbol{\sigma}'$ *with* $h_{\hat{\mathbf{r}}}(\boldsymbol{\sigma}') > h_{\hat{\mathbf{r}}}(\boldsymbol{\sigma})$ *are ruled out by Algorithm 1 then*

$$\overline{\beta}(\hat{\mathbf{r}}) = h_{\hat{\mathbf{r}}}(\boldsymbol{\sigma}).$$

[3] Topological cancellation, which can also cause a drop in the limsup neighborhood exponential rate, is harder to study. See [2] for one example of this.

Proof. First we prove (i). If $G = \sum_{j=1}^s f_j \ell_j^{p_j}$ where the f_j are locally analytic functions on a neighborhood of σ then we can write F as a sum of meromorphic functions whose pole sets no longer contain the stratum S. In generic directions this implies that $\Phi_{\boldsymbol{\sigma}}(\mathbf{r})$ can be written as a sum of integrals over imaginary fibers with basepoints arbitrarily close to critical points of lower height than $h_{\mathbf{r}}(\boldsymbol{\sigma})$, giving the stated result.

To prove (ii) we make an analytic change of coordinates $\mathbf{z} = (\mathbf{u}, \mathbf{v})$ by setting $v_j = \ell_{k_j}(\mathbf{z})$ for $j = 1, \dots, s$ and letting $\mathbf{u}$ be $d - s$ remaining coordinates parametrizing the flat L containing S near $\mathbf{s}$. We can rewrite G in these coordinates as

$$G = \sum_{\mathbf{m} \in \mathbb{N}^s : m_j < p_j} c_{\mathbf{m}}(\mathbf{u}) \mathbf{v}^{\mathbf{m}} + \sum_{j=1}^{s} v_j^{p_j} \tilde{c}_j(\mathbf{u}, \mathbf{v}), \tag{16}$$

where each $c_{\mathbf{m}}(\mathbf{u})$ is an analytic function on L and each $\tilde{c}_j(\mathbf{u})$ is an analytic function in a vicinity of $\boldsymbol{\sigma}$ in $\mathbb{C}^d$.

Reasoning as in part (i), the contribution of each term in the second sum of (16) has an exponential rate strictly less than $h_{\hat{\mathbf{r}}}(\boldsymbol{\sigma})$, while the first sum of (16) identically vanishes if and only if G lies in the ideal $\mathcal{J}(\boldsymbol{\sigma})$.

Assume for now that $\mathbf{p} = \mathbf{1}$, so that the first sum in (16) contains only the term $c_{\mathbf{0}}(\mathbf{u})$. If $c_{\mathbf{0}}(\mathbf{0}) \neq 0$ then the leading term of $\Phi_{\boldsymbol{\sigma}}(\mathbf{r})$ given by Proposition 4.14 is nonzero, and we have found the correct exponential rate $h_{\hat{\mathbf{r}}}(\boldsymbol{\sigma})$. Similarly, $c_{\mathbf{0}}(\mathbf{u}) \neq 0$ for some point $\mathbf{u}$ in any sufficiently small neighbourhood of the origin then, because the non-degenerate critical point $\boldsymbol{\sigma}(\mathbf{r})$ on the stratum S varies smoothly with $\mathbf{r}$, the neighbourhood exponential rate will be $h_{\hat{\mathbf{r}}}(\boldsymbol{\sigma})$. The only other case occurs when $c_{\mathbf{0}}$ vanishes in a neighbourhood of zero, but then $c_{\mathbf{0}}$ is identically zero by analyticity, meaning $G \in J(\boldsymbol{\sigma})$.

Finally, let $\mathbf{p}$ be general and suppose the neighbourhood exponential rate of $\Phi_{\boldsymbol{\sigma}}(\mathbf{r})$ is less than $h_{\hat{\mathbf{r}}}(\boldsymbol{\sigma})$. Lemma 4.24 implies that multiplying $\Phi_{\boldsymbol{\sigma}}(\mathbf{r})$ by any Laurent polynomial $\mathbf{v}^{\mathbf{m}}$ does not increase its neighbourhood exponential growth. By choosing various $\mathbf{m} < \mathbf{p}$, and repeating the reasoning for $\mathbf{p} = \mathbf{1}$, we see that all $c_{\mathbf{m}} \equiv 0$, implying that G is in $\mathcal{J}$. ☐

5 Non-Simple Arrangements in Generic Directions

Any rational function whose singular set forms a non-simple hyperplane arrangement can be decomposed into a finite sum of rational functions, each of which has a singular set that defines a simple hyperplane arrangements. To make this explicit we use a known, canonical partial fraction expansion in terms of the no-broken-circuit basis for the matroid $\{\mathbf{b}^{(1)}, \dots, \mathbf{b}^{(k)}\}$ for each flat with some codimension ℓ and $k \geq \ell$ hyperplanes. We begin with an example.

Example 5.1. Consider $F(x, y) = 1/(\ell_1 \ell_2 \ell_3)$ as shown in Figure 8, where

$$\ell_1(x,y) = 1 - x/3 - 2y/3 \qquad \ell_2(x,y) = 1 - 2x/3 - y/3 \qquad \ell_3(x,y) = 1 - 3x/5 - 2y/5\,.$$

In the main diagonal direction $\mathbf{r} = (1, 1)$, F admits contributing points $(3/2, 3/4), (3/4, 3/2)$, and $(5/6, 5/4)$ on the flats $\mathcal{V}_1, \mathcal{V}_2$, and $\mathcal{V}_3$, respectively, and contributing point $(1, 1)$ which is the single point in all the flats $\mathcal{V}_{1,2}, \mathcal{V}_{1,3}, \mathcal{V}_{2,3}$, and $\mathcal{V}_{1,2,3}$. As the three lines ℓ_1, ℓ_2, and ℓ_3 in two dimensions have common intersection point $(1, 1)$, they are linearly dependent. Basic linear algebra allows one to derive

$$(1/5)\ell_1(x,y) + (4/5)\ell_2(x,y) = \ell_3(x,y)\,.$$

Dividing this equation by $\ell_3(x, y)$ gives

$$(1/5)\frac{\ell_1(x,y)}{\ell_3(x,y)} + (4/5)\frac{\ell_2(x,y)}{\ell_3(x,y)} = 1,$$

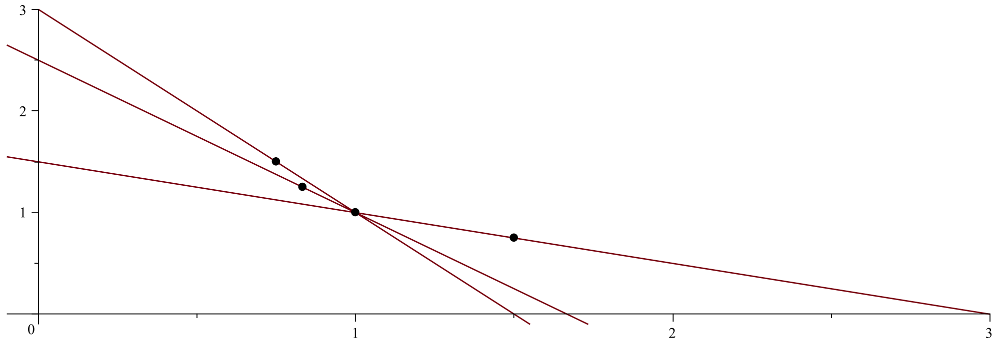

Figure 8: Real singularities and contributing points of the rational function in Example 5.1. All three factors intersect at the point $(1,1)$, meaning F is not simple.

so that

$$F(x,y) = \frac{1}{\ell_1(x,y)\ell_2(x,y)\ell_3(x,y)} = \underbrace{\frac{1/5}{\ell_2(x,y)\ell_3(x,y)^2}}_{F_1} + \underbrace{\frac{4/5}{\ell_1(x,y)\ell_3(x,y)^2}}_{F_2}.$$

Both $F_1(x,y)$ and $F_2(x,y)$ are simple, so we can apply Theorem 4.16 to obtain dominant asymptotics. For instance, in the direction $\mathbf{r} = (1,1)$ the function F_2 admits the point $(1,1)$ as its contributing singularity determining dominant asymptotics, while $(1,1)$ is not a contributing singularity of F_1. The contributing singularity of F_1 which determines dominant asymptotics is $(5/6, 5/4)$, meaning the diagonal coefficients of F_1 decay exponentially. Thus, Theorem 4.16 gives

$$\begin{aligned}
[x^n y^n]F(x,y) &= (1/5)[x^n y^n]F_1(x,y) + (4/5)[x^n y^n]F_2(x,y) \\
&= \frac{15n}{4}\left(1 + O\left(\frac{1}{n}\right)\right).
\end{aligned}$$

$\triangleleft$

To generalize this approach, we introduce some definitions. The **support** of a rational function is the set of divisors appearing in the denominator when in lowest terms. In other words,

$$\sup\left(\frac{G(\mathbf{z})}{\ell_1(\mathbf{z})^{q_1}\cdots\ell_m(\mathbf{z})^{q_m}}\right) := \{\ell_j(\mathbf{z}) : q_j > 0\}$$

when no ℓ_j divides G.

As in the theory of matroids, we call any minimal linearly dependent set $\{\ell_{i_1}, \dots, \ell_{i_s}\}$ a **circuit**. A **broken circuit** is the independent collection obtained from any circuit by removing the element ℓ_j with largest index j. A collection is said to be **χ-independent** if it does not contain a broken circuit; note that any χ-independent set is also linearly independent.

In Example 5.1 the set $\{\ell_1, \ell_2, \ell_3\}$ is the only circuit, so $\{\ell_1, \ell_2\}$ is the only broken circuit. Note that the supports of the two rational functions which we analyze, $\ell_1\ell_3^2$ and $\ell_2\ell_3^2$, correspond to χ-independent sets $\{\ell_1, \ell_3\}$ and $\{\ell_2, \ell_3\}$. This is no coincidence: Proposition 5.2 below shows that one can always decompose a rational function into a sum of rational functions whose supports are χ-independent, and the following shows in general one can make no further simplifications.

Proposition 5.2. *Let $\ell_1, \dots, \ell_m$ be any m linear functions of d variables.*

(i) The set of rational functions

$$\left\{ \frac{1}{\ell_{i_1}(\mathbf{z}) \cdots \ell_{i_s}(\mathbf{z})} : \{\ell_{i_1}, \ldots, \ell_{i_s}\} \textit{ is } \chi\textit{-independent} \right\}$$

is linearly independent over $\mathbb{C}$.

(ii) The span over $\mathbb{C}$ *of the rational functions*

$$\left\{ \frac{1}{\ell_{i_1}(\mathbf{z})^{p_1} \cdots \ell_{i_s}(\mathbf{z})^{p_s}} : \{\ell_{i_1}, \ldots, \ell_{i_s}\} \textit{ is } \chi\textit{-independent and } \sum_{i=1}^{s} p_i = M \right\}$$

contains the inverses of all products of the ℓ_j *over multisets of cardinality* M.

(iii) Algorithm 2 terminates in a finite number of steps and outputs a sum of rational functions whose supports contain no broken circuits.

Proof. The first conclusion follows from standard results on hyperplane arrangements; see Orlik and Terao [31, Theorems 3.43, 3.126, 5.89] or Pemantle and Wilson [34, Prop. 10.2.10].

To prove the second, given any linear dependence

$$0 = a_1 \ell_{i_1}(\mathbf{z}) + \cdots + a_s \ell_{i_s}(\mathbf{z})$$

with each $a_i \neq 0$, one can divide by any $a_j l_{i_j}(\mathbf{z})$ and then by some product of all $\ell_k(\mathbf{z})$ to obtain

$$\frac{1}{\boldsymbol{\ell}^{\mathbf{q}}} = \frac{1}{\ell_1(\mathbf{z})^{q_1} \cdots \ell_m(\mathbf{z})^{q_m}} = \sum_{k \neq j} \frac{(-a_k/a_j)}{\boldsymbol{\ell}^{\mathbf{q} + \mathbf{e}^{(i_j)} - \mathbf{e}^{(i_k)}}} \tag{17}$$

for any $\mathbf{q} \in \mathbb{N}^m$, where $\mathbf{e}^{(\kappa)}$ is the elementary basis vector with a one in the κth coordinate and all other coordinates zero.

Given any rational function $F(\mathbf{z}) = G(\mathbf{z})/\boldsymbol{\ell}^{\mathbf{p}}$ whose support contains a broken circuit $\{i_1, \ldots, i_s\}$ with indices in increasing order, one can apply the base exchange equation (17) with any $i_j > i_s$ such that $\{i_1, \ldots, i_s, i_j\}$ is linearly dependent; such an i_j exists by the definition of a broken circuit. At each step, every vector $\mathbf{q} + e_{i_j} - e_{i_k}$ on the right hand side of Equation (17) has smaller lexicographical order than $\mathbf{q}$. Therefore, repeating the process formalized in Algorithm 2, one must eventually arrive at an expression for F as a sum of rational functions whose supports contain no broken circuits. The second while loop terminates because any time a summand $\tilde{F}$ is further decomposed into several summands, the supports of these new rational functions are subsets of the support of $\tilde{F}$, and the new functions have denominators of smaller total degree than the denominator of $\tilde{F}$. This proves part (*iii*), and thereby, part (*ii*). □

Example 5.3. If

$$\ell_1 = 1 - 2x - 2y + 2z, \qquad \ell_2 = 1 - 2x, \qquad \ell_3 = 1 - 2y, \qquad \ell_4 = 1 - 2z, \qquad \ell_5 = 1 - x - y$$

then the circuits are the sets

$$\{\ell_1, \ell_2, \ell_3, \ell_4\}, \quad \{\ell_1, \ell_4, \ell_5\}, \quad \{\ell_2, \ell_3, \ell_5\},$$

and the broken circuits are the sets

$$\{\ell_1, \ell_2, \ell_3\}, \quad \{\ell_1, \ell_4\}, \quad \{\ell_2, \ell_3\}.$$

The collection of χ-independent sets is

$$\varnothing$$

$$\{\ell_1\} \quad \{\ell_2\} \quad \{\ell_3\} \quad \{\ell_4\} \quad \{\ell_5\}$$
$$\{\ell_1,\ell_2\} \quad \{\ell_1,\ell_3\} \quad \{\ell_1,\ell_5\} \quad \{\ell_2,\ell_4\} \quad \{\ell_2,\ell_5\} \quad \{\ell_3,\ell_4\} \quad \{\ell_3,\ell_5\} \quad \{\ell_4,\ell_5\}$$
$$\{\ell_1,\ell_2,\ell_5\} \quad \{\ell_1,\ell_3,\ell_5\} \quad \{\ell_2,\ell_4,\ell_5\} \quad \{\ell_3,\ell_4,\ell_5\}\,.$$

Consider the rational function

$$F(x,y,z) = \frac{1}{\ell_1(x,y)l_2(x,y)l_3(x,y)l_4(x,y)l_5(x,y)^2}\,.$$

Running Algorithm 2 gives the decomposition

$$\begin{aligned} F(x,y,z) = \quad & \frac{1}{4\ell_2(x,y)\ell_4(x,y)\ell_5(x,y)^4} + \frac{1}{4\ell_3(x,y)\ell_4(x,y)\ell_5(x,y)^4} \\ & + \frac{1}{4\ell_1(x,y)\ell_2(x,y)\ell_5(x,y)^4} + \frac{1}{4\ell_1(x,y)\ell_3(x,y)\ell_5(x,y)^4}. \end{aligned}$$

Each of these summands is now simple, and we may use our Maple implementation of the results in Section 4 to find asymptotics in direction $\mathbf{r} = (1,2,3)$, say. The summands have dominant coefficient asymptotics

$$\left(\frac{81\sqrt{42}}{224}\right) 54^n n^{5/2}, \qquad (2/3)\,64^n n^3, \qquad \frac{-(-27648)^n}{5\pi n}, \qquad \frac{-(-27648)^n}{4\pi n},$$

respectively, so that

$$[x^n y^{2n} z^{3n}]F(x,y,z) = \frac{-9(-27648)^n}{20\pi n}\left(1 + O\left(\frac{1}{n}\right)\right).$$

◁

Part (i) of Theorem 4.27 continues to hold for non-simple arrangements. In particular, there is an ideal $\mathcal{J}$ such that $G \in \mathcal{J}$ implies $G/\prod_{j=1}^m \ell_j^{m_j}$ decomposes into a sum of rational functions whose denominators have poles on at most $k-1$ hyperplanes, k being the co-dimension of the flat containing a given critical point. Hence for $G \in \mathcal{J}$, $G/\prod_{j=1}^m \ell_j^{m_j}$ will have neighborhood exponential rate $\overline{\beta}(\hat{\mathbf{r}}) < h_{\hat{\mathbf{r}}}(\boldsymbol{\sigma})$, provided no other critical point produces a contribution with exponential rate $h_{\hat{\mathbf{r}}}(\boldsymbol{\sigma})$. The converse, however, fails for non-simple arrangements, as shown by the following example.

Example 5.4. Let $\ell_j = y - 1 - \lambda_j(x-1)$ for $1 \le j \le 4$ be four real lines through $\boldsymbol{\sigma} = (1,1)$, shown in Figure 9, whose normals have positive slopes $\nu_j := -1/\lambda_j$ that decrease in j.

The normal cone at $\boldsymbol{\sigma}$ contains the lines with slopes from ν_4 to ν_1. Let $G(x,y) := \ell_1(x,y)\ell_2(x,y) + \ell_3(x,y)\ell_4(x,y)$, so

$$\frac{G(x,y)}{\prod_{j=1}^4 \ell_j(x,y)} = \frac{1}{\ell_1(x,y)\ell_2(x,y)} + \frac{1}{\ell_3(x,y)\ell_4(x,y)}\,.$$

As we know from the simple case, this has an exponential rate of zero in directions with slopes in the two intervals $[\nu_1,\nu_2]$ and $[\nu_3,\nu_4]$, where $\boldsymbol{\sigma}$ is the contributing singularity determining asymptotics, and a negative exponential rate elsewhere. In particular, the exponential rate is negative along directions $\hat{\mathbf{r}}$ with slopes in (ν_2,ν_3), which are generic and have $\overline{\beta}(\hat{\mathbf{r}}) < 0 = h_{\hat{\mathbf{r}}}(\boldsymbol{\sigma})$. On the other hand, the numerator $\ell_1\ell_3 + \ell_2\ell_4$ leads to two non-canceling contributions rather than to none. Thus, any algebraic test for the numerator to create a drop in the neighborhood exponential rate must take into account specific normal cones. While this may seem trivial in low dimensions, computation of the chamber decomposition of the normal cone at $\boldsymbol{\sigma}$ is in general a high complexity computation, and the description of the set of numerators for which the neighborhood exponential rate drops is correspondingly difficult to compute. ◁

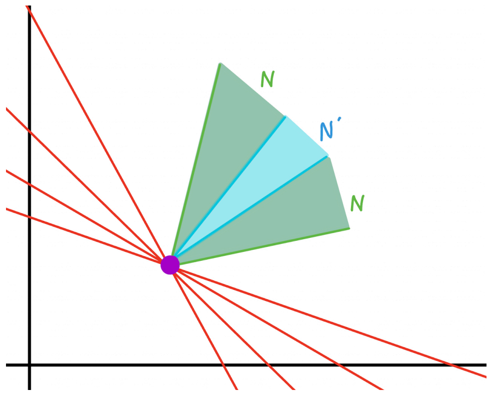

Figure 9: A non-simple arrangement where the exponential rate drops within a hole N' of the normal cone N at $\boldsymbol{\sigma} = (1,1)$.

6 Simple Arrangements in Non-Generic Directions

We now discuss the analysis in non-generic directions. First, we see how asymptotics behave in an exact non-generic direction. Then, we study transitions in asymptotic behaviour around non-generic directions.

6.1 Exact non-generic directions

Fix a direction $\hat{\mathbf{r}}$ and suppose F admits a unique contributing singularity $\boldsymbol{\sigma}$ of highest height, contained in the stratum $\mathcal{S}_{1,\ldots,t}$. Suppose $\boldsymbol{\sigma}$ is a non-generic direction such that

$$(-\nabla h)(\boldsymbol{\sigma}) = a_1 \mathbf{b}^{(1)} + \cdots + a_s \mathbf{b}^{(s)} + 0 \cdot \mathbf{b}^{(s+1)} + \cdots + 0 \cdot \mathbf{b}^{(t)} \tag{18}$$

for some $s < t$, where each $a_j \neq 0$ (and thus $a_j > 0$ by the definition of contributing point). We work in the simple case, so $\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(t)}$ are linearly independent, and up to permuting coordinates we may assume that the matrix

$$M = \begin{pmatrix} \mathbf{b}^{(1)} \\ \vdots \\ \mathbf{b}^{(t)} \\ \mathbf{e}^{(t+1)} \\ \vdots \\ \mathbf{e}^{(d)} \end{pmatrix}$$

has full rank, where $\mathbf{e}^{(j)}$ denotes the jth elementary basis vector.

Because $\boldsymbol{\sigma}$ is minimal and has highest height, asymptotic growth of $[\mathbf{z}^{\mathbf{r}}]F(\mathbf{z})$ is determined, up to an exponentially negligible error term, by the integral

$$I = \frac{1}{(2\pi i)^d} \int_{\boldsymbol{\sigma} - \epsilon \mathbf{m} + i\mathbb{R}^d} \frac{\tilde{G}(\mathbf{z})}{\ell_1(\mathbf{z})^{p_1} \cdots \ell_t(\mathbf{z})^{p_t}} \frac{d\mathbf{z}}{\mathbf{z}^{\mathbf{r}}},$$

where $\mathbf{m} = M^{-1}\left(\mathbf{e}^{(1)} + \cdots + \mathbf{e}^{(t)}\right)$ and

$$\tilde{G}(\mathbf{z}) = \frac{G(\mathbf{z})}{(z_1 \cdots z_d) \prod_{j>t} \ell_j(\mathbf{z})^{p_j}}.$$

Making the substitution $\mathbf{w} = M(\boldsymbol{\sigma} - \mathbf{z})$ gives

$$I = \frac{1}{|\det M|(2\pi i)^d} \int_{\epsilon(\mathbf{1}_t,\mathbf{0})+i\mathbb{R}^d} \frac{\tilde{G}(\boldsymbol{\sigma} - M^{-1}\mathbf{w})}{w_1^{p_1} \cdots w_t^{p_t} \left(\boldsymbol{\sigma} - M^{-1}\mathbf{w}\right)^{\mathbf{r}}} \, d\mathbf{w}, \tag{19}$$

where $\mathbf{1}_t$ denotes the t-dimensional all ones vector. Since the a_j in (18) are positive, replacing the domain of integration in (19) by any of the $2^s - 1$ imaginary fibers with basepoints $(\pm\mathbf{1}_s, \mathbf{1}_{t-s}, \mathbf{0})$ not equal to $(\mathbf{1}_t, \mathbf{0})$ results in an integral of exponentially smaller growth[4]. Taking a signed sum of these integrals thus introduces an exponentially negligible error and results in a $(d-s)$-dimensional integral obtained by taking univariate residues in the variables $w_1, \ldots, w_s$ at the origin.

We can be most explicit when $\mathbf{p} = \mathbf{1}$, when the residues of $w_1, \ldots, w_s$ at the origin are obtained by removing the factor $(w_1 \cdots w_s)$ from the denominator and setting these variables equal to zero in the remaining expression. If $\mathbf{p} = \mathbf{1}$ and we make the change of variables $y_j = i w_{s+j}$ for $j = 1, \ldots, d-s$, then dominant asymptotics of $[\mathbf{z}^{\mathbf{r}}]F(\mathbf{z})$ are given by the integral

$$I' := \frac{(-1)^{t-s}}{|\det M|(2\pi)^{d-s}\, i^{t-s}} \int_{\mathbb{R}^{d-s}+i\epsilon(\mathbf{1}_{t-s},\mathbf{0})} \frac{\tilde{G}\left(\boldsymbol{\sigma} + iM^{-1}\binom{\mathbf{0}}{\mathbf{y}}\right)}{y_1 \cdots y_{t-s}} \, e^{-\mathbf{r}\cdot\log\left(\boldsymbol{\sigma} + iM^{-1}\binom{\mathbf{0}}{\mathbf{y}}\right)} d\mathbf{y}, \tag{20}$$

where $\mathbf{y} = (y_1, \ldots, y_{t-s})$. For general $\mathbf{p}$, repeated differentiation shows [34, Theorem 10.2.6] that the residue has the form

$$\left. \frac{\tilde{G}(\boldsymbol{\sigma} - M^{-1}\mathbf{w})}{w_{s+1}^{p_{s+1}} \cdots w_t^{p_t} \left(\boldsymbol{\sigma} - M^{-1}\mathbf{w}\right)^{\mathbf{r}}} \; P(\mathbf{r}, \mathbf{w}) \right|_{w_j = 0, 1 \le j \le s},$$

where P is a multivariate polynomial in $\mathbf{r}$ of degree $p_1 + \cdots + p_s - s$ having leading term

$$R(\mathbf{w}) = \prod_{k=1}^{s} \frac{1}{(p_k - 1)!} \left(\sum_{j=1}^{d} \frac{r_j M^{-1}_{jk}}{(\boldsymbol{\sigma} - M^{-1}\mathbf{w})_j} \right)^{p_k - 1}.$$

Thus,

$$\begin{aligned} I &= \frac{1}{|\det M|(2\pi i)^{d-s} \prod_{k=1}^{s}(p_k-1)!} \int_{\epsilon(\mathbf{1}_{t-s},\mathbf{0})+i\mathbb{R}^{d-s}} \left. \frac{\tilde{G}(\boldsymbol{\sigma} - M^{-1}\mathbf{w}) R(\mathbf{w})}{w_{s+1}^{p_{s+1}} \cdots w_t^{p_t} \left(\boldsymbol{\sigma} - M^{-1}\mathbf{w}\right)^{\mathbf{r}}} \right|_{w_j=0, 1\le j\le s} d\mathbf{w} \left(1 + O\left(\frac{1}{|\mathbf{r}|}\right)\right) \\ &= \frac{i^{p_{s+1}+\cdots+p_t}}{|\det M|(2\pi)^{d-s} \prod_{k=1}^{s}(p_k-1)!} \int_{\mathbb{R}^{d-s}+i\epsilon(\mathbf{1}_{t-s},\mathbf{0})} \frac{\tilde{G}\left(\boldsymbol{\sigma} + iM^{-1}\binom{\mathbf{0}}{\mathbf{y}}\right) R(\mathbf{0}, -i\mathbf{y})}{y_{s+1}^{p_{s+1}} \cdots y_t^{p_t} \left(\boldsymbol{\sigma} + iM^{-1}\binom{\mathbf{0}}{\mathbf{y}}\right)^{\mathbf{r}}} d\mathbf{y} \left(1 + O\left(\frac{1}{|\mathbf{r}|}\right)\right), \end{aligned} \tag{21}$$

where $\mathbf{y} = (y_1, \ldots, y_{t-s})$ and $|\mathbf{r}| = r_1 + \cdots + r_d$.

Example 6.1. Consider again the function

$$F(x, y) = \frac{1}{\left(1 - \frac{2x+y}{3}\right)\left(1 - \frac{x+2y}{3}\right)}.$$

The sequence

$$[x^{2n} y^n] F(x, y) = \frac{1}{(2\pi i)^2} \int_{\mathcal{T}} \frac{1}{\left(1 - \frac{2x+y}{3}\right)\left(1 - \frac{x+2y}{3}\right)} \frac{dx dy}{x^{2n+1} y^{n+1}} \tag{22}$$

is in the non-generic direction $\mathbf{r} = (2, 1)$, meaning its asymptotic behaviour can be quantitatively different than what happens when $1/3 < \hat{r}_1 < 2/3$ (limit to a constant with exponential error) and $2/3 < \hat{r}_1 < 1$ (limit

[4]If $\hat{\mathbf{r}}$ were a generic direction, we would be able to add all 2^t fibers and use univariate residues will get rid of all w_j in the integrand denominator.

to zero exponentially with polynomial error). The highest contributing singularity is now $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{1,2} = (1,1)$ on the stratum $\mathcal{S}_{1,2}$, which now also coincides with the critical point $\boldsymbol{\sigma}_1$ on the flat $\mathcal{V}_1$.

Since there are no bounded components of $\mathcal{M}_{\mathbb{R}}$ in any quadrant except the positive quadrant, the Cauchy integral in (22) equals

$$\begin{aligned}[x^{2n}y^n]F(x,y) &= \frac{1}{(2\pi i)^2}\int_{(\epsilon,\epsilon)+i\mathbb{R}^2} \frac{1}{\left(1-\frac{2x+y}{3}\right)\left(1-\frac{x+2y}{3}\right)}\frac{dxdy}{x^{2n+1}y^{n+1}} \\ &= \frac{1}{(2\pi i)^2}\int_{(1-\epsilon,1-\epsilon)+i\mathbb{R}^2} \frac{1}{\left(1-\frac{2x+y}{3}\right)\left(1-\frac{x+2y}{3}\right)}\frac{dxdy}{x^{2n+1}y^{n+1}}.\end{aligned}$$

Making the change of variables described above shows

$$[x^{2n}y^n]F(x,y) = \frac{3}{(2\pi i)^2}\int_{(\epsilon,\epsilon)+i\mathbb{R}^2} \frac{1}{w_1w_2}\frac{dw_1dw_2}{(1-2w_1+w_2)^{2n+1}(1+w_1-2w_2)^{n+1}}.$$

If

$$\tilde{\omega} = \frac{1}{w_1w_2}\frac{dw_1dw_2}{(1-2w_1+w_2)^{2n+1}(1+w_1-2w_2)^{n+1}},$$

then

$$[x^{2n}y^n]F(x,y) = \frac{3}{(2\pi i)^2}\left[\int_{(\epsilon,\epsilon)+i\mathbb{R}^2}\tilde{\omega} - \int_{(-\epsilon,\epsilon)+i\mathbb{R}^2}\tilde{\omega}\right] + O(\tau^n)$$

for some $\tau \in (0,1)$. Taking the residue in w_1 at the origin then yields

$$\begin{aligned}[x^{2n}y^n]F(x,y) &= \frac{3}{2\pi i}\int_{\epsilon+i\mathbb{R}} \frac{1}{w_2}\frac{1}{(1+w_2)^{2n+1}(1-2w_2)^{n+1}}dw_2 + O(\tau^n) \\ &= \frac{-3}{2\pi i}\int_{\mathbb{R}+i\epsilon}\frac{1}{y(1+iy)(1-2iy)}e^{-n[2\log(1+iy)+\log(1-2iy)]}dy + O(\tau^n).\end{aligned}$$

$\triangleleft$

The idea behind an asymptotic analysis is to replace the analytic functions in the integrand under consideration with the leading terms of their power series expansions at the origin. Doing this rigorously requires extending the asymptotic bounds of [34, Chapter 5] to integrals which are singular at the origin due to a division by monomial terms. When $(-\nabla h_{\mathbf{r}})(\boldsymbol{\sigma})$ lies on a k-dimensional facet of $N(\boldsymbol{\sigma})$, asymptotics of the dominant term in the residue integral are obtained by analyzing an integral of the form

$$\int_{\mathbb{R}^k+i\boldsymbol{\epsilon}} \mathbf{y}^{\mathbf{c}} \cdot e^{-\mathbf{y}^tA\mathbf{y}}d\mathbf{y},$$

where A is a $k\times k$ positive-definite matrix and $\mathbf{c}\in\mathbb{Z}^k$. This can be viewed as a Gaussian *negative*-moment integral; although positive moments ($\mathbf{c}\in\mathbb{N}^k$) of the Gaussian distribution can be determined through the Wick-Isserlis Theorem [21, Theorem 3.2.5], such negative moments do not seem to have received previously attention in the literature.

We postpone the general discussion of this theory to future work and focus here on the $s = d-1$ codimension 1 case, where it is easy to be explicit. In this setting, asymptotics boil down to an analysis of the integral

$$\int_{\mathbb{R}+i\epsilon}\frac{e^{-at^2}}{t^r}dt,$$

which is characterized by the following result.

Proposition 6.2. *Let r be a positive integer and $a \geq 0$. Then*

$$\int_{\mathbb{R}+i\epsilon} \frac{e^{-at^2}}{t^r} dt = \frac{(-i)^r a^{(r-1)/2} \pi}{\Gamma\left(\frac{r+1}{2}\right)} .$$

Proof. Suppose that r is odd and $a = 1$. Then as the integrand under consideration is odd, Cauchy's Integral Theorem implies

$$\int_{\mathbb{R}+i\epsilon} \frac{e^{-t^2}}{t^r} dt = -\int_{-\mathbb{R}+i\epsilon} \frac{e^{-t^2}}{t^r} dt = -\int_{\mathcal{C}} \frac{e^{-t^2}}{t^r} dt,$$

where $\mathcal{C}$ is the positively oriented unit half-circle above the x-axis. As $\mathcal{C}$ is a compact domain of integration, Fubini's Theorem implies that we can expand the integrand as a power series, integrate each term using an explicit parametrization, and sum to obtain

$$\int_{\mathcal{C}} \frac{e^{-at^2}}{t^r} dt = \frac{(\pi i)(-1)^{(r-1)/2}}{((r-1)/2)!} = \frac{(-1)^{r+1} i^r a^{(r-1)/2} \pi}{\Gamma\left(\frac{r+1}{2}\right)} .$$

The case when r is odd and a is arbitrary is reduced to this case by a linear change of variables.

Now suppose that r is even, and let $J_r(a) := \int_{\mathbb{R}+i\epsilon} \frac{e^{-at^2}}{t^r} dt$. Differentiating under the integral sign with respect to a implies

$$(\partial J_r/\partial a)(a) = \int_{\mathbb{R}+i\epsilon} -\frac{e^{-at^2}}{t^{r-2}} dt = -J_{r-2}(a)$$

for $r \geq 2$, while

$$J_r(0) = \int_{\mathbb{R}+i\epsilon} \frac{1}{t^r} dt = 0$$

when $r \geq 2$ and

$$J_0(a) = \int_{\mathbb{R}+i\epsilon} e^{-at^2} dt = \sqrt{\pi/a}.$$

Solving this recurrence implies

$$J_r(a) = \frac{(-1)^{r/2} a^{(r-1)/2} \sqrt{\pi}}{(r/2 - 1/2)(r/2 - 3/2) \cdots (1/2)} = \frac{(-1)^r i^r a^{(r-1)/2} \pi}{\Gamma\left(\frac{r+1}{2}\right)} ,$$

as the difference equation satisfied by the gamma function shows

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n-1)(2n-3)\cdots(3)(1)}{2^n} \Gamma\left(\frac{1}{2}\right) = \frac{(2n-1)(2n-3)\cdots(3)(1)}{2^n} \sqrt{\pi}$$

for any non-negative integer n. ◻

This allows for the following asymptotic determination.

Proposition 6.3. *Suppose $F(\mathbf{z})$ is simple and $\mathbf{r}$ is a non-generic direction with a unique contributing singularity of maximal height where $G(\boldsymbol{\sigma}) \neq 0$. If $(-\nabla h_{\mathbf{r}})(\boldsymbol{\sigma})$ lies on the codimension 1 face of $N(\boldsymbol{\sigma})$ with*

$$(-\nabla h_{\mathbf{r}})(\boldsymbol{\sigma}) = a_1 \cdot \mathbf{b}^{(1)} + \cdots + a_{d-1} \cdot \mathbf{b}^{(d-1)} + 0 \cdot \mathbf{b}^{(d)}$$

for $\boldsymbol{\sigma}$ in the stratum defined by $\ell_1 = \cdots = \ell_d = 0$, then as $\mathbf{r} \to \infty$,

$$[\mathbf{z}^{\mathbf{r}}]F(\mathbf{z}) = \boldsymbol{\sigma}^{-\mathbf{r}} \cdot \left(C + O\left(\frac{1}{n}\right)\right),$$

where

$$C = \prod_{j=1}^{d-1} \frac{\left(\left(\frac{r_1}{\sigma_1} \quad \cdots \quad \frac{r_d}{\sigma_d}\right) \cdot M^{-1}\right)_j^{p_j - 1}}{(p_j - 1)!} \cdot \frac{G(\boldsymbol{\sigma})}{\prod_{j>d} \ell_j(\boldsymbol{\sigma})^{p_j}} \cdot \frac{(\mathbf{q}^T\mathbf{q}/2)^{(p_d-1)/2}}{2(\sigma_1 \cdots \sigma_d)\,|\det M|\,\Gamma\left(\frac{p_d+1}{2}\right)},$$

M is the matrix with rows $\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(d)}$, and $\mathbf{q}$ is the rightmost column of the matrix

$$Q = \begin{pmatrix} \sqrt{r_1}/\sigma_1 & 0 & \mathbf{0} & 0 \\ 0 & \sqrt{r_2}/\sigma_2 & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\ 0 & 0 & \mathbf{0} & \sqrt{r_d}/\sigma_d \end{pmatrix} M^{-1}.$$

If $\mathbf{r} = n\hat{\mathbf{r}}$ as $n \to \infty$ then the order of growth of $[\mathbf{z}^{\mathbf{r}}]F(\mathbf{z})$ is $\boldsymbol{\sigma}^{-\mathbf{r}} n^{p_1+\cdots+p_{d-1}+p_d/2-d+1/2}$. In the simple pole case, when $\mathbf{p} = \mathbf{1}$, the leading constant here is half what it would be if $\mathbf{r}$ was generic (i.e., if $(-\nabla h_{\mathbf{r}})(\boldsymbol{\sigma})$ was in the interior of $N(\boldsymbol{\sigma})$).

Proof. In this one-dimensional case, the (now univariate) integral in (21) equals

$$I' = \int_{\mathbb{R}+i\epsilon} \frac{\tilde{G}\left(\boldsymbol{\sigma} + iM^{-1}\binom{\mathbf{0}}{y}\right)}{y_d^{p_d}\left(\boldsymbol{\sigma} + iM^{-1}\binom{\mathbf{0}}{y}\right)^{\mathbf{r}}} \prod_{k=1}^{d-1} \left(\sum_{j=1}^{d} \frac{r_j M^{-1}_{jk}}{(\boldsymbol{\sigma} + M^{-1}\binom{\mathbf{0}}{y})_j} \right)^{p_k - 1} dy.$$

Note that

$$I' = \int_{\mathbb{R}+i\epsilon} \frac{A(y)}{y^{p_d}} e^{-\psi(y)} dy,$$

where $A(y)$ and $\psi(y)$ are analytic functions at the origin with

$$A(y) = \underbrace{\tilde{G}(\boldsymbol{\sigma}) \prod_{j=1}^{d-1} \left(\left(\frac{r_1}{\sigma_1} \quad \cdots \quad \frac{r_d}{\sigma_d}\right) \cdot M^{-1}\right)_j^{p_j-1}}_{C_1} + O(y)$$

$$\psi(y) = \mathbf{r} \cdot \log(\boldsymbol{\sigma} + M^{-1}\tbinom{\mathbf{0}}{y}) = \log(\boldsymbol{\sigma}) + \underbrace{(\mathbf{q}^T\mathbf{q}/2)}_{C_2} y^2 + O(y^3).$$

The desired result is given by an application of the following lemma. ☐

Lemma 6.4. *For some $\epsilon > 0$ and $k \in \mathbb{N}$ let*

$$I = \int_{\mathbb{R}+i\epsilon} y^{-k} A(y) e^{-n\psi(y)},$$

where

- *A, ψ are analytic in $\mathbb{R} + (-2\epsilon, 2\epsilon)i \subset \mathbb{C}$;*
- *$A(y) = a_0 + O(y)$ and $\psi(y) = b_2 y^2 + O(y^3)$ at 0;*
- *$\psi'(y) = 0$ implies $y = 0$;*
- *$\Re(\psi) \geq 0$ with equality only when $y = 0$, and $|\Re(\psi)| \to \infty$ as $|y| \to \infty$.*

Then

$$I \sim a_0 \int_{\mathbb{R}+i\epsilon} y^{-k} e^{-nb_0 y^2} dy = \frac{a_0(-i)^k (nb_0)^{(k-1)/2}}{\Gamma(\frac{k+1}{2})}.$$

Proof. For $n > 0$ let $\mathcal{C}_1 = (-n^{-2/5}, n^{-2/5})$ and $\mathcal{C}_2 = \mathbb{R} \setminus \mathcal{C}_1$, so that

$$I = \int_{\mathcal{C}_1+i\epsilon} y^{-k} A(y) e^{-n\psi(y)} dy + \int_{\mathcal{C}_2+i\epsilon} y^{-k} A(y) e^{-n\psi(y)} dy.$$

The hypotheses on the real part of ψ imply that the integrand under consideration decays exponentially as y grows, meaning the value of I is independent of $\epsilon > 0$. For the rest of this proof we take $\epsilon = \epsilon(n) = 1/\sqrt{n}$.

Step 1 (Prune Tails). Our first goal is to show that the integral over $\mathcal{C}_2 + i\epsilon$ is negligible. To do this we bound the integral over $(n^{-2/5}, \infty) + i\epsilon$, with the integral over $(-\infty, -n^{-2/5}) + i\epsilon$ bounded with an analogous argument. Since $\psi'(y) = 0$ has no solution other than $y = 0$, repeatedly integrating by parts shows that for any fixed $a > 0$

$$\int_{(a,\infty)+i\epsilon} y^{-k} A(y) e^{-n\psi(y)} dy$$

decays to zero faster than any negative integer power of n (see, for instance, the argument in Stein [37, Section VIII.1.1]). Because of the power series expansions of A and ψ at the origin, there exists $\sigma > 0$ and $K_1, K_2 > 0$ such that for all $|t| \leq \sigma$ and n sufficiently large,

$$|A(t+i\epsilon)| \leq K_1 \quad \text{and} \quad \Re(\psi(t+i\epsilon)) \geq K_2 \Re((t+i\epsilon)^2) = K_2(t^2 - \epsilon^2).$$

When n is sufficiently large then $n^{-2/5} < \sigma$, and we see that

$$\left| \int_{(n^{-2/5},\infty)+i\epsilon} y^{-k} A(y) e^{-n\psi(y)} dy - \int_{(n^{-2/5},\sigma)+i\epsilon} y^{-k} A(y) e^{-n\psi(y)} dy \right| \to 0$$

faster than any negative integer power of n. Furthermore, with $\epsilon = 1/\sqrt{n}$ we have

$$\left| \int_{(n^{-2/5},\sigma)+i\epsilon} y^{-k} A(y) e^{-n\psi(y)} dy \right| \leq K_1 \int_{n^{-2/5}}^{\sigma} |t + i\epsilon|^{-k} \, e^{-nK_2(t^2-\epsilon^2)} dt \leq K_1 e^{K_2} \int_{n^{-2/5}}^{\sigma} t^{-k} e^{-nK_2 t^2} dt,$$

and making the change of variables $w = nK_2 t^2$ implies

$$\begin{aligned}
\int_{n^{-2/5}}^{\sigma} t^{-k} e^{-nK_2 t^2} dt &= O\left(n^{(k-1)/2} \int_{K_2 n^{1/5}}^{\infty} w^{-(k+1)/2} e^{-w} dw \right) \\
&= O\left(n^{(k-1)/2-(k+1)/10} \int_{K_2 n^{1/5}}^{\infty} e^{-w} dw \right) \\
&= O\left(n^{(2k-3)/5} e^{-K_2 n^{1/5}} \right).
\end{aligned}$$

Putting everything together, and repeating the argument for the elements of $\mathcal{C}_2$ which are less than zero, we have shown that

$$I \sim \int_{\mathcal{C}_1+i\epsilon} y^{-k} A(y) e^{-n\psi(y)} dy,$$

up to an error that decays faster than any negative integer power of n.

Step 2 (Approximate). We now show that A and ψ can be replaced by their leading power series terms while introducing an error which will not affect asymptotics. Note that

$$y^{-k} A(y) e^{-n\psi(y)} = a_0 y^{-k} e^{-nb_0 y^2} \left(1 + O(y + ny^3)\right),$$

and $|y| \leq 2n^{-2/5}$ when $y \in \mathcal{C}_1 + i\epsilon$. Thus,

$$\begin{aligned}\left|\int_{\mathcal{C}_1+i\epsilon} y^{-k}A(y)e^{-n\psi(y)}dy - a_0\int_{\mathcal{C}_1+i\epsilon} y^{-k}e^{-nb_0y^2}dy\right| &= O\left(n^{-1/5}\int_{\mathcal{C}_1+i\epsilon}\left|y^{-k}e^{-nb_0y^2}\right|dy\right)\\ &= O\left(n^{-1/5}\int_{\mathcal{C}_1}\frac{1}{(t^2+\epsilon^2)^{k/2}}e^{-n(t^2-\epsilon^2)}dt\right)\\ &= O\left(n^{k/2-1/5}\int_{\mathbb{R}}\frac{1}{(nt^2+1)^{k/2}}e^{-nt^2}dt\right),\end{aligned}$$

where we have made the substitution $y = t + i\epsilon$ and used $\epsilon = 1/\sqrt{n}$. Substituting $w = nt^2$ then implies

$$O\left(n^{k/2-1/5}\int_{\mathbb{R}}\frac{1}{(nt^2+1)^{k/2}}e^{-nt^2}dt\right) = O\left(n^{(k-1)/2-1/5}\int_0^\infty\frac{1}{\sqrt{w}(w+1)^{k/2}}e^{-w}dw\right) = O\left(n^{(k-1)/2-1/5}\right),$$

since

$$\int_0^\infty\frac{1}{\sqrt{w}(w+1)^{k/2}}e^{-w}dw \leq \int_0^\infty\frac{1}{\sqrt{w}}e^{-w}dw = \sqrt{\pi}$$

is finite for all $k > 0$.

Step 3 (Add Tails Back). Following the reasoning of Step 1, the integral

$$a_0\int_{\mathcal{C}_2+i\epsilon} y^{-k}e^{-nb_0y^2}dy$$

is asymptotically negligible, meaning

$$I = a_0\int_{\mathbb{R}+i\epsilon} y^{-k}e^{-nb_0y^2}dy + O\left(n^{(k-1)/2-1/5}\right).$$

Proposition 6.2 gives the value of this integral, showing it grows as a constant times $n^{(k-1)/2}$. □

Remark 6.5. Our proof of Lemma 6.4 mirrors the standard presentation of the saddle-point method for integrals of the form $\int_{\mathbb{R}} A(y)e^{-n\psi(y)}dy$, where A and ψ are analytic (or, more generally, smooth). Our arguments are slightly more involved because the negative powers of y introduced in our situation necessitates working off the real line.

Example 6.6. Returning to Example 6.1, we see that

$$\begin{aligned}\frac{-3}{2\pi i}\int_{\mathbb{R}+i\epsilon}\frac{1}{y(1+iy)(1-2iy)}e^{-n[2\log(1+iy)+\log(1-2iy)]}dy &= \frac{-3}{2\pi i}\int_{\mathbb{R}+i\epsilon}\frac{e^{-n\left[3y^2+O(y^3)\right]}}{y}(1+O(y))\,dy + O(\tau^n)\\ &\sim \frac{-3}{2\pi i}\left(\int_{\mathbb{R}+i\epsilon}\frac{e^{-3ny^2}}{y}dy\right),\end{aligned}$$

so Proposition 6.2 implies

$$[x^{2n}y^n]F(x,y) \sim \frac{3}{2}.$$

◁

Explicit expressions for non-generic directions where $(-\nabla h_{\mathbf{r}})(\boldsymbol{\sigma})$ lies on higher codimensional faces of $N(\boldsymbol{\sigma})$ follow from a similar approximation procedure, however deriving the necessary bounds is slightly harder due to the presence of additional variables. A rigorous investigation of these cases is in progress.

6.2 Bridging the Exponential Gaps

Now we consider asymptotic transitions around non-generic directions. Suppose we again have a contributing point $\boldsymbol{\sigma}$ for the non-generic direction $\hat{\mathbf{r}}$ under the assumptions of Section 6.1 and $\mathbf{p} = \mathbf{1}$, so we have only simple poles.

To study such transitions, we write $\mathbf{r} = n\hat{\mathbf{r}} + \sqrt{n}\theta_1\mathbf{v}_1 + \cdots + \sqrt{n}\theta_{t-s}\mathbf{v}_{t-s}$ where

$$\mathbf{v}_j = \sigma \odot \mathbf{b}^{(j)} = \left(\sigma_1\mathbf{b}_1^{(j)}, \ldots, \sigma_d\mathbf{b}_d^{(j)}\right)$$

and work from (20). Then

$$\mathbf{r} \cdot \log\left(\boldsymbol{\sigma} - iM^{-1}\left(\begin{smallmatrix}\mathbf{0}\\ \mathbf{r}\end{smallmatrix}\right)\right) = n\phi(\mathbf{y}) + \sqrt{n}\theta_1\psi_1(\mathbf{y}) + \cdots + \sqrt{n}\theta_{t-s}\psi_{t-s}(\mathbf{y})$$

where

$$\begin{aligned}\phi(\mathbf{y}) &= \hat{\mathbf{r}} \cdot \log\left(\boldsymbol{\sigma} + iM^{-1}\left(\begin{smallmatrix}\mathbf{0}\\ \mathbf{y}\end{smallmatrix}\right)\right)\\ \psi_j(\mathbf{y}) &= \mathbf{v}_j \cdot \log\left(\boldsymbol{\sigma} + iM^{-1}\left(\begin{smallmatrix}\mathbf{0}\\ \mathbf{y}\end{smallmatrix}\right)\right).\end{aligned}$$

Since the derivative of the jth coordinate of $\log\left(\boldsymbol{\sigma} + iM^{-1}\left(\begin{smallmatrix}\mathbf{0}\\ \mathbf{y}\end{smallmatrix}\right)\right)$ with respect to y_k is $\sigma_j^{-1}iM^{-1}_{j,k+s}$,

$$\begin{aligned}(\nabla\phi)(\mathbf{0}) &= (-i)\sum_{j=1}^{d}\frac{r_j}{\sigma_j}M^{-1}_{j,k+s}\\ &= (-i)\left[\left(\frac{r_1}{\sigma_1}, \ldots, \frac{r_d}{\sigma_d}\right)\cdot M^{-1}\right]_{[s+1,\ldots,t]}\\ &= i\left[\left(a_1\mathbf{b}^{(1)} + \cdots + a_s\mathbf{b}^{(s)}\right)\cdot M^{-1}\right]_{[s+1,\ldots,t]}\\ &= 0\end{aligned}$$

as the first t rows of M are $\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(t)}$, where the notation $P_{[a,\ldots,b]}$ refers to the a through bth columns of a matrix P. Similarly,

$$(\nabla\psi_j)(\mathbf{0}) = (-i)\left[\mathbf{b}^{(s+j)}\cdot M^{-1}\right]_{[s+1,\ldots,t]} = (-i)\mathbf{e}^{(s+j)}.$$

In the same codimension 1 case from the last section, the modified saddle-point method described in the proof of Proposition 6.3 shows that when $\theta = O(n^c)$ for some $c < 1/2$ then

$$I_\theta \sim \frac{\boldsymbol{\sigma}^{-\mathbf{r}}i\tilde{G}(\boldsymbol{\sigma})}{|\det M|(2\pi)}\underbrace{\int_{\mathbb{R}+i\epsilon}\frac{1}{y}e^{-n(\mathbf{q}^T\mathbf{q}/2)y^2 - i\sqrt{n}\theta y}\,d\mathbf{y}}_{J(\theta)},$$

where M and $\mathbf{q}$ are the same as above. This implies the following result, where we recall the Gaussian error function $\Psi(z) = \frac{2}{\pi}\int_0^z e^{-u^2/2}\,du$.

Proposition 6.7. *Suppose*

$$F(\mathbf{z}) = \frac{G(\mathbf{z})}{\ell_1(\mathbf{z})\cdots\ell_d(\mathbf{z})}$$

is simple and $\mathbf{r}$ *is a non-generic direction with a unique contributing singularity* $\boldsymbol{\sigma}$ *of maximal height, which lies on the stratum* $\ell_1 = \cdots = \ell_d = 0$. *Suppose also that* $G(\boldsymbol{\sigma}) \neq 0$ *and* $(-\nabla h_{\mathbf{r}})(\boldsymbol{\sigma})$ *lies on a codimension 1 face of* $N(\boldsymbol{\sigma})$, *in such a way that*

$$(-\nabla h_{\mathbf{r}})(\boldsymbol{\sigma}) = \lambda_1\cdot\mathbf{b}^{(1)} + \cdots + \lambda_{d-1}\cdot\mathbf{b}^{(d-1)} + 0\cdot\mathbf{b}^{(d)}$$

for some $\lambda_j > 0$. *If* $\mathbf{v} = \left(\sigma_1 b_1^{(d)}, \ldots, \sigma_n b_n^{(d)}\right)$ *and* $\theta = O(n^c)$ *for some* $c < 1/2$, *then*

$$\left[\mathbf{z}^{n\mathbf{r}+\theta\sqrt{n}\mathbf{v}}\right] F(\mathbf{z}) \sim \boldsymbol{\sigma}^{-n\mathbf{r}-\theta\sqrt{n}\mathbf{v}} \frac{G(\boldsymbol{\sigma})}{2\sigma_1\cdots\sigma_d|\det M|}\left(\Psi\left(\frac{\theta}{\sqrt{2\mathbf{q}^T\mathbf{q}}}\right)+1\right),$$

where M *is the matrix whose rows are the* $\mathbf{b}^{(j)}$ *and* $\mathbf{q}$ *is the right-most column of*

$$Q = \begin{pmatrix} \sqrt{r_1}/\sigma_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sqrt{r_2}/\sigma_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \sqrt{r_d}/\sigma_d \end{pmatrix} M^{-1}.$$

Proof. The integral $J(\theta)$ is actually easier to analyze than the integrals from the previous section. In fact, differentiating under the integral sign with respect to θ simplifies to remove the factor of y in the denominator, giving

$$J'(\theta) = i\sqrt{n}\int_{\mathbb{R}+i\epsilon} e^{-n(\mathbf{q}^T\mathbf{q}/2)y^2 - i\sqrt{n}\theta y}\, d\mathbf{y} = i\sqrt{n}\int_{\mathbb{R}} e^{-n(\mathbf{q}^T\mathbf{q}/2)y^2 - i\sqrt{n}\theta y}\, d\mathbf{y} = i\sqrt{\frac{2\pi}{\mathbf{q}^T\mathbf{q}}}e^{-\frac{\theta^2}{2\mathbf{q}^T\mathbf{q}}}.$$

Integrating with respect to θ then yields

$$J(\theta) = -\pi i\Psi\left(\frac{\theta}{\sqrt{2\mathbf{q}^T\mathbf{q}}}\right) + J(0),$$

where

$$J(0) = \int_{\mathbb{R}+i\epsilon} \frac{e^{-n(\mathbf{q}^T\mathbf{q}/2)y^2}}{y}\, d\mathbf{y} = -\pi i$$

by Proposition 6.2. □

The general codimension case reduces to an analysis of integrals of the form

$$J(\boldsymbol{\theta}) = \int_{\mathbb{R}^b + i\epsilon\mathbf{1}} \frac{1}{y_1\cdots y_a} e^{-n(\mathbf{y}^T\mathcal{H}\mathbf{y}) + i\sqrt{n}(\theta_1 y_1 + \cdots + \theta_a y_a)} d\mathbf{y},$$

where $\mathcal{H}$ is a positive definite matrix. A suitable change of variables splits this integral into a product of integrals, allowing one to reduce to the case $a = b$ where every variable appears in the denominator. This results in a so-called *hydrodynamic* scaling limit, which will be rigorously covered in future work.

References

[1] M. F. Atiyah, R. Bott, and L. Gårding. Lacunas for hyperbolic differential operators with constant coefficients. I. *Acta Math.*, 124:109–189, 1970.

[2] Y. Baryshnikov, S. Melczer, and R. Pemantle. Asymptotics of multivariate sequences in the presence of a lacuna. *Submitted, arxiv.org/abs/1905.04187*, 2021.

[3] Y. Baryshnikov, S. Melczer, and R. Pemantle. Stationary points at infinity for analytic combinatorics. *Foundations of Computational Mathematics*, 2021.

[4] Y. Baryshnikov, S. Melczer, and R. Pemantle. Stationary points at infinity for analytic combinatorics. *Foundations Comp. Math.*, 2021.

[5] Y. Baryshnikov and R. Pemantle. Asymptotics of multivariate sequences, part iii: quadratic points. *Adv. Math.*, 228:3127–3206, 2011.

[6] E. A. Bender and L. B. Richmond. Central and local limit theorems applied to asymptotic enumeration. ii. multivariate generating functions. *J. Combin. Theory Ser. A*, 34:255–265, 1983.

[7] Edward A. Bender and L. Bruce Richmond. Multivariate asymptotics for products of large powers with applications to Lagrange inversion. *Electron. J. Combin.*, 6:Research Paper 8, 21 pp. (electronic), 1999.

[8] Andrea Bertozzi and James McKenna. Multidimensional residues, generating functions, and their application to queueing networks. *SIAM Rev.*, 35(2):239–268, 1993.

[9] T. Brylawski. The broken-circuit complex. *Trans. AMS*, 234(2):417–433, 1977.

[10] Gilles Christol. Diagonales de fractions rationnelles et equations différentielles. In *Study group on ultrametric analysis, 10th year: 1982/83, No. 2*, pages Exp. No. 18, 10. Inst. Henri Poincaré, Paris, 1984.

[11] J. de Loera and B. Sturmfels. Algebraic unimodular counting. *Math. Program.*, 96:183–203, 2003.

[12] T. DeVries, J. van der Hoeven, and R. Pemantle. Automatic asymptotics for coefficients of smooth, bivariate rational functions. *Online J. Anal. Comb.*, 6:24 pages, 2011.

[13] P. Flajolet and R. Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009.

[14] Zhicheng Gao and L. Bruce Richmond. Central and local limit theorems applied to asymptotic enumeration. IV. Multivariate generating functions. *J. Comput. Appl. Math.*, 41(1-2):177–186, 1992. Asymptotic methods in analysis and combinatorics.

[15] A. Gilbert and H. Karloff. On the fractal behavior of tcp. In *Proceedings of the 35th Annual ACM Symposium on Theory of Computing*, pages 297–306, New York, 2003. ACM Press.

[16] Mark Goresky and Robert MacPherson. *Stratified Morse theory*, volume 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988.

[17] Peter Henrici. *Applied and computational complex analysis. Vol. 2.* Wiley Interscience [John Wiley & Sons], New York-London-Sydney, 1977. Special functions—integral transforms—asymptotics—continued fractions.

[18] Lars Hörmander. *The analysis of linear partial differential operators. I*, volume 256 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1990. Distribution theory and Fourier analysis.

[19] Hsien-Kuei Hwang. Large deviations of combinatorial distributions. II. Local limit theorems. *Ann. Appl. Probab.*, 8(1):163–181, 1998.

[20] Pierre Lairez. Computing periods of rational integrals. *Math. Comp.*, 85(300):1719–1752, 2016.

[21] Sergei K. Lando and Alexander K. Zvonkin. *Graphs on surfaces and their applications*, volume 141 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. With an appendix by Don B. Zagier, Low-Dimensional Topology, II.

[22] L. Lipshitz. The diagonal of a D-finite power series is D-finite. *J. Algebra*, 113(2):373–378, 1988.

[23] M. Lladser. Uniform formulae for coefficients of meromorphic functions in two variables. *SIAM J. Dis. Math.*, 20(4):811–828, 2006.

[24] Stephen Melczer. *An Invitation to Analytic Combinatorics in One and Several Variables*. Springer, New York, 2021.

[25] Stephen Melczer and Bruno Salvy. Effective coefficient asymptotics of multivariate rational functions via semi-numerical algorithms for polynomial systems. *J. Symbolic Comput.*, 103:234–279, 2021.

[26] Marc Mezzarobba. NumGfun: a package for numerical and analytic computation with D-finite functions. pages 139–146, 2010.

[27] Marc Mezzarobba. Rigorous multiple-precision evaluation of D-finite functions in SageMath. Technical Report 1607.01967, arXiv, 2016. Extended abstract of a talk at the 5th International Congress on Mathematical Software.

[28] A. M. Odlyzko. Asymptotic enumeration methods. In *Handbook of combinatorics, Vol. 1, 2*, pages 1063–1229. Elsevier, Amsterdam, 1995.

[29] P. Orlik and L. Solomon. Combinatorics and topology of complements of hyperplanes. *Invent. Math.*, 56:167–189, 1980.

[30] P. Orlik and H. Terao. Commutative algebras for arrangements. *Nagoya Math J.*, 134:65–73, 1994.

[31] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[32] R. Pemantle and M. C. Wilson. Asymptotics of multivariate sequences: I. Smooth points of the singular variety. *J. Comb. Theory, Ser. A*, 97(1):129–161, 2002.

[33] R. Pemantle and M. C. Wilson. Asymptotics of multivariate sequences. II. Multiple points of the singular variety. *Combin. Probab. Comput.*, 13(4-5):735–761, 2004.

[34] R. Pemantle and M. C. Wilson. *Analytic Combinatorics in Several Variables*. Cambridge University Press, 2013.

[35] Robin Pemantle. Generating functions with high-order poles are nearly polynomial. In *Mathematics and computer science (Versailles, 2000)*, Trends Math., pages 305–321. Birkhäuser, Basel, 2000.

[36] A. Raichev and M.C. Wilson. Asymptotics of coefficients of multivariate generating functions: improvements for multiple points. *Online Journal of Analytic Combinatorics*, 6(0), 2011.

[37] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993.

[38] Joris van der Hoeven. Fast evaluation of holonomic functions near and in regular singularities. *J. Symbolic Comput.*, 31(6):717–743, 2001.

[39] A. Varchenko and I.M. Gelfand. Combinatorics and topology of configuration of affine hyperplanes in real space. *Funk. Analiz i ego Prilozh.*, 21:11–22, 1987.