Biased random walks on Galton–Watson trees

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Summary. We consider random walks with a bias toward the root on the family tree $T$ of a supercritical Galton–Watson branching process and show that the speed is positive whenever the walk is transient. The corresponding harmonic measures are carried by subsets of the boundary of dimension smaller than that of the whole boundary. When the bias is directed away from the root and the extinction probability is positive, the speed may be zero even though the walk is transient; the critical bias for positive speed is determined.

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1 Introduction

Consider a supercritical Galton–Watson branching process with generating function $f(s) = \sum_{k=0}^{\infty} p_k s^k$, i.e., each individual has $k$ offspring with probability $p_k$, and $m := f'(1) \in (1, \infty)$. Started with a single progenitor, this process yields a random infinite family tree $T$, called a Galton–Watson tree, on the event of nonextinction. We assume throughout that no $p_k$ is equal to 1.

Simple random walk gives some information on the structure of a tree; to explore this structure further, random walks with a bias toward the root have been used (e.g., Beretti and Sokal (1985), Lawler and Sokal (1988), Lyons (1990)). The rate of escape (speed) of a random walk indicates how much of the tree a single path explores, while the dimension of harmonic measure indicates how much of the tree is explored by the ensemble of almost all paths.

For $\lambda \geq 0$, the $\lambda$-biased random walk on a locally-finite rooted tree $T$, denoted $\text{RW}_\lambda$, is the time-homogeneous Markov chain $(X_n; n \geq 0)$ on the

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vertices of $T$ such that if $u$ is a vertex with $k \geq 1$ children $v_1, \ldots, v_k$ and parent $u$, then $P[X_{n+1} = v_i \mid X_n = u] = 1/(k + \lambda)$ for $i = 1, \ldots, k$ and $P[X_{n+1} = u \mid X_n = u] = \lambda/(k + \lambda)$; from the root all transitions to its children are equally likely. In case $k = \lambda = 0$, then $P[X_{n+1} = u \mid X_n = u] = 1$. Normally, we fix the initial state $X_0$ to be the root, $\rho$. See Fig. 1.

For almost every Galton–Watson tree $T$ on the event of nonextinction, $\text{RW}_\lambda$ is transient for $1 \leq \lambda < m$ (Lyons 1990). Here we show that for $1 < \lambda < m$, the random walk escapes at a positive speed and the corresponding harmonic measure has Hausdorff dimension less than that of the whole boundary. For $\lambda = 1$, i.e., the case of simple random walk, this was shown in Lyons et al. (1995) by using an explicit stationary measure on the space of trees. We know of no such direct construction when $\lambda > 1$; instead, the proof is based on some a priori bounds on the Green function and a regeneration argument. The speed of the random walk is the almost sure limit (if it exists) of $|X_n|/n$, where $|x|$ denotes the distance from the root to the vertex $x$. In Sect. 5, we use positivity of the speed (and, in particular, the finiteness of the mean time between regenerations) to establish the existence of a finite measure on the space of trees which is absolutely continuous with respect to Galton–Watson measure and is stationary for the $\lambda$-harmonic flow. This is the key to the “dimension drop” of harmonic measure. In Corollary 5.3, we deduce that there exists a.s. a subtree $T^{(0)}$ of $T$ with smaller exponential growth such that $\text{RW}_\lambda$ on $T$ is confined to $T^{(0)}$ with overwhelming probability.

When the bias is away from the root, i.e., $0 < \lambda < 1$, the walk is obviously transient on any infinite tree, but the walk may have zero speed when too much time is spent at leaves. In Theorem 4.1, we show that for Galton–Watson trees, the speed is positive iff $\lambda > f'(q)$, where $q$ is the extinction probability.

2 Linear growth of the range

For the speed of $\text{RW}_\lambda$ to be positive, certainly the range of $\text{RW}_\lambda$ must grow linearly in the number of steps taken. In this section, we establish that when $\lambda > 1$, the range grows linearly for any tree on which $\text{RW}_\lambda$ is transient; this is false for $\lambda = 1$. We begin with an a priori bound on the Green function.
Let \( G(x, y) := \sum_{n=0}^{\infty} P_{x}[X_n = y] \) be the Green function of RW_\lambda on \( T \), i.e., the expected number of visits to \( y \) when the walk starts at \( x \). Let \( d(x) \) denote the number of children of a vertex \( x \).

**Proposition 2.1** Let \( \lambda > 1 \) and let \( T \) be any tree on which RW_\lambda is transient. Then for every vertex \( x \in T \), we have

\[
G(x, x) \leq \frac{d(x) + \lambda}{\lambda - 1} G(\rho, \rho).
\]  

(2.1)

*Proof.* Let \( \tilde{G}(x, x) \) denote the expected number of visits to \( x \) before visiting \( \rho \) when starting from \( x \). Let \( f(x, y) := P_{x}[\exists n > 0 X_n = y] \) denote the probability of visiting \( y \) when starting at \( x \) (ignoring the initial visit if \( x = y \)), and let \( \tilde{f}(x, y) \) denote the probability of visiting \( y \) before visiting \( \rho \) when starting at \( x \).

By considering separately the path before and after the first visit to \( \rho \), we see that

\[
G(x, x) \leq \tilde{G}(x, x) + f(x, \rho) f(\rho, x) G(x, x) \leq \tilde{G}(x, x) - f(\rho, x) G(x, x)
\]

and therefore

\[
G(x, x) \leq \frac{\tilde{G}(x, x)}{1 - f(\rho, x)} = \tilde{G}(x, x) G(\rho, \rho).
\]  

(2.2)

(This is valid for any transient Markov chain.) Denote by \( x_* \) the parent of the vertex \( x \) (i.e., the neighbor of \( x \) that is closer to the root), and observe that

\[
1 - \tilde{f}(x, x) \geq \frac{\lambda}{d(x) + \lambda} \left( 1 - \tilde{f}(x_*, x) \right).
\]

By comparing the steps of RW_\lambda on the path connecting \( \rho \) and \( x \) to a simple asymmetric random walk on the integers, and using a standard result on gambler's ruin, we find that \( \tilde{f}(x_*, x) \leq 1/\lambda \). Therefore

\[
1 - \tilde{f}(x, x) \geq \frac{\lambda}{d(x) + \lambda} \left( 1 - \frac{1}{\lambda} \right) = \frac{\lambda - 1}{d(x) + \lambda}.
\]  

(2.3)

Since \( \tilde{G}(x, x) = 1/(1 - \tilde{f}(x, x)) \), combining (2.3) and (2.2) yields (2.1). \( \square \)

Let \( R_n \) be the number of distinct vertices visited by time \( n \). Our next proposition is interesting in itself.

**Proposition 2.2** Let \( \lambda > 1 \) and let \( T \) be any tree on which RW_\lambda is transient. Then for all \( n \geq 1 \),

\[
\frac{E[R_n]}{n} \leq \frac{1}{n} + \frac{\lambda - 1}{2\lambda G(\rho, \rho) + (\lambda - 1)}.
\]

*Proof.* For every \( k \leq n \), we have

\[
P[\forall j \in (k, n) X_j = X_1 | X_1] \leq G(X_1, X_n)^{-1}.
\]
Since $R_n$ is the number of epochs at which a vertex is visited for the last time, it follows that
\[
\mathbb{E}[R_n] = 1 + \mathbb{E} \left[ \sum_{k=0}^{n-1} \mathbb{1}(\{\mathbb{E}(X_0) + X_k\}) \right] \geq 1 + \mathbb{E} \left[ \sum_{k=0}^{n-1} G(X_k, X_{k+1}) \right] \\
\geq 1 + (\lambda - 1)G(\rho, \rho)^{-1} \mathbb{E} \left[ \sum_{k=0}^{n-1} \frac{1}{d(X_k) + \lambda} \right] \tag{2.4}
\]
by Proposition 2.1. This bound is effective when the typical degrees are small.
To handle large degrees, note that for $x \neq \rho$, the drift at $x$ is
\[
\mathbb{E}(|X_{n+1}| - |X_n| \mid X_n = x) = \frac{d(x) - \lambda}{d(x) + \lambda}.
\]
Therefore,
\[
\mathbb{E}[R_n] \geq 1 + \mathbb{E}[|X_n|] \geq 1 + \mathbb{E} \left[ \sum_{k=0}^{n-1} \frac{d(X_k) - \lambda}{d(X_k) + \lambda} \right]. \tag{2.5}
\]
Now multiply (2.4) by $2\lambda G(\rho, \rho)/(\lambda - 1)$ and add to (2.5). After a small amount of algebra, we obtain the proposition. \(\square\)

Remark. The expected range can grow linearly even when $\text{RW}_1$ is recurrent, as can be checked for the case $\lambda = 2$ on the binary tree.

3 Speed

Our aim in this section is to prove the following theorem.

**Theorem 3.1** For $1 < \lambda < m$ and for a.e. Galton–Watson tree $T$ upon non-extinction, the limit $\lim_{n \to \infty} |X_n|/n$ exists a.s. and is a positive constant depending only on $\lambda$ and the offspring distribution. A lower bound is
\[
\frac{(1 - \lambda^{-1})^3}{12} (1 - q_1)^3,
\]
where $q_1$ is the smallest nonnegative number satisfying
\[
f(1 - \lambda^{-1}(1 - q_1)) = q_2.
\]

Our proof relies on the existence of infinitely many regeneration epochs, where, given a path $(X_0, X_1, \ldots)$, we call $n > 0$ a fresh epoch if $X_n \neq X_k$ for all $k < n$ and a regeneration epoch if, in addition, $X_{n-1} \neq X_k$ for all $k > n$. Define $\gamma(T)$ to be the probability that, for the tree $T'$ gotten by adjoining a new vertex to the root of $T$ and designating it the root of $T'$, the walk $\text{RW}_1$ on $T'$ never returns to its root. This is the same as the effective conductance from the root of $T'$ to infinity when edges at distance $n$ from the root of $T'$ have conductance $\lambda^{-n}$. To establish that there are infinitely many regeneration epochs, we work on the space of trees, not, as in Sect. 2, on only one tree. At first reading, we recommend that the reader consider only the case $p_0 = 0$. 

Random walks on Galton-Watson trees

For this and other proofs, let $P_{\text{non}}$ and $E_{\text{non}}$ denote probability and expectation conditional on nonextinction.

**Lemma 3.2** Let $A$ be a measurable set of infinite trees and $\mathcal{F}_n$ be the $\sigma$-field generated by the events $\{X_i = X_j\}$ for $0 \leq i < j \leq n$. Let $\alpha$ be a stopping time with respect to $\{\mathcal{F}_n\}$ such that $\alpha$ is a fresh epoch and let $T^\alpha$ denote the descendant subtree of $X_\alpha$. Then

$$P_{\text{non}}[T^\alpha \in A | \mathcal{F}_n] = P_{\text{non}}[T \in A].$$

**Proof.** This lemma expresses a strong Markov property, which is evident without the conditioning on nonextinction. Since each of the events $T^\alpha \in A$ and $T \in A$ implies nonextinction of $T$, we have

$$P_{\text{non}}[T^\alpha \in A | \mathcal{F}_n] = \frac{P[T^\alpha \in A | \mathcal{F}_n]}{1 - q} = \frac{P[T \in A]}{1 - q} = P_{\text{non}}[T \in A].$$

**Lemma 3.3** Let $1 < \lambda < m$. For a.e. Galton-Watson tree $T$ upon nonextinction and a.e. sample path of RW$_h$, there are infinitely many regeneration epochs.

**Proof.** Condition throughout on nonextinction. It suffices to show that for any $N$, there is a.s. a regeneration epoch $n \geq N$. Since $T$ is infinite, there is a.s. a fresh epoch $n \geq N$; let $\alpha$ be the first such. From Lemma 3.2, with the same notation, we have

$$P_{\text{non}}[\exists \text{ a regeneration epoch } \geq N | \mathcal{F}_N] \geq P_{\text{non}}[\alpha \text{ is a regeneration epoch}| \mathcal{F}_N] = E_{\text{non}}[\gamma(T)].$$

Denote by $\mathcal{F}_\infty$ the join of all the $\sigma$-fields $\mathcal{F}_n$. By martingale convergence, the conditional probability of a regeneration epoch after $N$ given $\mathcal{F}_\infty$ is almost surely

$$\lim_{k} P_{\text{non}}[\exists \text{ regeneration } \geq N | \mathcal{F}_{N+k}] \geq \lim_{k} \inf P_{\text{non}}[\exists \text{ regeneration } \geq N + k | \mathcal{F}_{N+k}] \geq E_{\text{non}}[\gamma(T)].$$

Since the regeneration epochs are $\mathcal{F}_\infty$-measurable, there is a.s. a regeneration epoch after each $N$. □

Let the regeneration epochs be $0 < \tau_1 < \tau_2 < \ldots$. These are defined only on the event of nonextinction.

**Proposition 3.4** For $1 < \lambda < m$, on the event of nonextinction, the differences between successive regeneration epochs $\{\tau_{n+1} - \tau_n\}_{n \geq 1}$ are i.i.d. as are the increments $\{|X_{\tau_{n+1}} - X_{\tau_n}|\}_{n \geq 1}$.

**Proof.** The proof of this intuitively clear assertion requires more formal notation. Label the edges from each vertex $x$ to its children by the integers $1, \ldots, d(x)$ so that each vertex is identified with the sequence of labels leading to it from the root. This identifies the tree $T$ with a set $[T]$ of finite sequences
of positive integers. For every vertex $x$, let $T(x)$ denote the tree of descendants of $x$, rooted at $x$; we identify $T(x)$ with the set $[T(x)]$ of sequences which, when appended to the sequence identifying $x$, correspond to vertices in $T$. A (finite or infinite) path $X := (X_k; k \geq 0)$ is described by the sequence of non-negative integers $X := (X_k; k \geq 0)$, where $X_k$ is 0 if $X_k$ is the parent of $X_{k-1}$ and is otherwise the label on the edge from $X_{k-1}$ to $X_k$. Here, as in the sequel, we use angle brackets $\langle \cdots \rangle$ to denote a sequence (rather than a set).

Conditional on the event of nonextinction, the sequence of fresh trees $T(X_{\tau_n})$ seen at regeneration epochs is clearly stationary, but not i.i.d. However, as we establish below, the part of a tree between regeneration epochs, together with the path taken through this part of the tree, is independent of the rest of the tree and of the rest of the walk. We call this part a slab (see Fig. 2):

$$\text{Slab}_n := \{(T(X_{\tau_n}) \setminus T(X_{\tau_{n+1}}) \cup X_{\tau_{n+1}}); (X_{\tau_{n+1}}, \ldots, X_{\tau_{n+1}})\}. \quad (3.2)$$

(These are defined only on the event of nonextinction. Note that Slab$_n$ is rooted at $X_{\tau_n}$.) The stationarity of the sequence of fresh trees seen at regeneration epochs implies that the random variables Slab$_n$ are identically distributed.

Now we demonstrate that the slabs are mutually independent given nonextinction, which implies the proposition.

Note that for $k \leq n$, the variables $X_k$ are measurable with respect to $\{X_k; k < \tau_n\}$; in particular, $\tau_n$ is just the length of this sequence. Thus it suffices to show that for $n \geq 1$, the fresh tree $T(X_{\tau_n})$ and the remaining walk $\langle X_{\tau_{n+1}}; k \geq 1 \rangle$ are independent of $\{T\setminus T(X_{\tau_n}) \cup \{X_{\tau_n}\}\}$ and $\langle X_{\tau_{n+1}}; k \leq \tau_n \rangle$ given nonextinction. Define the maps $\phi$, and $\psi$ by

$$\phi((T); X) := ((T \setminus T(X)) \cup X); \langle X_{\tau_{n+1}}; 1 \leq k \leq \tau_n \rangle)$$

and

$$\psi((T); X) := ((T(X)); \langle X_{\tau_{n+1}}; k \geq 1 \rangle).$$
Let $GW$ be the measure on trees given by the Galton–Watson process and let $P = RW_t \times GW$ be the associated probability measure defined on a space $\Omega$ of paths in trees. Let $T$ be a Galton–Watson tree and let $X'$ be a sample from $RW_t$ on the enlarged tree $T'$ started, however, at the root of $T$. Let $Q$ be the distribution of the pair $(T'; X')$ [not $(T'; X')$], so that $Q$ is a probability measure on a space $\Omega'$ which contains $\Omega$, in the sense that the set of pairs $(T'; X') \in \Omega'$ such that $X'$ remains in $T$ may be identified with $\Omega$. Note that $Q(\Omega') = E[\gamma(T')]$.

Likewise, for any time $t$, we have $P[\psi_t \in \Omega \mid t$ fresh, $\phi_t] = Q(\Omega)$. More generally, for any event $B \subseteq \Omega$, we have

$$P[\psi_t \in B \mid t$ fresh, $\phi_t] = Q(B).$$

For $1 \leq k < t$, denote by $C'_t$ the event that $t$ is a fresh epoch and that there are exactly $k$ regeneration epochs before time $t$ when the walk is killed at time $t$. Let $\Omega_{non}$ be the intersection of $\Omega$ and the event of nonextinction. Then for any time $t$, any positive integer $n$, and any events $B \subseteq \Omega_{non}$ and $F$, we have by (3.3) that

$$P[\psi_t \in B, \phi_t \in F, \tau_n = t] = P[\psi_t \in B, \phi_t \in F, C'_n] = Q(B)P[C'_n, \phi_t \in F].$$

Therefore,

$$P[\psi_t \in B, \phi_t \in F] = \sum_{i \in \mathbb{N}} Q(B)P[C'_n, \phi_t \in F]$$

$$= \frac{Q(B)}{Q(\Omega_{non})} \sum_{i \in \mathbb{N}} P[C'_n, \phi_t \in F]Q(\Omega_{non})$$

$$= \frac{Q(B)}{Q(\Omega_{non})} \sum_{i \in \mathbb{N}} P[\psi_t \in F, \tau_n = t]$$

$$= \frac{Q(B)}{Q(\Omega_{non})} P[\phi_{\tau_n} \in F].$$

In the case that $F$ is the whole universe, $\{\phi_{\tau_n} \in F\}$ is the event of nonextinction and we get $P[\psi_n \in B] = (1 - q)Q(B)/Q(\Omega_{non})$. Substitution into (3.4) yields

$$P[\psi_n \in B, \phi_{\tau_n} \in F] = \frac{P[\psi_n \in B]}{1 - q} \cdot \frac{P[\phi_{\tau_n} \in F]}{1 - q},$$

which establishes the desired independence. \(\square\)

**Corollary 3.5** For $1 < \lambda < m$, the differences between successive regeneration epochs, $\{\tau_{n+1} - \tau_n\}_{n \geq 1}$, have finite means conditional on the event of nonextinction. An upper bound on their mean is the reciprocal of (3.1).

**Proof.** The expected number of regeneration epochs in $[1, n]$ is the sum over $k \in [1, n]$ of the probability that $k$ is a regeneration epoch. For each $k$, this is $E[\gamma(T)]$ times the probability that $k$ is a fresh epoch. The sum over $[1, n]$ of the probabilities that $k$ is a fresh epoch equals $E[R_n]$. Therefore, by Proposition 2.2,
the expected number of regeneration epochs grows linearly in time with a lower bound of
\[
\lim_{n \to \infty} \mathbb{E}[\gamma(T)] \mathbb{E} \left[ \frac{R_n}{n} \right] \geq \mathbb{E}[\gamma(T)] \mathbb{E} \left[ \frac{\lambda - 1}{3\lambda G(\rho, \rho)} \right] = \frac{\lambda - 1}{3\lambda} \mathbb{E}[\gamma(T)]^2. \tag{3.5}
\]

Since the times between regeneration epochs are i.i.d. given nonextinction, it follows by the strong law of large numbers that \( \mathbb{E}_{\text{non}}[\tau_2 - \tau_1] < \infty \). Moreover, according to (3.5), an upper bound for their mean is \( 3\lambda/[(\lambda - 1)\mathbb{E}[\gamma(T)]^2] \).

In order to make this bound more explicit, we use the connection between random walks and percolation of Lyons (1992). Define \( \gamma'(T) \) to be the effective conductance from the root of \( T' \) to infinity when the edge from the root of \( T' \) to the root of \( T \) has unit conductance, while edges at distance \( n \geq 1 \) from the root of \( T' \) have conductance \( \lambda^{1-n}/(\lambda - 1) \). Also, let \( p(T) \) be the probability that the component of the root of \( T \) is infinite when the edges of \( T \) are removed independently with probability \( 1 - \lambda^{-1} \) each. Then the inequality at the bottom of p. 2047 of Lyons (1992) says that
\[
\gamma'(T) \leq p(T) \leq 2\gamma'(T).
\]

It is easy to calculate that \( \gamma(T) \geq (\lambda - 1)\gamma'(T)/\lambda \), whence
\[
\mathbb{E}[\gamma(T)] \geq \frac{\lambda - 1}{2\lambda} \mathbb{E}[p(T)] = \frac{\lambda - 1}{2\lambda} (1 - q_1),
\]
since \( \mathbb{E}[p(T)] \) is the probability of nonextinction of a Galton-Watson branching process with probability generating function \( s \mapsto f(1 - \lambda^{-1} + \lambda^{-1}s) \).

**Proof of Theorem 3.1.** Condition on nonextinction. By the strong law of large numbers, \( \tau_n/n \to \mathbb{E}_{\text{non}}[\tau_2 - \tau_1] \) a.s. and \( |X_\tau_n|/n \to \mathbb{E}_{\text{non}}[|X_2| - |X_1|] \) a.s. Therefore,
\[
\frac{|X_\tau_n|}{\tau_n} \to \frac{\mathbb{E}_{\text{non}}[|X_2| - |X_1|]}{\mathbb{E}_{\text{non}}[\tau_2 - \tau_1]} \text{ a.s.}
\tag{3.6}
\]

Since \( \lim \tau_n/n \) exists and is finite by Corollary 3.5, we have \( \tau_{n+1}/\tau_n \to 1 \) and the theorem follows. The lower bound arises from the upper bound in Corollary 3.5 and the observation that the numerator of (3.6) is at least 1.

### 4 Outward-biased random walks

If \( \lambda < 1 \) and \( \rho_0 = 0 \), the argument of the preceding section works to give the existence and positivity of the speed of \( \text{RW}_\lambda \), provided we substitute the easy (2.5) for Proposition 2.2. Thus, when \( \lambda < 1 \), the most interesting possibility occurs when \( \rho_0 > 0 \); the walk may have zero speed by spending too much time at leaves. Recall that \( q \) is the extinction probability of the Galton-Watson process.

**Theorem 4.1** Suppose that \( \rho_0 > 0 \). Let \( T \) be a Galton-Watson tree conditioned on nonextinction. The speed of \( \text{RW}_\lambda \) exists and is constant a.s. It is positive if \( f'(q) < \lambda < 1 \) and zero if \( 0 \leq \lambda \leq f'(q) \).
Random walks on Galton–Watson trees

Fig. 3. Part of the tree $T_f$ decomposed as the tree $T_{g}$ (solid lines) together with bushes (dashed lines)

Proof: Since the case $\lambda = 0$ is obvious, we assume that $\lambda > 0$. Let $\rho(s) := f(s) - f(qs)/(1 - q)$ and $h(s) := f(qs)/q$. Then an $f$-Galton–Watson tree $T_f$ conditioned on nonextinction may be generated by first generating a $g$-Galton–Watson tree $T_g$ and then appending to each vertex $x$ of $T_g$ a random number $N_x$ of $h$-Galton–Watson shrubs, where $N_x$ has a distribution dependent on $d_g(x)$ only and, given $T_g$ and the numbers $N_x$, the shrubs are i.i.d. We shall not need the explicit form of the distribution of $N_x$ (see Lyons (1992)). Call the union of the $N_x$ shrubs at $x$ a bush (see Fig. 3).

If we observe $\text{RW}_1$ on $T_f$ only at the times $\sigma_n$ that it makes a transition along an edge of $T_g$, then we see a sample $Y_n := X_{\sigma_n}$ of $\text{RW}_1$ on $T_g$. Between these observations, there are excursions of random lengths, possibly zero. To determine the lengths of these excursions, we consider a single bush. The expected length of time that $\text{RW}_1$ takes to return to the root on a fixed finite tree $\Gamma$ is equal to the reciprocal of the stationary probability of the root of $\Gamma$. Since $\text{RW}_1$ is reversible, this is easily calculated to be $2\sum_{n=1}^\infty \Gamma_n \lambda^{1-n}/\Gamma_1$, where $\Gamma_n$ is the number of vertices in generation $n$. In particular, for $h$-Galton–Watson bushes, this sum has expectation

$$2\sum_{n=1}^\infty h'(1)^{n-1}\lambda^{1-n} = \begin{cases} \frac{2}{1-f'(q)\lambda^{-1}} & \text{if } \lambda > f'(q), \\ \infty & \text{otherwise}. \end{cases} \quad (4.1)$$

When $0 < \lambda \leq f'(q)$, it follows that the expected time between regeneration epochs on $T_f$ is infinite, whence by the strong law of large numbers, the speed is a.s. zero. (Note that the expected distance between successive regeneration loci on $T_f$ is the same as on $T_g$, hence is finite.)

Now assume that $f'(q) < \lambda < 1$. Between times $\sigma_n$ and $\sigma_{n+1}$, the walk $\langle X_t \rangle$ makes a random number of excursions into the bush at $Y_n$. The number of excursions has a geometric distribution minus 1 with mean $(d_T(Y_n) - d_T(Y_n))/(\lambda + d_T(Y_n))$. In conjunction with (4.1), this implies that

$$E_{\text{non}}[\sigma_{n+1} - \sigma_n | Y_n] \leq c d_T(Y_n) \quad (4.2)$$

for some constant $c$ depending only on $\lambda$ and $f$. Let $Z_1, \ldots, Z_{K_n}$ be the distinct vertices among $Y_1, \ldots, Y_n$. Let $U_i = \sum_{j=1}^n 1_{\{Y_j = Z_i\}}$. Then

$$\sum_{i=1}^n d_T(Y_i) \leq \sum_{k=1}^{K_n} U_k d_T(Z_k),$$
so that
\[ E_{\text{non}} \left[ \sum_{i=1}^{n} d_{\mathcal{H}}(Y_i) \right] \leq E_{\text{non}} \left[ \sum_{k=1}^{n} U_k d_{\mathcal{H}}(Z_k) \right]. \]

For each \( k \), comparison to asymmetric simple random walk and use of Lemma 3.2 gives
\[ E_{\text{non}}[U_k d_{\mathcal{H}}(Z_k)] = E_{\text{non}}[d_{\mathcal{H}}(Z_k)] E_{\text{non}}[U_k d_{\mathcal{H}}(Z_k)] \]
\[ \leq E_{\text{non}} \left[ d_{\mathcal{H}}(Z_k) \right] \frac{1 + \lambda}{1 - \lambda} = \frac{m}{1 - q} \frac{1 + \lambda}{1 - \lambda}. \]

Therefore,
\[ E_{\text{non}} \left[ \sum_{i=1}^{n} d_{\mathcal{H}}(Y_i) \right] \leq n \frac{m}{1 - q} \frac{1 + \lambda}{1 - \lambda}. \]

In conjunction with (4.2), this yields
\[ E_{\text{non}}[\sigma_n/n] \leq \frac{cm(1 + \lambda)}{(1 - q)(1 - \lambda)}, \]

whence by Fatou's lemma
\[ \liminf_{n \to \infty} \sigma_n/n < \infty \text{ a.s.} \]

Because regenerations occur with positive frequency on \( T_{\alpha} \), it follows that \( \liminf_{n \to \infty} \tau_k/k < \infty \text{ a.s.} \),
where \( \tau_k \) are the regeneration epochs of \( X \). By the strong law of large numbers,

it follows that \( E[\tau_{k+1} - \tau_k] < \infty \), and the above limit is a limit a.s. with

constant value \( E[\tau_2 - \tau_1] \). Now for \( \tau_k \leq n < \tau_{k+1} \), we have

\[ |X_n| \leq |X_{\tau_k}| \leq |X_{\tau_k} + n - \tau_k| \leq |X_{\tau_k} + \tau_{k+1} - \tau_k| \text{ since } \lim \frac{\tau_{k+1}}{\tau_k} = 1, \]

it follows that
\[ \lim_{n \to \infty} \frac{|X_n|}{n} = \lim_{k \to \infty} \frac{|X_{\tau_k}|}{\tau_k} \geq \lim_{k \to \infty} k/\tau_k > 0. \]

5 Dimension of harmonic measure

Recall that the Hausdorff dimension of a Borel measure \( \nu \) on a metric space

is defined as the infimum of Hausdorff dimensions of Borel sets with full \( \nu \)-measure.

Given a rooted tree \( T \), let \( \partial T \) denote the set of infinite self-avoiding paths

from the root of \( T \). This becomes a compact metric space when equipped with

the standard metric that assigns distance \( e^{-n} \) to any pair of self-avoiding paths

with exactly \( n \) edges in common. The Hausdorff dimension of \( \partial T \) is \( \log m \) for

a.e. Galton–Watson tree \( T \) (Hawkes 1981). Let \( \text{UNIF}_T \) denote the measure

on \( \partial T \) which is the weak limit of measures uniform on the vertices in the

nth generation of \( T \); this limit exists on a.e. Galton–Watson tree \( T \); see, e.g.,

Eq. (6.2) in Lyons et al. (1995). When the random walk \( RW \) is transient

and cycles are erased from the path, the path converges almost surely to an

element of \( \partial T \) whose law is denoted \( \text{HARM}_T \). Let \( \text{HARM}^4 \) be the function

which assigns to every tree \( T \) the probability measure on its first generation

corresponding to \( \text{HARM}_T \), i.e.,

\[ \text{HARM}^4(T)(x) = \text{HARM}_T \{ \text{paths passing through } x \} \]
for a vertex $x$ in the first generation of $T$. This gives transition probabilities for a Markov chain on the space of trees if we let $\text{HARM}^4_T(x)$ be the transition probability from $T$ to the descendant tree $T(x)$.

Call $t$ an exit epoch for the path $(X_k; k \geq 0)$ if $X_{k-1}$ is the parent of $X_k$ and $X_k \neq X_{k-1}$ for all $k > t$. Let $(t_k)$ be the successive exit epochs. Then $(X_{t_k})$ forms a random ray of $T$ with distribution $\text{HARM}^4_T$ by definition. Therefore,

$$\text{The subtrees } T(X_{t_k}) \text{ form a } \text{HARM}^4_T-\text{Markov chain.} \quad (5.1)$$

For a fixed offspring distribution, let $GW$ denote the resulting Galton–Watson measure on the space of trees.

**Theorem 5.1** For $0 \leq \lambda < m$, conditional on nonextinction, the Hausdorff dimension of $\text{HARM}^4_T$ is $GW$-a.s. strictly less than $\log m$. For $0 \leq \lambda_1 < \lambda_2 < m$, the measures $\text{HARM}^{\lambda_1}_T$ and $\text{HARM}^{\lambda_2}_T$ are $GW$-a.s. mutually singular. (We allow $\lambda = 0$ only if $p_0 = 0$.)

The proof depends on the following lemma.

**Lemma 5.2** Assume $p_0 = 0$. For $0 \leq \lambda < m$, there is a finite stationary measure for the $\text{HARM}^4_T$-Markov chain, denoted $\mu_{\text{HARM}}$, that is absolutely continuous with respect to $GW$.

**Proof of Theorem 5.1.** Because of the decomposition described in the previous section, the theorem reduces to the case $p_0 = 0$. Theorem 7.1 of Lyons et al. (1995) shows that the dimension of $\text{HARM}^4_T$ will be a.s. less than $\log m$ as long as $\text{HARM}^4_T$ has a stationary measure absolutely continuous with respect to $GW$, and as long as $\text{HARM}^4_T$ is not a.s. equal to $\text{UNIF}_T$. The argument of Proposition 8.3 in that paper applies in the present case to show that $\text{HARM}^4_T$ is not a.s. equal to $\text{UNIF}_T$, and Lemma 5.2 of the present work thus shows that $\text{dim}(\text{HARM}^4_T) < \log m$ a.s. Theorem 7.1 of Lyons et al. (1995) also shows that $\text{HARM}^{\lambda_1}_T$ and $\text{HARM}^{\lambda_2}_T$ are a.s. mutually singular if they are not a.s. equal. To see that they are a.s. unequal, note that a.s. equality would force the vector

$$\begin{bmatrix} \gamma_{\lambda_1}(T(x)) \\ \gamma_{\lambda_2}(T(x)) \end{bmatrix}_{x \in \Lambda} \quad (5.2)$$

to be a multiple of the constant vector $1$ since

$$\text{HARM}^4_T(x) = \frac{\gamma_{\lambda}(T(x))}{\sum_{y=1}^{\Lambda} \gamma_{\lambda}(T(y))}.$$ 

For Galton–Watson trees, each component of this vector has the same law as that of $\gamma_{\lambda_1}(T)/\gamma_{\lambda_2}(T)$. Thus, the independence of $T(x)$ and $T(y)$ for two distinct children $x$ and $y$ of the root implies that the random vector (5.2) is, in fact, constant $GW$-a.s. Thus, $\gamma_{\lambda_1}(T)/\gamma_{\lambda_2}(T)$ is a constant $GW$-a.s. This is
easily seen to imply that some \( p_k \) equals 1, which contradicts our standing assumption. □

Proof of Lemma 5.2. The case \( \lambda = 1 \) was done in Lyons et al. (1995), so assume that \( \lambda \neq 1 \). We provide only a sketch due to space restrictions. Let \( \Psi_n := \langle T(X_{n0}), T(X_{n1+1}), \ldots, T(X_{n+1}) \rangle \) be the sequence of forward trees seen by the walk during the \( n \)th slab. Then \( \langle \Psi_n; n \geq 1 \rangle \) is a stationary Markov chain. There is at least one exit epoch occurring in each slab, namely, \( \tau_n \). For each \( n \), let \( \Phi_n \) be the finite sequence of trees \( \langle T(X_i); i \text{ an exit epoch in the } n \text{th slab} \rangle \). Thus, \( \langle \Phi_n; n \geq 1 \rangle \) is a factor of \( \langle \Psi_n; n \geq 1 \rangle \). Let \( h(\langle \Phi_n \rangle) \geq 1 \) be the length of the sequence \( \Phi_1 \). The tower over \( \langle \Phi_n \rangle \) with height function \( h \) yields a shift-invariant distribution for \( \langle T(X_n) \rangle \). Examination of the tower construction shows that this last sequence is a \( \text{HARM}_r^d \)-Markov chain. It is necessarily stationary, with some initial distribution \( \mu_{\text{HARM}} \).

It remains to prove that \( \mu_{\text{HARM}} \) is absolutely continuous with respect to \( \text{GW} \). Now for any Borel subset \( A \) of trees,

\[
\mu_{\text{HARM}}(A) \leq \int \sum_{n=1}^{\tau_2-1} 1_A(T(X_n)) \, d\text{GW} =: \nu(A).
\]

Thus, it suffices to show that if \( \text{GW}(A) = 0 \), then \( \nu(A) = 0 \). Indeed,

\[
\nu(A) \leq \int \sum_{v \in T} 1_A(T(v)) \, d\text{GW}.
\]

For each vertex \( v \) in a Galton-Watson tree \( T \), the forward tree \( T(v) \) is also a Galton-Watson tree, so the last integral vanishes. □

We now demonstrate how the drop in dimension of harmonic measure implies the confinement of \( \text{RW}_\lambda \) to a smaller subtree. Given a tree \( T \) and positive integer \( n \), let \( T_n \) be the vertices of \( T \) at distance \( n \) from the root and \( |T_n| \) be the cardinality of \( T_n \). We remark that the following proof is both easier and more general than the analogous proof of Theorem 9.9 in Lyons et al. (1995).

Corollary 5.3 Assume that \( p_0 = 0 \). Fix an offspring distribution and \( \lambda \in [0,m) \). For \( \text{GW} \)-almost all trees \( T \) and for every \( \varepsilon > 0 \), there is a subtree \( T^{(\varepsilon)} \subseteq T \) such that

\[
\text{RW}_\lambda \{ X_n \in T^\varepsilon \text{ for all } n \} \geq 1 - \varepsilon \quad (5.3)
\]

and

\[
\frac{1}{n} \log |T^{(\varepsilon)}| \to \dim(\lambda),
\]

where \( \dim(\lambda) < \log m \) is the dimension of \( \text{HARM}_r^d \). Furthermore, any subtree \( T^{(\varepsilon)} \) satisfying (5.3) must have growth

\[
\liminf \frac{1}{n} \log |T^{(\varepsilon)}| \geq \dim(\lambda).
\]
Random walks on Galton–Watson trees

Proof. Let \( t_k := 1 + \max \{ t; |X_t| = k \} \) be the \( k \)th exit epoch and \( D(x,k) \) be the set of descendants \( y \) of \( x \) with \( |y| \leq |x| + k \). We shall use three sample path properties of RW\(_k\) on a fixed tree:

\[
speed: \lim_{n \to \infty} \frac{|X_n|}{n} = \text{speed}(\lambda) > 0 \text{ a.s.} \tag{5.4}
\]

\[
\text{H\ölder exponent: } \lim_{n \to \infty} \frac{1}{k} \log \frac{1}{\text{HARM}_k^*(X_n)} = \dim(\lambda) \text{ a.s.} \tag{5.5}
\]

\[
\text{neighborhood size: } \forall \delta > 0 \limsup_{n \to \infty} \frac{\log |D(X_n, \delta|X_n|)|}{|X_n|} \leq \delta \log m \text{ a.s.} \tag{5.6}
\]

(In fact, the limit in (5.6) exists and equals the right-hand side, but this is not needed.) The first property (5.4) was proved in Sect. 2 and the second (5.5) follows from a result of Billingsley and an idea of Furstenberg once the absolute continuity in Lemma 5.2 has been established; see Lyons et al. (1995), Lemma 4.1 and Sect. 5. In order to see that (5.6) holds for GW-a.e. tree, denote by \( Y_k \) the \( k \)th fresh point visited by RW\(_k\). Then (5.6) can be written as

\[
\forall \delta > 0 \limsup_{k \to \infty} \frac{|Y_k|\delta^{-1} \log |D(Y_k, \delta|Y_k|)|}{|Y_k|} \leq \delta \log m
\]

and since \( |Y_k|/k \) has a positive a.s. limit, this is equivalent to

\[
\forall \delta^* > 0 \limsup_k \frac{k^{-1} \log |D(Y_k, \delta^*k)|}{|Y_k|} \leq \delta^* \log m. \tag{5.7}
\]

Now the random variables \( |D(Y_k, \delta^*k)| \) are identically distributed, though not independent. Indeed, the descendant subtree of \( Y_k \) has the law of GW. Since the expected number of descendants of \( Y_k \) at generation \( |X_k| + j \) is \( m^j \) for every \( j \), we have

\[
P(|D(Y_k, \delta^*k)| \geq m^{\delta^*k}) \leq m^{-\delta^*k} \sum_{j=0}^m m^j.
\]

If \( \delta' > \delta^* \), then the right-hand side decays exponentially in \( k \), so by the Borel–Cantelli lemma, we get (5.7), hence (5.6).

Now (5.5) alone implies the last assertion of Corollary 5.3.

Applying Egorov's theorem to the two almost sure asymptotics (5.4) and (5.5), we see that for each \( \varepsilon > 0 \), there is a set of paths \( A_\varepsilon \) with \( \text{RW}_k(A_\varepsilon) > 1 - \varepsilon \) and such that the convergence is uniform on \( A_\varepsilon \). Thus, we can choose \( (\delta_n) \) decreasing to 0 such that on \( A_\varepsilon \), for all \( k \) and all \( n \),

\[
\text{HARM}_k^*(X_n) > e^{-k(\dim(\lambda) + \delta_k)} \text{ and } \left| \frac{|X_n|}{n \text{speed}(\lambda)} - 1 \right| < \delta_n. \tag{5.8}
\]

Now since \( \delta_n \) is eventually less than any fixed \( \delta \), (5.6) implies that

\[
\limsup_{n \to \infty} |X_n|^{-1} \log |D(X_n, 3\delta|X_n|)| = 0 \text{ a.s.},
\]
so applying Egorov's theorem again and replacing $A_\epsilon$ by a subset thereof (which we continue to denote $A_\epsilon$), we may assume that there exists a sequence $\langle n_k \rangle$ decreasing to $0$ such that

$$|D(X_{n_k}, 3\delta|X_{n_k}|)| \leq e^{\frac{|X_{n_k}|}{n_k}}$$

for all $n_k$ (5.9)

on $A_\epsilon$.

Define $F_0^{(e)}$ to consist of all vertices $v \in T$ such that either $\delta|v| \geq 1/3$ or both

$$\text{HARM}_T^4(v) \geq e^{-|v|(\dim(\lambda)+\delta|v|)} \quad \text{and} \quad |D(v, 3\delta|v|)| \leq e^{\frac{|v||v|}{2}}.$$

Finally, let

$$F^{(e)} = \bigcup_{v \in F_0^{(e)}} D(v, 3\delta|v|)$$

and denote by $T^{(e)}$ the component of the root in $F^{(e)}$. Since the number of vertices $v \in T_n$ satisfying $\text{HARM}_T^4(v) \geq e^{-|v|(\dim(\lambda)+\delta|v|)}$ is at most $e^{\frac{\dim(\lambda)}{\delta|v|}}$, the bound on $|D(v, 3\delta|v|)|$ bounds the growth rate from above as asserted in the statement of the corollary. It remains to establish that $\text{RW}_2$ stays inside $F^{(e)}$ forever on the event $A_\epsilon$, since that will imply that the walk is confined to $T^{(e)}$ on this event. The points visited at exit epochs $t_k$ are in $F_0^{(e)}$ by the first part of (5.8) and (5.9). Fix a path $\langle X_t \rangle$ in $A_\epsilon$ and a time $n$, and suppose that the last exit epoch before $n$ is $t_k$, so that $t_k \leq n < t_{k+1}$. Denote by $N := t_{k+1} - 1$ the time preceding the next exit epoch, and observe that $X_N = X_{t_k}$. If $\delta_n \geq 1/3$, then $X_n$ is in $F_0^{(e)}$ since $\delta_1 \geq \delta_n$, so consider the case that $\delta_n < 1/3$. By the second part of (5.8), we have

$$\frac{|X_n|}{n \text{ speed}(\lambda)} < 1 + \delta_n \quad \text{and} \quad \frac{|X_N|}{N \text{ speed}(\lambda)} \geq \frac{|X_n|}{N \text{ speed}(\lambda)} > 1 - \delta_n \geq 1 - \delta_n.$$

Dividing, we find that

$$\frac{|X_n|}{|X_N|} \leq \frac{1 + \delta_n}{1 - \delta_n} \leq (1 + 3\delta_n)|X_N|.$$

It follows that $X_n$ is in $D(X_{n_k}, 3\delta|X_{n_k}|)$ and this completes the proof. □

6 Dependence on the bias parameter $\lambda$

Fix an offspring distribution, and recall that speed $(\lambda)$ denotes the a.s. constant speed of $\text{RW}_2$ on Galton–Watson trees upon nonextinction. Similarly, denote by $\dim(\lambda)$ the a.s. constant dimension of the harmonic measure $\text{HARM}_T^4$. The methods of this paper are not well suited to analyze the dependence of speed$(\lambda)$ and $\dim(\lambda)$ on the parameter $\lambda$. We state explicitly two questions in this direction, and refer to the survey Lyons et al. (1996) for further questions and relevant examples.
**Question 1** Assume that the offspring distribution satisfies $p_0 = 0$. Is $	ext{speed}(\lambda)$ monotonic nonincreasing for $\lambda \in [0, m]$?

Though a positive answer is intuitively compelling, the evidence available indicates that if monotonicity holds, it is a special property of Galton–Watson trees. The calculations in Sect. 4 show that the assumption $p_0 = 0$ cannot be dropped. Even if we restrict attention to trees without leaves, there exist family trees of two-type Galton–Watson processes for which $	ext{speed}(\lambda)$ is not monotonic in $\lambda$ (see Lyons et al. 1996).

**Question 2** Determine the smoothness properties of $	ext{speed}(\lambda)$ and $	ext{dim}(\lambda)$ for $\lambda \in [0, m]$.

In particular, the methods of the present paper do not yield the intuitively "obvious" inequality

$$\liminf_{\lambda \to 1} \text{speed}(\lambda) > 0,$$

since the a priori bound for the Green function in Proposition 2.1 blows up as $\lambda \to 1$. Of course, continuity of the speed at $\lambda = 1$ would immediately imply (6.1).

Continuity for $\lambda < 1$ is easier to establish, since comparison with simple asymmetric random walk on the integers is possible.

**Proposition 6.1** If $p_0 = 0$, then $	ext{speed}(\lambda)$ is continuous for $\lambda \in [0, 1]$.

**Proof.** We construct a richer probability space on which random walks with laws $\text{RW}_\lambda$ are simultaneously defined for all $\lambda \geq 0$. Pick a tree $T$ according to Galton–Watson measure. Label the edges of $T$ as in the proof of Proposition 3.4. Let $\langle U_n \rangle$ be a sequence of i.i.d. random variables uniformly distributed on $[0, 1]$. For every $\lambda \geq 0$, we define inductively a sequence of vertices $\langle X_n^\lambda \rangle$ as follows. First, let $X_0^\lambda$ be the root of $T$. For $n \geq 1$, denote by $d_{n-1}(\lambda)$ the number of children of $X_{n-1}^\lambda$. If $X_n^\lambda$ is the root, then define $X_n^\lambda := [d_{n-1}(\lambda) \cdot U_n]$. Otherwise, let

$$X_n^\lambda := [(\lambda + d_{n-1}(\lambda)) \cdot U_n]$$

if the right-hand side is at most $d_{n-1}(\lambda)$, and $X_n^\lambda := 0$ if the right-hand side of (6.2) is strictly greater than $d_{n-1}(\lambda)$. This defines the path $\langle X_n^\lambda \rangle$ as in the proof of Proposition 3.4.

Given $T$, the sequence $\langle X_n^\lambda \rangle$ is clearly a sample from $\text{RW}_\lambda$. For any fixed $\lambda_0 \geq 0$ and $n \geq 1$, we clearly have pointwise convergence:

$$X_n^\lambda \to X_n^{\lambda_0} \text{ almost surely as } \lambda \to \lambda_0.$$

Pick $\lambda_{\text{max}} < 1$. Denote by $\tau_k(\lambda)$ the $k$th regeneration epoch of $\langle X_n^\lambda \rangle$. We shall show continuity of speed for $\lambda \in [0, \lambda_{\text{max}}]$ by using the formula

$$\text{speed}(\lambda) = \frac{\mathbb{E}[|X_{\tau_2(\lambda)} - X_{\tau_1(\lambda)}|]}{\mathbb{E}[	au_2(\lambda) - \tau_1(\lambda)]}.$$
Using the random variables $U_n$, we also define an asymmetric simple random walk $(Y_n)$ on the integers. Let $Y_0 := 0$ and for $n \geq 1$, let

$$Y_n := Y_{n-1} + \text{sign} \left( \frac{1}{1 + \lambda_{\text{max}}} - U_n \right).$$

Whenever $Y_n > Y_{n-1}$, necessarily $|X_n^\lambda| > |X_{n-1}^\lambda|$ for all $\lambda \in [0, \lambda_{\text{max}}]$. Therefore every regeneration epoch for the process $(X_n^\lambda)$ is also a regeneration epoch for each of the processes $(X_n^\lambda)$ with $\lambda \leq \lambda_{\text{max}}$. Denoting the $k$th regeneration epoch by $\tau_k^n$, we see that $\tau_k^n(\lambda) \leq \tau_k^n(\lambda_0)$ for all $\lambda \leq \lambda_{\text{max}}$, and therefore $\tau_k^n(\lambda) \to \tau_k^n(\lambda_0)$ when $\lambda \to \lambda_0 \leq \lambda_{\text{max}}$. Because the speed of $(Y_n)$ is positive, $\tau_k^n$ is integrable for each $k$ (indeed, it has an exponentially decaying tail – see, e.g., Lemma 5.1 in Dembo et al. (1995)). Thus, continuity of speed($\lambda$) in the interval $[0, \lambda_{\text{max}}]$ follows from (6.3), (6.4) and Lebesgue’s dominated convergence theorem.

Remark. Similarly, if $p_i = 0$ for $i < N$, then speed($\lambda$) is continuous for $\lambda \in [0, N]$.

Remark. Very similar methods allow us to deduce Theorem 3.1 for $1 < \lambda < \rho$ for positive-regular nonsingular multitype branching processes such that each particle has at least one child (a stronger condition than a.s. nonextinction, but analogous to $p_0 = 0$), where $\rho$ is the maximal eigenvalue of the mean matrix. We do not know how to prove that the speed of simple random walk ($\lambda = 1$) is positive on multitype trees.

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References