

ON WHICH GRAPHS ARE ALL RANDOM WALKS IN RANDOM ENVIRONMENTS TRANSIENT?

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SUMMARY:

An infinite graph G has the property that a random walk in random environment on G defined by i.i.d. resistances with any common distribution is almost surely transient, if and only if for some $p < 1$, simple random walk is transient on a percolation cluster of G under bond percolation with parameter p .

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Simple random walks on infinite graphs have been widely studied with particular attention to the dichotomy between recurrence and transience. In the last two decades there has been interest in random walks in random environments (RWRE's), which are mixtures of Markov chains in which the transition probabilities are picked according to some prior distribution. For reversible Markov chains, the transition probabilities may be described by associating resistances $R(vw)$ to each edge (vw) , the transition probabilities out of any vertex being proportional to the inverses of the resistances on the edges incident to that vertex:

$$p(v, w) = \frac{R(vw)^{-1}}{\sum_z R(vz)^{-1}} \quad (1)$$

where the sum is over all vertices z adjacent to v . One natural model⁴, considered in Grimmett and Kesten (1984), involves taking the resistances as i.i.d. positive random variables. A result of Adams and Lyons (1991) implies that if a tree Γ has positive Hausdorff dimension, then for an arbitrary distribution of the resistances the resulting network is almost surely transient. (See Lyons (1990) where the term *branching number* is used for the exponential of the Hausdorff dimension.) The surprising aspect of this is that no conditions are imposed on the tail of the distribution. This note proves a converse of this implication, but on a general graph. Specifically, the implication $(a) \Rightarrow (b)$ in part (ii) of the theorem below shows that almost sure transience for an arbitrary resistance distribution on a tree Γ implies $p_c < 1$ for IID bond percolation. By results in Lyons (1990), this implies that Γ has positive dimension. Part (i) is immediate from the well known connections between random walks and electrical networks (c.f. Doyle and Snell (1984)) and is included for completeness.

Theorem: *Let G be a locally finite connected graph. Suppose that a positive random variable $R(e)$ is attached to every edge e of G so that the collection $\{R(e)\}$ is i.i.d.. Let μ denote their common distribution. Consider the RWRE given by (1).*

(i) If $\int x d\mu < \infty$ and simple random walk on G is transient, then the resistor network $\{R(e)\}$ has finite net resistance almost surely, thus defining almost surely a transient random walk.

⁴see Remark 1 for another RWRE model relevant to the present discussion

(ii) The following conditions on G are equivalent: (a) for any distribution μ on $(0, \infty)$ the resistor network $\{R(e)\}$ is almost surely transient; (b) there exists some $p < 1$ such that independent bond percolation on G with parameter p contains with positive probability a cluster on which simple random walk is transient.

PROOF: We rely on two facts from the theory relating reversible random walks to the corresponding resistor networks. Firstly, the random walk is transient if and only if the network has finite resistance from some (hence every) vertex to infinity, i.e., there is some nonzero flow $\{F(e)\}$, satisfying Kirchoff's laws except at one source vertex, such that the energy

$$\sum_e F(e)^2 R(e)$$

is finite. Secondly, Rayleigh's principle asserts that the resistance of a network is monotone in each of the individual resistances. Both facts may be found in Doyle and Snell (1984). The proof of (i) follows immediately: G is transient for simple random walk, so some flow $\{F(e)\}$ on G has finite energy

$$\sum_e F(e)^2$$

with respect to unit resistances; the expected energy of this flow in the network of random resistances is

$$\mathbf{E} \sum_e R(e) F(e)^2 = \int x d\mu \sum_e F(e)^2 < \infty$$

so the resistance is almost surely finite and hence the network is almost surely transient.

(ii) The direction (b) \Rightarrow (a) is easy: assume (b) for some parameter p and consider the random subgraph of G consisting of those edges e for which $R(e) \leq Q_p$, where $1 > \mu(0, Q_p) \geq p$. Resistances have the following linearity: multiplying every resistor by C multiplies the resistance of the network by C . Thus our assumption (b) implies there is a

positive probability that some component of this subgraph will have finite resistances when all edges have resistance Q_p . By Rayleigh's law, decreasing the resistances on edges in this component from Q_p to $R(e)$ and decreasing all other resistances from ∞ to $R(e)$ gives a network with finite resistance, and therefore random walk on G with resistances $\{R(e)\}$ is transient with positive probability. Since this is a tail event for the countable collection $\{R(e)\}$, the probability is one.

Now we verify that (a) \Rightarrow (b). Assume that for every $p < 1$, simple random walk on p -percolation clusters is almost surely recurrent. Let $p_k \uparrow 1$ and construct a probability measure μ on $(0, \infty)$ as follows. Fix a vertex $v \in G$ and set $\gamma_1 = 1$. Define a measure μ_1 on $\mathbf{R} \cup \{\infty\}$ by $\mu_1(\{\gamma_1\}) = p_1$ and $\mu_1(\{\infty\}) = 1 - p_1$. The RWRE on G with resistance distribution μ_1 is just simple random walk on the p_1 -percolation clusters and is thus recurrent. Therefore there exists N_1 such that when choosing a random environment according to μ_1 , the probability is at least $1/2$ that either all edges adjacent to v have infinite resistance (i.e. v is isolated) or else the resulting RWRE started from v returns to v in the first N_1 steps. Let D_1 be the maximal degree of any vertex in G within distance N_1 of v .

For the induction step, let $k \geq 2$ and assume that γ_j, μ_j, N_j and D_j have been defined for $1 \leq j < k$ and $\mu_j(\infty) = 1 - p_j$. Pick $\gamma_k > 2N_{k-1}D_{k-1}\gamma_{k-1}$. Define μ_k by moving mass $p_k - p_{k-1}$ from infinity to γ_k , i.e.

$$\mu_k = \mu_{k-1} + (p_k - p_{k-1})(\delta_{\gamma_k} - \delta_{\infty}).$$

The RWRE with resistance distribution μ_k is recurrent since it yields an environment whose resistances on the p_k -percolation cluster are bounded between 1 and γ_k [use the aforementioned linearity together with Rayleigh's law]. Thus we may choose an integer N_k , such that if a random environment is chosen according to μ_k , the probability is at most 2^{-k} that v is not isolated and the resulting RWRE started from v fails to return to v within the first N_k steps. Finally, define D_k to be the maximal degree of any vertex of G within distance N_k of v .

Finally, set $\mu(\{\gamma_k\}) = p_k - p_{k-1}$ for all $k \geq 2$, with $\mu(\{\gamma_1\}) = p_1$, so that μ is the weak limit of the μ_k . We construct a μ -RWRE $\{S_n\}$ on the same probability space as a sequence

$(\{S_n^{(k)}\} : k = 1, 2, 3, \dots)$ of μ_k -RWRE's as follows. Let the resistances $\{R(e)\}$ be i.i.d. with common distribution μ and define

$$R^{(k)}(e) = \begin{cases} R(e) & \text{if } R(e) \leq \gamma_k \\ \infty & \text{if } R(e) > \gamma_k \end{cases}.$$

Observe that this makes $\{R^{(k)}(e)\}$ i.i.d. with distribution μ_k . Let $\{S_n\}$ be a random walk starting from v with transition probabilities determined by the resistances $\{R(e)\}$. Note that if $R(wz) \leq \gamma_k$ then the transition probability from w to z with resistances $\{R^{(k)}(e)\}$ is greater than or equal to the transition probability from w to z with resistances $\{R(e)\}$. Thus for each k , we may define a random sequence $\{S_n^{(k)}\}$ so as to have transition probabilities determined by $\{R^{(k)}\}$ and so that $S_n^{(k)} = S_n$ for all $n \leq T_k$, where T_k is the least time t for which $R(S_t S_{t+1}) > \gamma_k$.

Define events

$$\begin{aligned} A_k &= \{R(vw) > \gamma_k \text{ for all } w \text{ neighboring } v\} \subseteq \{T_k = 0\} \\ B_k &= \{R(S_j, S_{j+1}) \geq \gamma_{k+1} \text{ for some } 0 \leq j < N_k\} \setminus A_k = \cup \{0 \leq T_k < N_k\} \setminus A_k \\ C_k &= \{S_n^{(k)} \neq v \text{ for all } 1 \leq n \leq N_k\}. \end{aligned}$$

The event $G_k = \{S_n \neq v \text{ for all } 1 \leq n \leq N_k\}$ is contained in $A_k \cup B_k \cup C_k$, since on $G_k \setminus C_k$ the time T_k is less than N_k . For $1 \leq j < N_k$, the probability of the event $\{T_k = j\}$ is at most $D_k \gamma_k / \gamma_{k+1}$, since the sum of $R(e)^{-1}$ over edges incident to S_j with resistance greater than γ_k is at most D_k / γ_{k+1} and there is at least one edge incident to S_j with $R(e)^{-1} \geq \gamma_k^{-1}$. Similarly, $\mathbf{P}(\{T_k = 0\} \setminus A_k) \leq D_k \gamma_k / \gamma_{k+1}$. Thus $\mathbf{P}(B_k) \leq N_k D_k \gamma_k / \gamma_{k+1} \leq 2^{-k}$ by construction of γ_{k+1} . But N_k is defined so that $\mathbf{P}(C_k \setminus A_k) \leq 2^{-k}$ and clearly $\mathbf{P}(A_k) \leq 1 - p_k$. Therefore

$$\mathbf{P}(G_k) \leq (1 - p_k) + 2^{1-k} \rightarrow 0$$

as $k \rightarrow \infty$ and the μ -RWRE is recurrent. □

Remarks:

1. The recent paper of Grimmett, Kesten and Zhang (1991) shows that for Euclidean lattices \mathbf{Z}^d ($d \geq 3$), random walk on supercritical percolation clusters is almost surely transient, so the theorem is applicable, and all RWRE's with i.i.d. positive resistances are transient. Note that the RWRE models discussed in Durrett (1986) and Sunyach (1987) may be recurrent even when $d \geq 3$. Resistances in Durrett's model are stationary under \mathbf{Z}^d -shifts, so his model is close to the present one.

2. By Lyons (1990), the critical probability for bond percolation on a tree Γ is $p_c(\Gamma) = e^{-\dim\Gamma}$ and if $\dim(\Gamma) > 0$ and $p > p_c$, then simple random walk on any infinite cluster is transient (since the cluster must have positive dimension). For a tree, property (b) is thus equivalent to $\dim(\Gamma) > 0$. The Max Flow Min Cut Theorem (as applied in Lyons 1990) allows us to formulate this purely in terms of flows:

On a tree, property (a) is equivalent to the existence of a flow $\{F(e)\}$ satisfying Kirchoff's laws (except at the root) for unit resistors, and having the exponential decay property:

$$F(e) \leq C\rho^{\text{dist}(e)}$$

where $\text{dist}(e)$ is the distance from e to the root of Γ .

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