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WHEN ARE TOUCHPOINTS LIMITS FOR GENERALIZED PÓLYA URNS?

ROBIN PEMANTLE

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ABSTRACT. Hill, Lane, and Sudderth (1980) consider a Pólya-like urn scheme in which \( X_0, X_1, \ldots \) are the successive proportions of red balls in an urn to which at the \( n \)th stage a red ball is added with probability \( f(X_n) \) and a black ball is added with probability \( 1 - f(X_n) \). For continuous \( f \) they show that \( X_n \) converges almost surely to a random limit \( X \) which is a fixed point for \( f \) and ask whether the point \( p \) can be a limit if \( p \) is a touchpoint, i.e. \( p = f(p) \) but \( f(x) > x \) for \( x \neq p \) in a neighborhood of \( p \). The answer is that it depends on whether the limit of \( (f(x) - x)/(p - x) \) is greater or less than \( 1/2 \) as \( x \) approaches \( p \) from the side where \( (f(x) - x)/(p - x) \) is positive.

Hill, Lane, and Sudderth (1980), hereafter referred to as [HLS], consider the following urn scheme. Let \( f : [0, 1] \to [0, 1] \) be any function and let an urn begin with \( l \) balls of which a proportion \( X_{l-1} \in (0, 1) \) are red and the remainder black. Add a new ball to the urn, whose color is red with probability \( f(X_{l-1}) \) and black otherwise. Let \( X_l \) be the new proportion of red balls and iterate the procedure, producing a sequence of proportions \( X_{l-1}, X_l, X_{l+1}, \ldots \). In the case where \( f \) is continuous, they show that \( X_n \) converges almost surely to some random variable \( X \). Furthermore, \( f(X) = X \) almost surely [HLS, Theorem 2.1 and Corollary 3.1]. Categorize points \( p \in (0, 1) \) for which \( p = f(p) \) by calling them upcrossings if \( (y - p)(f(y) - y) \) is positive for all \( y \) in some neighborhood of \( p \), and downcrossings if \( (y - p)(f(y) - y) \) is negative for all \( y \) in some neighborhood of \( p \). The terminology comes from the way the graph \( y = f(x) \) crosses the graph \( y = x \). The next results of [HLS] are that \( \text{prob}(X_n \to p) > 0 \) if \( p \) is a downcrossing and \( f \) maps \( (0, 1) \) into itself, while \( \text{prob}(X_n \to p) = 0 \) if \( p \) is an upcrossing. The only other kind of isolated point, \( p \), in the set \( \{ x : x = f(x) \} \) is a touchpoint where \( f(y) > y \) for all \( y \neq p \) in a neighborhood of \( p \), or else \( f(y) < y \) for all \( y \neq p \) in a neighborhood of \( p \). They ask whether touchpoints can be in the support of the limiting random variable \( X \).

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This note answers their question both ways for continuous \( f \), giving a condition on \( f \) near \( p \) implying \( \text{prob}(X_n \to p) > 0 \) and another condition that implies \( \text{prob}(X_n \to p) = 0 \). These conditions almost meet, in the sense that they cover all cases where \( (f'(x) - x)/(p - x) \) has a limit as \( x \to p \) except for the case where the limit is equal to \( 1/2 \). By symmetry between red and black balls, there is no loss of generality in considering only touchpoints of the first kind, where \( f(y) > y \) for \( y \neq p \) in a neighborhood of \( p \). Therefore, the proofs will be given only for the touchpoints of the first kind. Furthermore, whether \( X_n \) converges to \( p \) with positive probability depends only on the germ of \( f \) at \( p \) [HLS, Lemma 4.1], so the arguments below will assume without loss of generality that \( f(y) > y \) for all \( y \neq p \), as well as assuming that \( f \) maps \((0, 1)\) into itself.

Let \( \mathcal{F}_n \) be the \( \sigma \)-algebra generated by \( \{X_i; i \leq n\} \), and let \( \mathcal{F}_\tau \) be defined similarly for any stopping time \( \tau \). The key to the proof of both conditions will be the decomposition of the submartingale \( \{X_n, \mathcal{F}_n\} \) into a martingale and an increasing process. Write \( X_{n+1} = X_n + A_n + Y_n \), where

\[
A_n = \mathbb{E}(X_{n+1} - X_n | \mathcal{F}_n)
\]

is \( \mathcal{F}_n \)-measurable and \( Y_n = X_{n+1} - X_n - A_n \), so \( \mathbb{E}(Y_n | \mathcal{F}_n) = 0 \). Then calculate the following conditional probabilities given \( \mathcal{F}_n \):

\[
X_{n+1} = \begin{cases} 
\frac{nX_n + 1}{n+1} = X_n + \frac{1-X_n}{n+1} & \text{with probability } f(X_n), \\
\frac{nX_n}{n+1} = X_n - \frac{X_n}{n+1} & \text{with probability } 1 - f(X_n).
\end{cases}
\]

This gives \( A_n = (f(X_n) - X_n)/(n + 1) \), which is nonnegative by assumption; and hence

\[
Y_n = \begin{cases} 
\frac{1-f(X_n)}{n+1} & \text{with probability } f(X_n), \\
\frac{-f(X_n)}{n+1} & \text{with probability } 1 - f(X_n).
\end{cases}
\]

Also, \( Y_n \) is a mean zero random variable given \( \mathcal{F}_n \), with the conditional distribution of \( Y_n \) given \( \mathcal{F}_n \) satisfying \( \min(f(X_n), 1 - f(X_n))^2(n + 1)^{-2} = \inf \mathbb{E}(Y_n^2 | \mathcal{F}_n) \leq \sup \mathbb{E}(Y_n^2 | \mathcal{F}_n) \leq (n + 1)^{-1} \), where the inf is over \( \omega \) in the \( \mathcal{F}_n \)-measurable set for which \( X_n \) has the given value. Defining

\[
Z_{n,m} = \sum_{i=n}^{m-1} Y_i
\]

yields for each fixed \( n \) a martingale \( \{Z_{n,m}, \mathcal{F}_m\} \) with an \( L^2 \)-bound \( \mathbb{E}Z_{n,\infty}^2 \leq \sum_{i=n}^{\infty} (i + 1)^{-2} \leq 1/n \). If \( f \) is bounded away from 0 and 1 near \( p \), then a lower \( L^2 \)-bound is gotten by stopping the process \( X_n \) when it exits an interval on which \( \min(f(X_n), 1 - f(X_n)) > b \). If \( \tau \) is any stopping time bounded above by the exit time of the interval, then the above lower bound on \( \mathbb{E}Y_m^2 \) gives

\[
\mathbb{E}(Z_{n,\infty}^2 | \mathcal{F}_n) \geq \mathbb{E}(Z_{n,\tau}^2 | \mathcal{F}_n) \geq \text{prob}(\tau = \infty | \mathcal{F}_n) b^2(n + 1)^{-1}.
\]
The idea will be that if \( f(x) - x \) is less than \( (p - x)/2 \), then the increasing part \( A \) pushes \( X \) toward \( p \) so slowly that by the time \( X \) gets close to \( p \), the increments of \( Z \) are very very small, and \( Z \) cannot push \( X \) above \( p \). So, in fact, one gets convergence to \( p \) from below. On the other hand, if \( f(x) - x \) is greater than \( (p - x)/2 \), then the increasing part pushes \( X \) toward \( p \) fast enough so that the increments of \( Z \) are big enough compared with \( p - X \), so that, eventually, the addition of \( Z \) puts \( X \) over \( p \). A result along the lines of Pemantle [P1, P2] then implies that \( X_n \) cannot converge to \( p \).

Remark. It will be shown that convergence to a touchpoint near which \( f(x) > x \) is always from the left. Thus the behavior of the function to the right of the touchpoint is irrelevant.

**Theorem 1.** Let \( f \) be continuous in a neighborhood of a touchpoint \( p \) and suppose that \( f \) maps \((0, 1)\) into itself. Further suppose that \( x < f(x) \leq x + k(p-x) \) for some \( k < 1/2 \) and all \( x \) in some left neighborhood, \((p-e, p)\), of \( p \). Then \( \text{prob}(X_n \to p) > 0 \). [Similarly, if \( x > f(x) \geq x - k(x-p) \) for some \( k < 1/2 \) and all \( x \) in a right neighborhood, \((p, p+e)\) of \( p \), then also \( \text{prob}(X_n \to p) > 0 \).]

**Corollary 2.** If \( f \) is differentiable at a touchpoint \( p \) and continuous in a neighborhood of \( p \), then \( \text{prob}(X_n \to p) > 0 \) under the same nontriviality assumption \( f((0, 1)) \subseteq (0, 1) \).

**Proof.** Since \( f(x) - x \) does not change sign at \( p \), the derivative of \( f(x) - x \) must be zero at \( p \) and Theorem 1 applies. \( \Box \)

**Proof of Theorem 1.** Replacing \( f \) by a function agreeing with \( f \) on a neighborhood of \( p \), there is no loss of generality in assuming that \( f \) is continuous and that \( f(x) > x \) for all \( x \in [0, 1) \setminus \{p\} \). Thus it will suffice to prove that with positive probability there is an \( N \) for which \( n > N \) implies \( X_n < p \), since \( X_n \) converges to a fixed point of \( f \) [HLS, Corollary 3.1], which must then be \( p \). Pick a \( k \) for which the hypothesis is satisfied and pick \( k_1 \) with \( k < k_1 < 1/2 \). Pick a constant \( \gamma \) just barely greater than 1 so that \( \gamma k_1 < 1/2 \). The function \( g(r) = re^{(1-r)/2k_1\gamma} \) has value 1 at \( r = 1 \) and derivative \( g'(1) = 1 - 1/2k_1\gamma < 0 \), so there is an \( r \in (0, 1) \) for which \( g(r) > 1 \). Fix such an \( r \). Define

\[
T(n) = e^{n(1-r)/\gamma k_1}, \quad \text{so } g(r)^n = r^n T(n)^{1/2} > 1.
\]

Choose \( M \) big enough so that \( \gamma r^M < \varepsilon \) and define

\[
\tau_M = \inf\{j > T(M) : X_{j-1} < p - r^M < X_j\}
\]

if such a \( j \) exists, and \( \tau_M = -\infty \) otherwise. By the nontriviality assumption that \( f \) maps \((0, 1)\) into itself, \( \text{prob}(\tau_M > T(M)) > 0 \). For each \( n \geq M \), define \( \tau_{n+1} = \inf\{j \geq \tau_n : X_j > p - r^{n+1}\} \). Note that if \( X_j \geq p \) for some \( j > T(M) \), then \( \tau_n \leq j \) for all \( n \geq M \). The theorem will be proved by showing that \( \text{prob}(\tau_n > T(n) \text{ for all } n \geq M) > 0 \), which will imply that with
nonzero probability, \( X_n \) is eventually less than \( p \), proving the theorem. Begin by assuming that \( \tau_n > T(n) \) and calculate \( \text{prob}(\tau_{n+1} > T(n+1) | \tau_n > T(N)) \) as follows. Let \( \mathcal{B} \) be the event \( \{ \inf_{j > \tau_n} X_j \geq p - \gamma r^n \} \) and estimate

\[
\text{prob}(\mathcal{B}^c | \tau_n > T(N)) = \text{prob}\left( \inf_{j > \tau_n} X_j < p - \gamma r^n | \tau_n > T(N) \right)
\leq \text{prob}\left( \inf_{j > \tau_n} Z_{\tau_n,j} < -(\gamma - 1)r^n | \tau_n > T(N) \right)
\leq \mathbb{E}(Z_{\tau_n,\infty}^2 | \tau_n > T(N))/((\gamma - 1)r^n)^2
\leq e^{-n(1-r)/k_i \gamma (\gamma - 1)^{-2} r^{-2n}}
= (\gamma - 1)^{-2} [g(r)]^{-2n}.
\]

Next, note that if \( \mathcal{B} \) holds, then

\[
\sum_{T(n) < j < T(n+1)} A_j = \sum_{T(n) < j < T(n+1)} (f(X_j) - X_j)/(j + 1)
< (\ln[T(n+1)] - \ln[T(n)])(k \gamma r^n)
\leq (k \gamma r^n)[(1-r)/\gamma k_i + 1/T(n)]
= (k/k_i)(r^n - r^{n+1}) + k \gamma r^n / T(n).
\]

But then if \( \mathcal{B} \) holds and \( \tau_{n+1} = L \leq T(n+1) \), it must be the case that

\[
Z_{\tau_n,L} = X_L - X_{\tau_n} - \sum_{j=\tau_n}^{L-1} A_j
\geq X_L - X_{\tau_n} - \sum_{T(n) < j < T(n+1)} A_j
\geq r^n - r^{n+1} - \xi_n - (k/k_i)(r^n - r^{n+1}) - k \gamma r^n / T(n)
= r^n(1-r)(1 - (k/k_i)) - \xi_n - k \gamma r^n / T(n)
= r^n(1-r)(1 - (k/k_i)) - \tilde{\xi}_n.
\]

The term \( \xi_n \) comes from the fact that \( X_{\tau_n} \) may overshoot the stopping point \( p - r^n \), and \( \tilde{\xi}_n \) denotes the sum of \( \xi_n \) and the \( k \gamma r^n/T(n) \) term. Then \( \xi_n \) is bounded by \( X_{\tau_n} - X_{\tau_n-1} < \tau_n^{-1} < T(n)^{-1} \) by assumption. Since \( T(n)^{-1} \) is of order less than \( r^{2n} \), the \( \tilde{\xi}_n \) contribution vanishes asymptotically in the sense that

\[
\frac{r^n(1-r)(1 - (k/k_i)) - \tilde{\xi}_n}{r^n(1-r)(1 - (k/k_i))} \to 1.
\]
Now $E(Z_{\tau_n, \infty}^2 | \tau_n > T(N)) < T(n)^{-1}$, so

$$\begin{align*}
\text{prob}(\tau_{n+1} \leq T(n+1) | \tau_n > T(N)) & \leq \text{prob}(B^c | \tau_n > T(N)) \\
& + \text{prob} \left( B \text{ and } \sup_L Z_{\tau_n, L} \geq r^n(1-r) \left( 1 - \frac{k}{k_1} \right) - \xi_n | \tau_n > T(N) \right) \\
& \leq (1 - \gamma)^{-2} [g(r)]^{-2n} + T(n)^{-1} \left/ \left[ r^n(1-r) \left( 1 - \frac{k}{k_1} \right) - \xi_n \right]^2 \right. \\
& \leq (1 - \gamma)^{-2} [g(r)]^{-2n} + \left[ (1-r)(1-(k/k_1)) \right]^{-2} [g(r)]^{-2n} \\
& \times \left[ \frac{r^n(1-r)(1-(k/k_1)) - \xi_n}{r^n(1-r) \left( 1 - \frac{k}{k_1} \right)} \right]^2.
\end{align*}$$

Because the last term of the numerator vanishes asymptotically, the sum of these probabilities converges. Then $\text{prob}(\tau_n > T(n) \text{ for all } n > M) = \text{prob}(\tau_M > T(M)) \prod_{n \geq M} (1 - \text{prob}(\tau_{n+1} \leq T(n+1) | \tau_n > T(N))) > 0$ since each factor is positive and $\sum \text{prob}(\tau_{n+1} \leq T(n) | \tau_n > T(N))$ is finite. In this case, $X_n$ must converge to $p$ from below. \qed

**Theorem 3.** Suppose that $f(x) \geq x + k(p-x)$ for some $k > 1/2$ and all $x$ in some left neighborhood, $(p-\epsilon, p)$, of $p$. Then $\text{prob}(X_n \to p) = 0$. [Similarly, if $f(x) \leq x - k(x-p)$ for some $k > 1/2$ and all $x$ in a right neighborhood, $(p, p+\epsilon)$ of $p$, then also $\text{prob}(X_n \to p) = 0$.]

**Remark.** No continuity assumptions are needed this time.

**Proof.** Again there is no loss of generality in assuming that $f(x) \geq x$ for all $x$; similarly, assume $f(x) \geq \min(1, x + k|p-x|)$ on $[0, p]$. Furthermore, Lemma 2.2 of [HLS] says that replacing $f$ by a pointwise smaller function gives a process which can be defined on the same probability space so as always to be smaller. Thus replacing $f$ by the minimum of $1$ and $x + k|p-x|$ on $[0, p]$ and by $x$ on $[p, 1]$ gives a process which converges to $p$ whenever the original process does, so it suffices to prove the theorem for this choice of $f$. The importance of assuming this lies only in getting $f$ bounded away from $0$ and $1$ near $p$ (without assuming continuity) so that there will be a lower $L^2$-bound on $Z$.

The following argument is self-contained, but the reader may wish to look at Pemantle [P2, Lemmas 1 and 2] to see the template from which this proof was constructed.

**Lemma 4.** There are constants $a, c > 0$ and a neighborhood $\mathcal{N}$ of $p$ such that for any $n$

$$\text{prob}(Z_{n, \infty} > c n^{-1/2} \text{ or } X_{n+j} \notin \mathcal{N} \text{ for some } j|\mathcal{F}_n) > a.$$
Proof. Pick \( b > 0 \) and \( \mathcal{N} \) a neighborhood of \( p \) such that \( f(\mathcal{N}) \subseteq [b, 1-b] \).
Assume that \( X_n \in \mathcal{N} \) or else the result is trivially true. For \( k > 0 \), let \( \tau \leq \infty \) be the first time \( X_j \) exits \( \mathcal{N} \) or \( Z_{n,j} \) exits \((-kn^{-1/2}, kn^{-1/2})\). Then equation (1) gives \( \mathbb{E}(Z_{n,j}^2 | \mathcal{F}_n) \geq \text{prob}(\tau = \infty | \mathcal{F}_n) b^2(n+1)^{-1} \). On the other hand, \( \mathbb{E}(Z_{n,j}^2 | \mathcal{F}_n) \leq \mathbb{E}(X_t - X_n)^2 \leq k^2/n \), since \( Z \) is just the martingale part of \( X \). Putting these together gives \( \text{prob}(\tau = \infty | \mathcal{F}_n) \leq k^2(n+1)/b^2n \), and choosing \( k \) small enough makes this at most 1/3. Let
\[
q = \text{prob}(\tau < \infty, X_t \notin \mathcal{N} | \mathcal{F}_n),
\]
so that the conditional probability of \( Z_{n,j} \) exiting \((-kn^{-1/2}, kn^{-1/2})\) given \( \mathcal{F}_n \) is at least \( 2/3 - q \). Any martingale \( \mathbb{M} \) started at zero that exits an interval \((-L, L)\) with probability at least \( r \) and has increments bounded by \( L/2 \) satisfies \( \text{prob}(\text{sup } \mathbb{M} \geq L/2) \geq (3r - 1)/4 \); stopping \( \mathbb{M} \) upon exiting \((-L, L/2)\) and letting \( s = \text{prob}(\text{sup } \mathbb{M} > L/2) \) gives \( 0 = \mathbb{E}\mathbb{M} \leq sL + (r-s)(-L) + (1-r)(L/2) = 2L(s - (3r - 1)/4) \). Thus \( Z_{n,j} \geq k/2\sqrt{n} \) for some \( j \) with probability at least \((1 - 3q)/4 \).

Now for any \( j \), condition on the event \( Z_{n,j} \geq k/2\sqrt{n} \); then the conditional probability of the event \( Z_{n,\infty} < k/4\sqrt{n} \) can be bounded away from 1 using the following one-sided Tschebysheff estimate:

**Lemma 5.** If \( \mathbb{M} \) has mean zero and \( L < 0 \), then
\[
\text{prob}(\mathbb{M} \leq L) \leq \mathbb{E}\mathbb{M}^2 / (\mathbb{E}\mathbb{M}^2 + L^2).
\]

**Proof.** Write \( w \) for \( \text{prob}(\mathbb{M} \leq L) \). From
\[
0 = \mathbb{E}\mathbb{M}^2 = w \mathbb{E}(\mathbb{M} | \mathbb{M} \leq L) + (1-w) \mathbb{E}(\mathbb{M} | \mathbb{M} > L)
\]
and \( \mathbb{E}(\mathbb{M} | \mathbb{M} \leq L) \leq L \), it is immediate that
\[
\mathbb{E}(\mathbb{M} | \mathbb{M} > L) \geq -L \frac{w}{1-w}.
\]
Then
\[
\mathbb{E}\mathbb{M}^2 = w \mathbb{E}(M^2 | M \leq L) + (1-w) \mathbb{E}(\mathbb{M}^2 | \mathbb{M} > L)
\]
\[
\geq wL^2 + (1-w)(\mathbb{E}(\mathbb{M} | \mathbb{M} > L))^2
\]
\[
\geq wL^2 + (1-w)L^2(w^2/(1-w)^2)
\]
\[
= L^2 w/(1-w),
\]
from which the desired conclusion follows. \( \Box \)

Apply this to the process \( Z_{j,i} \) stopped at the entrance time \( \tau \) of the interval \((-\infty, -k/4\sqrt{n}) \) to get
\[
\text{prob}(Z_{n,\infty} \leq k/4\sqrt{n} | \mathcal{F}_j) \leq \text{prob}(Z_{j,\tau} \leq -k/4\sqrt{n} | \mathcal{F}_j)
\]
\[
\leq \mathbb{E}Z_{j,\tau}^2 / (\mathbb{E}Z_{j,\tau}^2 + k^2/16n)
\]
\[
\leq \mathbb{E}Z_{n,\infty}^2 / (\mathbb{E}Z_{n,\infty}^2 + k^2/16n)
\]
\[
\leq 16/(k^2 + 16).
\]
Combining this with the previous result shows that the conditional probability of 
$Z_{n,\infty} > k/4\sqrt{n}$ given $\mathcal{F}_n$ is at least $(1 - 3q)k^2/64 + 4k^2$. Recall that $q$ 
is the conditional probability of the process exiting $\mathcal{N}$ given $\mathcal{F}_n$, so that 
the probability we are trying to bound below is at least the maximum of $q$ and $(1 - 3q)k^2/(64 + 4k^2)$. For any value of $q$ the maximum is at least 
k^2/(64 + 7k^2), thus the statement of the lemma is proved with $c = k/4$ and 
a = k^2/(64 + 7k^2). □

Let $\tau$ be any finite stopping time. Conditioning on $\mathcal{F}_\tau$ then gives a stopping 
time version of the previous lemma:

\[(2) \quad \text{prob}(Z_{\tau,\infty} > c\tau^{-1/2} \text{ or } X_{\tau+j} \not\in \mathcal{N} \text{ for some } j|\mathcal{F}_\tau) > a.\]

A corollary of this is a sort of converse to the proof of Theorem 1, saying that if 
$X_n \to p$ then it does so from the left.

**Corollary 6.** Let $p$ be a touchpoint of the first kind, i.e., $f(y) > y$ for all $y \neq p$ 
in a neighborhood of $p$. Then the probability of the event that either $X_n > p$ 
infinitely often or $X_n$ does not converge to $p$ is 1.

**Proof.** Suppose to the contrary that the probability that $X_n$ converges to $p$ 
and is greater than or equal to $p$ infinitely often is nonzero. Then there are 
n, $M$, and some event $\mathcal{B} \in \mathcal{F}_n$ such that $n < M$ and conditional on $\mathcal{B}$, 
the probability of $X_j$ converging to $p$ and being greater than $p$ some time before 
$M$ but never leaving $\mathcal{N}$ after time $n$ is at least $1 - a/3$. Define $\tau$ to be the 
minimum of $M$ and the least $j \geq n$ such that $X_j > p$. Then letting $\mathcal{C}$ be the 
event that $X_j$ converges to $p$ without leaving $\mathcal{N}$ after time $n$,

\[
\text{prob}(\tau < M|\mathcal{B})\text{prob}(\mathcal{C}|\mathcal{B}, \tau < M) + \text{prob}(\tau = M|\mathcal{B})\text{prob}(\mathcal{C}|\mathcal{B}, \tau = M)
\]

\[= \text{prob}(\mathcal{C}|\mathcal{B}) \geq 1 - a/3.\]

So

\[
\text{prob}(\mathcal{C}|\mathcal{B}, \tau < M) \geq 1 - a/3 - \text{prob}(\tau = M|\mathcal{B}) \geq 1 - 2a/3.
\]

Now $\tau < M$ implies that $X_\tau > p$. But since $A_n$ is an increasing process, it 
follows that $X_\tau \to p$ and $X_\tau > p$ together imply $Z_{\tau,\infty} < 0$. Thus

\[
\text{prob}(Z_{\tau,\infty} < 0 \text{ and } X_{n+j} \in \mathcal{N} \text{ for all } j|\mathcal{B}, \tau < M) \geq 1 - 2a/3,
\]

and hence

\[
\text{prob}(Z_{\tau,\infty} > c\tau^{-1/2} \text{ or } X_{\tau+j} \not\in \mathcal{N} \text{ for some } j|\mathcal{B}, \tau > M) \leq 2a/3.
\]

But this contradicts (2), since the events $\mathcal{B}$ and $\tau < M$ are both in $\mathcal{F}_\tau$. □

**Continuation of the proof of Theorem 3.** It remains to show that under the 
hypothesis of the theorem, the probability is zero that $X_n$ eventually resides in 
$(p - \varepsilon, p)$. If the probability were nonzero, then for any $\delta$ there would be an
event $\mathcal{B}$ in some $\mathcal{F}_M$ for which $\text{prob}(X_{M+j} \in (p-\epsilon, p))$ for all $j \geq 0 \mid \mathcal{B}) > 1 - \delta$. In fact, conditioning on $X_M$, $\mathcal{B}$ may be taken to determine $X_M$. So it suffices to show that the probability of the event $X_{M+j} \in (p-\epsilon, p)$ for all $j \geq 0$ given $X_M$ is bounded away from 1. For what follows condition on $\mathcal{F}_M$ and on $X_M \in (p-\epsilon, p)$. Also choose $M$ large enough so that for any $n > M$, $n^{-k/2k_i} < cn^{-1/2}$ where $c$ is chosen as in Lemma 4, and choose $\epsilon$ small enough so that $(p-\epsilon, p)$ is a subset of a neighborhood $\mathcal{N}$ to which Lemma 4 applies.

Begin by setting up constants and stopping times: pick a $k < 3/4$ for which the hypothesis of the theorem is satisfied and pick $k_i$ so that $k > k_i > 1/2$. For $n \geq M$ define

$$V_n = (k/k_i) \ln(n) + 2\ln(p-X_n) \text{ for } X_n < p \text{ and } -\infty \text{ otherwise.}$$

By assumption on $X_M$, $V_M > -\infty$. Let $\tau$ be the least $n \geq M$ such that $X_n \not\in (p-\epsilon, p)$ or $V_n < 0$. Observe that if $V_n > 0$ then $1/n < (p-X_n)^{2k_i/k} \leq (p-X_n)^{4/3}$, so $|X_{n+1} - X_n|$ is small compared to $p-X_n$, so $V_{1\wedge n}$ can never reach $-\infty$ and is in fact bounded below by $\min(-1, V_M)$. Now for $n < \tau$ calculate

$$\begin{align*}
E(\ln(p-X_{n+1}) \mid \mathcal{F}_n) &\leq \ln E(p-X_{n+1} \mid \mathcal{F}_n) \\
&= \ln(p-X_n - A_n) \\
&\leq \ln((p-X_n)(1 - k/(n+1))) \\
&= \ln(p-X_n) + \ln(1 - k/(n+1));
\end{align*}$$

so

$$E(V_{n+1} \mid \mathcal{F}_n) \leq V_n + (k/k_i)(\ln(n+1) - \ln(n)) + 2\ln(1 - k/(n+1))$$

$$= V_n + (k/k_i)(n^{-1} + o(n^{-1})) - 2k(n^{-1} + o(n^{-1}))$$

$$= V_n - (2 - 1/k_i)k + o(1))n^{-1} < V_n - Cn^{-1}$$

for large $n$ and some $C > 0$. So $V_{n\wedge \tau}$ is a supermartingale for large $n$, bounded below by $\min(-1, V_M)$, and hence converges almost surely. Clearly it cannot converge without stopping, since the increments of the expectation sum to $-\infty$, therefore the stopping time is reached almost surely.

In other words, conditional upon any event in any $\mathcal{F}_M$, the probability is 1 that for some $n > M$, either $X_n$ will leave $(p-\epsilon, p)$ or $(k/k_i)\ln(n) < -2\ln(p-X_n)$. Let $\sigma \leq \infty$ be the least $n > M$ for which $(k/k_i)\ln(n) < -2\ln(p-X_n)$. We have just shown that the conditional probability of some $X_n$ leaving $(p-\epsilon, p)$ given $\sigma = \infty$ is one. On the other hand, the conditional probability of some $X_{n+j}$ leaving $(p-\epsilon, p)$ given $\sigma = \infty$ is at least $a$ by Lemma 4 since $X_{n+j} \not\in \mathcal{N}$ trivially implies $X_{n+j} \not\in (p-\epsilon, p)$, while $Z_{n,\infty} > cn^{1/2}$ implies $Z_{n,n+j} > cn^{1/2} > n^{-k/2k_i} > p - X_n$ for some $j$, which implies $X_{n+j} > p$. □
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DEPARTMENT OF STATISTICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720

Current address: Department of Mathematics, Kidder Hall, Oregon State University, Corvallis, Oregon 97331-4005