1. Find the area of the surface obtained by rotating the curve

$$2x = y^2 + 4, \quad 2 \leq x \leq 3$$

about the $x$-axis.

First, draw a picture. You should have a parabola centered at $(2,0)$, opening towards the positive $x$-axis. Note that we are only interested in the top half of the parabola\(^1\), since when we rotate this about the $x$-axis, we get the same surface as when we rotate the whole parabola. But using the whole parabola would give us twice the area.

There are two ways we can find the area, depending on whether we describe the curve in the form $y = \text{function of } x$, or in the form $x = \text{function of } y$.

First method: Rewrite $2x = y^2 + 4$ as $y = \sqrt{2x - 4}$ (this gives the top half of the parabola). Since

$$\frac{dy}{dx} = \frac{1}{\sqrt{2x - 4}},$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{1}{2x - 4}} = \sqrt{\frac{2x - 3}{2x - 4}},$$

and so the area of the surface of revolution is

$$\int_{2}^{3} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = 2\pi \int_{2}^{3} \sqrt{2x - 4} \sqrt{\frac{2x - 3}{2x - 4}} \, dx$$

$$= 2\pi \int_{2}^{3} \sqrt{2x - 3} \, dx = 2\pi \left[ \frac{1}{3} (2x - 3)^{\frac{3}{2}} \right]_{2}^{3} = \frac{2\pi}{3} \left( 3\sqrt{3} - 1 \right).$$

Second method: Rewrite $2x = y^2 + 4$ as $x = \frac{y^2}{2} + 2$, so we get $\frac{dx}{dy} = y$. Before we can integrate, we need to know the bounds on $y$. We have $2 \leq x \leq 3$, so using $y = \sqrt{2x - 4}$, we see the bounds on $y$ are $0 \leq y \leq \sqrt{2}$. Therefore the area of the surface of revolution is

$$\int_{0}^{\sqrt{2}} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dy = 2\pi \int_{0}^{\sqrt{2}} y \sqrt{1 + y^2} \, dy$$

$$= 2\pi \left[ \frac{1}{3} (1 + y^2)^{\frac{3}{2}} \right]_{0}^{\sqrt{2}} = \frac{2\pi}{3} \left( 3\sqrt{3} - 1 \right).$$

\(^1\) Of course, you could just as well use only the bottom half.


1. Find the arc length of the function $f(x) = x^3 + \frac{1}{12x}$, $1 \leq x \leq 2$.

First, differentiate: $f'(x) = 3x^2 - \frac{1}{12x^2}$. Therefore

$$1 + f'(x)^2 = 1 + 9x^4 - \frac{1}{2} + \frac{1}{144x^4} = 9x^4 + \frac{1}{2} + \frac{1}{144x^4} = \left(3x^2 + \frac{1}{12x^2}\right)^2.$$

$$1 + f'(x)^2 = 1 + \left(3x^2 - \frac{1}{12x^2}\right)^2$$
$$= 1 + 9x^4 - \frac{1}{2} + \frac{1}{144x^4}$$
$$= 9x^4 + \frac{1}{2} + \frac{1}{144x^4}$$
$$= \left(3x^2 + \frac{1}{12x^2}\right)^2.$$

Note that the coefficients of $f(x)$ have been specifically chosen so that you can express $1 + f'(x)^2$ as the square of a polynomial. Doing this is the key step. It follows that

$$\sqrt{1 + f'(x)^2} = \left|3x^2 + \frac{1}{12x^2}\right| = 3x^2 + \frac{1}{12x^2}.$$

(We can drop the absolute value signs since $3x^2 + \frac{1}{12x^2}$ is always positive). The arc length is therefore

$$\int_1^2 \sqrt{1 + f'(x)^2} \, dx = \int_1^2 3x^2 + \frac{1}{12x^2} \, dx = \left(x^3 - \frac{1}{12x}\right)|_1^2 = 8 - \frac{1}{24} - \left(1 - \frac{1}{12}\right) = 7 \frac{1}{24}.$$