1. Find the radius of convergence and interval of convergence of the series:

\[ \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}. \]

We can determine the radius of convergence using the ratio test. The \( n \)-th term is \( a_n = \frac{x^n}{\sqrt{n}} \), and

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n} \right| = \lim_{n \to \infty} \left| x \sqrt{\frac{n}{n+1}} \right| = \lim_{n \to \infty} |x| \sqrt{\frac{1}{1+\frac{1}{n}}} = |x|,
\]

so the series converges if \( |x| < 1 \), and diverges if \( |x| > 1 \). Therefore the radius of convergence is 1.

We know the interval of convergence contains the interval \((-1, 1)\), and does not include anything outside the interval \([-1, 1]\), but we don’t yet know what happens when \( x = -1 \) or \( x = 1 \).

If \( x = -1 \), the series is \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \), which is an alternating series. Since the sequence \( \frac{1}{\sqrt{n}} \) is positive, decreasing (\( \sqrt{n} \) is increasing), and tends to 0, the conditions of the alternating series test are satisfied, and we can conclude that \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \) converges. Therefore the interval of convergence includes \(-1\).

If \( x = 1 \), the series is \( \sum_{n=1}^{\infty} \frac{1^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \), which is a \( p \)-series with \( p = \frac{1}{2} \leq 1 \). Therefore it diverges, and consequently the interval of convergence does not include 1. We conclude that the interval of convergence is \([-1, 1)\).
Wednesday Quiz 6
Maths 104 - Calculus I
April 6, 2011

Note: In order to receive full credit, you must show work that justifies your answer.

1. Test the series for convergence or divergence:

\[
\sum_{k=1}^{\infty} k^2 2^{-k}.
\]

As is usually the case, there is more than one way of doing this. We could use the integral test (using integration by parts twice), but an easier method is the ratio test.

The \( k \)-th term is \( a_k = k^2 2^{-k} \), so

\[
\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \left| \frac{(k + 1)^2 2^{-(k+1)}}{k^2 2^{-k}} \right| = \lim_{k \to \infty} \left( 1 + k^{-1} \right)^2 2^{-1} = \frac{1}{2} < 1.
\]

Therefore, by the ratio test, the series \( \sum_{k=1}^{\infty} k^2 2^{-k} \) converges.