Review Solutions

19 July

(1) Is \( \{(x, y, z) \in \mathbb{R}^3 : x - y = 3z\} \) a subspace of \( \mathbb{R}^3 \)? Yes.

(2) Is \( \{(x, y) \in \mathbb{R}^2 : x^2 = y^2\} \) a subspace of \( \mathbb{R}^2 \)? No, not closed under addition.

(3) Is \( \{(x, y) \in \mathbb{R}^2 : x^2 + 2xy = -y^2\} \) a subspace of \( \mathbb{R}^2 \)? Yes. Proving this will be easier if you notice that the condition \( x^2 + 2xy = -y^2 \) is equivalent to \( x + y = 0 \).

(4) For which value of \( k \) is \( \{(x, y, z) \in \mathbb{R}^3 : 2x - y + 5z = k\} \) a subspace of \( \mathbb{R}^3 \)? \( k = 0 \).

(5) Is \( \{f(x) \in P_2 : f(1) = 0\} \) a subspace of \( P_2 \)? Yes.

(6) Is \( \{f(x) \in P_3 : f'(1) = f(0)\} \) a subspace of \( P_3 \)? Yes.

(7) If \( A \) is a \( 2 \times 2 \) matrix such that \( AB = BA \) for every \( 2 \times 2 \) matrix \( B \), what can you say about \( A \)?

For every \( 2 \times 2 \) matrix \( B \), \( AB = BA \).

(8) Find the ranks of the following matrices:

\[
\begin{pmatrix}
5 & 0 & -4 \\
7 & 1 & 13 \\
-2 & 0 & 6
\end{pmatrix}, \quad
\begin{pmatrix}
3 & -3 & 1 \\
-1 & 2 & 4 \\
2 & -5 & -1 \\
1 & 2 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
2 & 5 & 1 & -4 \\
-7 & 0 & 4 & -1 \\
-4 & -3 & 1 & 2
\end{pmatrix}
\]

(a) 3 (you can see this by noticing that the determinant is non-zero), (b) 3, (c) 2

(9) Are the following sets of vectors linearly independent? Explain why.

\( \mathbf{a} \)

\[
\begin{pmatrix}
1 \\
1 \\
4
\end{pmatrix}, \quad
\begin{pmatrix}
3 \\
-1 \\
2
\end{pmatrix}, \quad
\begin{pmatrix}
-8 \\
6 \\
3
\end{pmatrix}
\]

\( \mathbf{b} \)

\[(0, 1, 3, -2), (2, 1, 1, 3), (-3, 2, 4, -1)\]

\( \mathbf{c} \)

\[
\begin{pmatrix}
0 \\
2 \\
-3
\end{pmatrix}, \quad
\begin{pmatrix}
1 \\
1 \\
2
\end{pmatrix}, \quad
\begin{pmatrix}
3 \\
1 \\
4
\end{pmatrix}, \quad
\begin{pmatrix}
-2 \\
3 \\
-1
\end{pmatrix}
\]

(a) Yes. The matrix with these vectors as columns has rank 3.

(b) Yes. Same reason as for (a).

(c) No, they are a set of 4 vectors in the three dimensional space \( \mathbb{R}^3 \).

(10) For which value(s) of \( a \) is the set

\( \{(3, 2, a), (a + 1, -1, 2), (2, 1, 0)\} \)

linearly dependent?

For \( a = -2 \) and \( -1 \). To be linearly dependent, the determinant of the matrix having these vectors as rows must be zero. This gives a polynomial in \( a \) which has roots \( a = -2 \) and \( -1 \).
(11) For which value(s) of $a$ is the set
\[
\{(a - 3, 1, -a), (1, -1, 0), (2a, 0, -a + 1), (1, 0, a)\}
\]
linearly dependent?

All values, since they are a set of 4 vectors in the three dimensional space $\mathbb{R}^3$.

(12) If $A$ is a $3 \times 4$ matrix, $B$ is a $4 \times 3$ matrix and $\text{rank}(AB) = 2$, what are the possible values of $\text{rank}(A)$?

Either 2 or 3. The rank is at most 3 since $A$ has 3 rows, and is at least 2 since $AB$ has rank 2. One way of seeing this is by noticing that the columns of $AB$ are linear combinations of the columns of $A$ (if this isn’t clear, do a random example), so if $A$ only had rank 1, all its columns would be scalar multiples of a single vector, so the same would be true of $AB$, and if $A$ had rank 0, it would be the zero matrix, and so would $AB$.

(13) If $A$ is a $22 \times 17$ matrix of rank 12 and $AX = b$ is consistent, how many free variables are there in the solution?

$17 - 12 = 5$.

(14) Find the determinants of the following matrices

(a) $\begin{pmatrix} 5 & 2 & 1 \\ 1 & 1 & -4 \\ -2 & -7 & 2 \end{pmatrix}$, (b) $\begin{pmatrix} 0 & 3 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 2 & 4 & -7 \\ 0 & 3 & -1 & 0 & 1 & 2 \\ -1 & -3 & 2 & 0 & 3 & -1 \\ 2 & 2 & -4 & 0 & -6 & 3 \\ 0 & 1 & 3 & 0 & 1 & -2 \end{pmatrix}$

(a) $-123$, (b) $-24$ (expand along rows and columns with many zeros, and perform row and/or column operations to reduce the amount of work you need to do)

(15) If

\[
A = \begin{pmatrix}
1 & 3 & -1 & 3 \\
1 & 7 & -2 & 0 \\
0 & 1 & 1 & -4 \\
2 & 5 & -2 & 2
\end{pmatrix}, \quad B = \begin{pmatrix}
2 & 0 & 0 & 0 \\
1 & 3 & 0 & 0 \\
6 & 7 & 1 & 0 \\
2 & -3 & 3 & -2
\end{pmatrix},
\]

find $\det(A^TBA^{-1})$.

$\det(A^TBA^{-1}) = \det(A^T) \det(B) \det(A^{-1}) = \det(A) \det(B) \det(A)^{-1} = \det(B) = -12$ (det of triangular matrix is product of diagonal entries)

(16) Find $\det(A^{-1})$ if

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 \\
4 & 1 & 6 & 0 \\
6 & 2 & 0 & -3
\end{pmatrix}
\]

$\det(A^{-1}) = \det(A)^{-1} = \frac{1}{18}$.

(17) Find the inverses of the following matrices

(a) $\begin{pmatrix} 5 & 6 \\ 4 & 7 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$, (c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 3 & 2 \end{pmatrix}$

(a) $\frac{1}{11} \begin{pmatrix} 7 & -6 \\ -4 & 5 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & -2 & -1 \\ -3 & 4 & 1 \\ -1 & 1 & 1 \end{pmatrix}$, (c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 4 & 1 & -3/2 & 1/2 \end{pmatrix}$

\[2\]
Given that

\[
\begin{pmatrix}
2 & -3 & -2 & 1 \\
-1 & 1 & 1 & 0 \\
1 & 2 & 0 & -1 \\
2 & -1 & 1 & 4 \\
\end{pmatrix}
\]

\[-1
\begin{pmatrix}
-6 & -13 & -2 & 1 \\
7 & 15 & 3 & -1 \\
-13 & -27 & -5 & 2 \\
8 & 17 & 3 & -1 \\
\end{pmatrix},
\]

solve the system of linear equations

\[-6w - 13x - 2y + z = 1
7w + 15x + 3y - z = 0
-13w - 27x - 5y + 2z = 1
8w + 17x + 3y - z = 2
\]

If \(A\) is an invertible matrix, the solution to \(AX = B\) is \(X = A^{-1}B\). Thus

\[
\begin{pmatrix}
w \\
x \\
y \\
z \\
\end{pmatrix}
= \begin{pmatrix}
2 & -3 & -2 & 1 \\
-1 & 1 & 1 & 0 \\
1 & 2 & 0 & -1 \\
2 & -1 & 1 & 4 \\
\end{pmatrix}^{-1}
\begin{pmatrix}
1 \\
0 \\
1 \\
2 \\
\end{pmatrix}
= \begin{pmatrix}
2 \\
0 \\
-1 \\
11 \\
\end{pmatrix}.
\]

Note: when you’re solving systems of equations, you can always check whether your answer is a solution by plugging it back in.

For the following matrices, find the eigenvalues, and for each eigenvalue, find a basis for the eigenspace. State whether the matrix is diagonalizable or not.

(a) Eigenvalues: 2 and 4, with \(\{(1,1)^T\}\) and \(\{(3,1)^T\}\) bases for the eigenspaces.
(b) Eigenvalues: 1+3i and 1-3i, with \(\{(1+i, -3)^T\}\) and \(\{(1-i, -3)^T\}\) bases for the eigenspaces (if you found different eigenvectors, your answers might still be correct).
(c) Eigenvalues: 1 and 2 (mult 2), with \(\{(2,1,1)^T\}\) and \(\{(3, -2,0)^T, (3,0,1)^T\}\) bases for the eigenspaces.
(d) Eigenvalues: -1 (mult 2) and 0, with \(\{(2,1,1)^T\}\) and \(\{(-6,4,3)^T\}\) bases for the eigenspaces.

The first three matrices are diagonalizable since the dimension of each eigenspace equals the multiplicity of the eigenvalue. The last matrix is not diagonalizable since -1 is an eigenvalue of multiplicity 2, but the eigenspace is 1 dimensional.

Suppose \(A\) is a \(3 \times 3\) matrix with eigenvector \((1,1,0)^T\) corresponding to the eigenvalue 0, and eigenvectors \((1,0,0)^T, (0,1,-1)^T, (-1,2,-2)^T\) corresponding to the eigenvalue 2. What is \(A\)?

\(A\) is diagonalizable since it has three linearly independent eigenvectors: \((1,1,0)^T, (1,0,0)^T, (0,1,-1)^T\) (we ignore \((-1,2,-2)^T\) since it is in the span of \((1,0,0)^T, (0,1,-1)^T\)). The corresponding eigenvalues are 0, 2, 2, so \(A = PDP^{-1}\), where

\[
P = \begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 0 & -1 \\
\end{pmatrix}, \quad D = \begin{pmatrix}
0 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2 \\
\end{pmatrix}.
\]

Therefore

\[
A = \begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 0 & -1 \\
\end{pmatrix} \begin{pmatrix}
0 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2 \\
\end{pmatrix} \begin{pmatrix}
0 & 1 & 1 \\
1 & -1 & -1 \\
0 & 0 & -1 \\
\end{pmatrix} = \begin{pmatrix}
2 & -2 & -2 \\
0 & 0 & -2 \\
0 & 0 & 2 \\
\end{pmatrix}.
\]
Let \( A = \begin{pmatrix} -12 & -11 \\ 22 & 21 \end{pmatrix} \). What is \( A^5 \)?

Diagonalize \( A \):

\[
A = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}^{-1},
\]

so

\[
A^5 = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & 10000 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -100002 & -100001 \\ 200002 & 200001 \end{pmatrix}.
\]

Determine if \( \{1, x - 1, (x - 1)^2, x^2\} \) is a linearly independent set of functions on \( \mathbb{R} \).

These four functions are solutions to the 4th order linear homogeneous DE \( y^{(4)} = 0 \), so we can check linear independence by computing the Wronskian. It is zero, so the functions are linearly dependent. Alternatively, use the fact that this is a set of four vectors in the three dimensional vector space \( P_2 \).

Find the general solution to the following differential equations:

(a) \( y^{(3)} - y'' + 4y' - 4y = 0 \)
\[ y = \mu_1 e^x + \mu_2 \cos(2x) + \mu_3 \sin(2x) \]
(b) \( y'' - y' - 12y = 0 \)
\[ y = \mu_1 e^{4x} + \mu_2 e^{-3x} \]
(c) \( y'' - 4y' + 5y = 0 \)
\[ y = \mu_1 e^{x} + \mu_2 e^{3x} + \mu_3 x e^{3x} \]
(d) \( y^{(3)} - 5y'' + 9y = 0 \)
\[ y = \mu_1 e^{x} + \mu_2 e^{-3x} + \mu_3 + \mu_4 x + \mu_5 x^2 \]
(e) \( y^{(5)} - 9y^{(3)} = 0 \)
\[ y = \mu_1 e^{3x} + \mu_2 e^{-3x} + \mu_3 + \mu_4 x + \mu_5 x^2 \]

Find particular solutions to

(a) \( y^{(3)} - y'' + 4y' - 4y = xe^{-x} \)
\[ y_p = -\frac{9}{100} e^{-x} - \frac{1}{16} xe^{-x} \]
(b) \( y'' - y' - 12y = 6e^{4x} - \cos(x) \)
\[ y_p = \frac{6}{7} xe^{4x} + \frac{13}{170} \cos(x) + \frac{17}{170} \sin(x) \]
(c) \( y'' - 4y' + 5y = \sin(x) e^{2x} \)
\[ y_p = -\frac{1}{2} x \cos(x) e^{2x} \]
(d) \( y^{(3)} - 5y'' + 9y = 4 \cos(2x) e^x - \sin(2x) e^x \)
\[ y_p = \left[ \frac{1}{3} \cos(2x) - \frac{5}{18} \sin(2x) \right] e^x \]
(e) \( y^{(5)} - 9y^{(3)} = x^2 + 3x - 5 \)
\[ y_p = -\frac{1}{50} x^5 - \frac{1}{16} x^4 + \frac{43}{480} x^3 \] (Don’t worry, if there’s a problem like this on the midterm, the coefficients won’t be so horrible.)
(25) Suppose \( y_p \) is a particular solution to
\[ y'' - 6y' + 10y = \ln(x^2 + 1) + x \]  
with \( y_p(0) = 1, y_p'(0) = -1 \). Find a solution to (*) satisfying \( y(0) = 2, y'(0) = 3 \) (your answer will be in terms of \( y_p \)).

The general solution is \( y = [\mu_1 \cos(x) + \mu_2 \sin(x)]e^{3x} + y_p \). Use the initial conditions to solve for \( \mu_1 \) and \( \mu_2 \). Final answer: \( y = [\cos(x) + \sin(x)]e^{3x} + y_p \).

(26) Find the general solution to the following differential equations:

(a) \( x^2y'' - 6xy' + 12y = 0 \)
\[ y = \mu_1 x^3 + \mu_2 x^4 \]

(b) \( x^3y'' + 6x^2y' + 9xy + 3y = 0 \)
\[ y = \mu_1 x^{-1} + \mu_2 x^{-1} \cos(\sqrt{2} \ln x) + \mu_3 x^{-1} \sin(\sqrt{2} \ln x) \]

(c) \( x^2y'' + 5xy' + 5y = 0 \)
\[ y = \mu_1 x^{-2} \cos(\ln x) + \mu_2 x^{-2} \sin(\ln x) \]

(d) \( x^3y'' + 3x^2y' + 17xy' = 0 \)
\[ y = \mu_1 + \mu_2 \cos(4 \ln x) + \mu_3 \sin(4 \ln x) \]

(e) \( x^4y'' + 3x^2y' - 5xy' + 5y = 0 \)
\[ y = \mu_1 x + \mu_2 (\ln x + 3x^2 \cos(\ln x) + \mu_4 x^2 \sin(\ln x) \]

(27) If \( y \) is a solution to \( x^3y'' + 6x^2y' + 4xy' - 4y = 0 \) satisfying \( y(1) = 0, y'(1) = 1, y''(1) = -2 \), find \( y(2) \).

General solution: \( y = \mu_1 x + \mu_2 x^{-2} + \mu_3 x^{-2} \ln x \). Initial conditions give \( \mu_1 = \frac{3}{4}, \mu_2 = -\frac{1}{4}, \mu_3 = 0 \), so \( y(2) = \frac{1}{4} - \frac{3}{2} = \frac{7}{12} \).

(28) Write down a homogeneous Cauchy-Euler equation of degree 2 that has \( y = x^2 \) and \( y = x^2 \ln x \) as solutions.

The auxiliary equation must have 2 as a repeated root, so is \( \lambda^2 - 4\lambda + 4 = \lambda(\lambda - 1) - 3\lambda + 4 \). Therefore the Cauchy-Euler equation is \( x^2y'' - 3xy' + 4y = 0 \).

(29) Suppose \( x^2 + 2x + 3 \) and \( x^3 + x^2 + 2 \) are solutions to a nonhomogeneous linear differential equation. Give a non-zero solution to the corresponding homogeneous linear differential equation.

The difference of solutions to a nonhomogeneous linear DE is a solution to the corresponding homogeneous DE. Thus \( x^3 + x^2 + 2 - (x^2 + 2x + 3) = x^3 - 2x - 1 \) is a solution.

(30) Suppose a 2kg mass is attached to a spring with spring constant 6N.m\(^{-1}\), and a damping force with damping constant 8N.s.m\(^{-1}\) retards the motion. If an external force of \( f(t) = 50e^{-2t} \) N is applied to the system, find the equation of the motion of the mass given that it starts from rest at the equilibrium position.

Complementary solution: \( x_c = \mu_1 e^{-3t} + \mu_2 e^{-t} \). Particular solution: guess \( x_p = Ae^{-2t} \). You should find \( A = -25 \). Thus \( x = \mu_1 e^{-3t} + \mu_2 e^{-t} - 25e^{-2t} \), and the initial conditions imply that \( \mu_1 = \mu_2 = \frac{25}{3} \), so the equation of motion is \( x = \frac{25}{3} e^{-3t} + \frac{25}{2} e^{-t} - 25e^{-2t} \).

(31) If a mass of 2kg is attached to a spring with spring constant 8N.m\(^{-1}\), and a damping force with damping constant \( \rho = 8N.s.m^{-1} \) is present, find the equation of motion if the mass is released with a downward velocity of 10m.s\(^{-1}\) from a point 1m above the equilibrium position. Find the time(s) at which the mass passes through the equilibrium position.

Equation of motion: \( x = (-1 + 8t)e^{-2t} \) (critical damping). Passes through equilibrium position at \( t = 1/8s \).

(32) In the absence of damping, a 3kg mass attached to a spring is found to oscillate with a period of \( 4\pi \) seconds. What is the spring constant?
\[ T = 2\pi/\omega, \text{ so } \omega = 2\pi/T = 1/2. \text{ } k/m = \omega^2, \text{ so } k = m\omega^2 = 3(1/2)^2 = 3/4 \text{ (in N.m}^{-1}\).
(33) Solve $X' = AX$ for the following matrices $A$:

(a) \[
\begin{pmatrix}
1 & 5 \\
-1 & -3
\end{pmatrix}, \quad
(b) \begin{pmatrix}
5 & -3 \\
3 & -1
\end{pmatrix}, \quad
(c) \begin{pmatrix}
4 & -3 \\
2 & -1
\end{pmatrix}, \quad
(d) \begin{pmatrix}
2 & -1 & 5 \\
-1 & 3 & -8 \\
0 & 1 & -2
\end{pmatrix}
\]

(a) $A$ has complex eigenvalues $-1 \pm i$. An eigenvector corresponding to $-1 + i$ is $(2 + i, -1)^T$, which has real part $(2, -1)^T$ and imaginary part $(1, 0)^T$. Therefore

$$X = \mu_1 \left[ \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cos(t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(t) \right] + \mu_2 \left[ \begin{pmatrix} 2 \\ -1 \end{pmatrix} \sin(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(t) \right] e^{-t}.$$  

(b) $A$ has a repeated eigenvalue and is not diagonalizable. $X = \mu_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \mu_2 \left[ \begin{pmatrix} 1/3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] e^{2t}.$  

Note that if $A$ has a repeated eigenvalue and is diagonal, you do not need to solve $(A - \lambda I)P = K$.

(c) $A$ has distinct real eigenvalues. $X = \mu_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \mu_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{2t}.$

(d) $A$ has an eigenvalue of multiplicity 3 and is not diagonalizable. $X = \mu_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^t + \mu_2 \left[ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \right] e^t + \mu_3 \left[ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \right] e^t.$