1. $r(t) = \langle t, t^2, \frac{2}{3}t^3 \rangle$ is a curve with parameter $t$.

(a) $(0, 0, 0)$ is an intersection point of this curve with the plane $x + y + z = 0$. Are there any other intersection points?

To find the value(s) of $t$ which give the intersection points, we substitute $x = t$, $y = t^2$ and $z = \frac{2}{3}t^3$ into $x + y + z = 0$: $t + t^2 + \frac{2}{3}t^3 = 0$, so $t(1 + t + \frac{2}{3}t^2) = 0$, so $t = 0$ or $1 + t + \frac{2}{3}t^2 = 0$. When $t = 0$, we get the point $(0, 0, 0)$, so the question is whether there exists a real value of $t$ such that $1 + t + \frac{2}{3}t^2 = 0$. Look at the discriminant:

$1^2 - 4(1)(\frac{2}{3}) = 1 - \frac{8}{3} = -\frac{5}{3} < 0$, so there are no real solutions. Therefore there are no points of intersection other than $(0, 0, 0)$.

(b) Is this curve perpendicular to the plane $x + y + z = 0$ at $(0, 0, 0)$? (Hint: check if its tangent vector is a normal vector of the plane.)

First we calculate the tangent vector at $(0, 0, 0)$ (this corresponds to $t = 0$). $r'(t) = \langle 1, 2t, 2t^2 \rangle$, so $r'(0) = \langle 1, 0, 0 \rangle$. A normal vector to the plane $x + y + z = 0$ is $\langle 1, 1, 1 \rangle$, so the tangent vector $\langle 1, 0, 0 \rangle$ is not a normal vector to the plane since it is not a scalar multiple of $\langle 1, 1, 1 \rangle$.

Note that if you found that the dot product of the tangent vector and the normal vector was non-zero, you could conclude that the tangent vector was not perpendicular to the normal vector, but not that the tangent vector was not perpendicular to the plane. For the tangent vector to be perpendicular to the plane, it has to be parallel to the normal vector of the plane.
1. At what point(s) does the curve \( \mathbf{r}(t) = \langle \sin(\pi t), 2t - 1, \cos(\pi t) - 1 \rangle \) (\( t \in \mathbb{R} \)) intersect the surface \(-x^2 + 4y^2 - (z + 1)^2 = 3\)?

We first need to find the values of \( t \) such that \( x = \sin(\pi t) \), \( y = 2t - 1 \) and \( z = \cos(\pi t) - 1 \) satisfy \(-x^2 + 4y^2 - (z + 1)^2 = 3\). Substitute and solve for \( t \):

\[
-(\sin(\pi t))^2 + 4(2t - 1)^2 - (\cos(\pi t) - 1 + 1)^2 = 3
\]

\[
\therefore 4(2t - 1)^2 = 3 + \sin^2(\pi t) + \cos^2(\pi t) = 4
\]

\[
\therefore (2t - 1)^2 = 1
\]

\[
\therefore 2t - 1 = \pm 1
\]

\[
\therefore 2t = 1 \pm 1 = 0 \text{ or } 2
\]

\[
\therefore t = 0 \text{ or } 1
\]

Therefore the points of intersection are \( \mathbf{r}(0) = (0, -1, 0) \) and \( \mathbf{r}(1) = (0, 1, -2) \).

2. Find the curvature of the curve \( x = t, y = e^t, z = \sin(t) \) at the point \((0,1,0)\).

The curve is parameterized by the vector function \( \mathbf{r}(t) = \langle t, e^t, \sin(t) \rangle \), and the point \((0,1,0)\) corresponds to \( t = 0 \). We work out the first and second derivatives of \( \mathbf{r}(t) \) at \( t = 0 \):

\[
\mathbf{r}'(t) = \langle 1, e^t, \cos(t) \rangle , \text{ so } \mathbf{r}'(0) = \langle 1, 1, 1 \rangle
\]

\[
\mathbf{r}''(t) = \langle 0, e^t, -\sin(t) \rangle , \text{ so } \mathbf{r}''(0) = \langle 0, 1, 0 \rangle
\]

Therefore \( \mathbf{r}'(0) \times \mathbf{r}''(0) = \langle 1, 1, 1 \rangle \times \langle 0, 1, 0 \rangle = \langle -1, 0, 1 \rangle \), so the curvature is

\[
\kappa = \frac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \frac{|\langle -1, 0, 1 \rangle|}{|\langle 1, 1, 1 \rangle|^3} = \frac{\sqrt{2}}{\sqrt{3}^3} = \frac{\sqrt{2}}{3\sqrt{3}} = \frac{\sqrt{6}}{9}.
\]