1. Find the volume of the solid under the surface \( x^2 + y - 2z = 0 \) and above the region in the \( xy \)-plane bounded by \( y = x^2 \) and \( y = 2 - x^2 \).

First, we need to know how to describe the region \( R \) in the \( xy \)-plane bounded by \( y = x^2 \) and \( y = 2 - x^2 \). These two curves intersect when \( x^2 = 2 - x^2 \), i.e. \( x = \pm 1 \), which gives \((-1, 1)\) and \((1, 1)\) as the points of intersection. Therefore \( R \) can be described as all points \((x, y)\) where \(-1 \leq x \leq 1\) and \( x^2 \leq y \leq 2 - x^2 \).

Next, we can rewrite the surface as \( z = \frac{1}{2}[x^2 + y] \), so the volume we want is given by

\[
\int \int_R \frac{1}{2}[x^2 + y] \, dA = \int_{-1}^{1} \int_{x^2}^{2-x^2} \frac{1}{2}[x^2 + y] \, dy \, dx
\]

\[
= \frac{1}{2} \int_{-1}^{1} \left[ x^2 y + \frac{1}{2}y^2 \right] \bigg|_{y=2-x^2}^{y=x^2} \, dx
\]

\[
= \frac{1}{2} \int_{-1}^{1} x^2(2 - x^2) - x^2(x^2) + \frac{1}{2}((2 - x^2)^2 - (x^2)^2) \, dx
\]

\[
= \frac{1}{2} \int_{-1}^{1} (2x^2 - 2x^4) + \frac{1}{2}(4 - 4x^2) \, dx
\]

\[
= \frac{1}{2} \int_{-1}^{1} 2 - 2x^4 \, dx
\]

\[
= \left[ x - \frac{x^5}{5} \right]_{x=-1}^{x=1}
\]

\[
= 8 - \frac{8}{5} = \frac{8}{5}
\]

2. Use a double integral to find the area of the region in the first and fourth quadrants bounded by \( r = 1 - \cos(\theta) \) and \( r = 1 + \cos(\theta) \).

The region we want is outside the region enclosed by \( r = 1 - \cos(\theta) \) and inside the region enclosed by \( r = 1 + \cos(\theta) \), so when we describe the region using polar coordinates, we have \( 1 - \cos(\theta) \leq r \leq 1 + \cos(\theta) \).
What are the bounds on $\theta$? At the points where the two curves intersect, we have $1 - \cos(\theta) = 1 + \cos(\theta)$, so $\cos(\theta) = 0$. The general solution to this is $\theta = \frac{\pi}{2} + k\pi$, where $k$ is an integer, so the bounds on $\theta$ will be $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

Therefore the area of the region $R$ is

$$
\int \int_R 1 dA = \int_{-\pi/2}^{\pi/2} \int_{1-\cos(\theta)}^{1+\cos(\theta)} r dr d\theta \\
= \int_{-\pi/2}^{\pi/2} \frac{1}{2} r^2 \bigg|_{r=1-\cos(\theta)}^{r=1+\cos(\theta)} d\theta \\
= \frac{1}{2} \int_{-\pi/2}^{\pi/2} [(1 + \cos(\theta))^2 - (1 - \cos(\theta))^2] d\theta \\
= \frac{1}{2} \int_{-\pi/2}^{\pi/2} 4 \cos(\theta) d\theta \\
= 2 \sin(\theta) \bigg|_{\theta=-\pi/2}^{\theta=\pi/2} \\
= 4.
$$
1. Find the volume of the solid under the surface \( z = 2xy \) and above the triangle with vertices \((0,0), (2,0)\) and \((0,2)\).

The triangle is bounded by the coordinate axes and the line \( y = 2 - x \), so we can describe the triangle as being all points \((x,y)\) such that \(0 \leq x \leq 2\) and \(0 \leq y \leq 2 - x\) (of course, we could also describe it as being all points \((x,y)\) such that \(0 \leq y \leq 2\) and \(0 \leq x \leq 2 - y\)). The volume of the solid is the double integral of the function \( f(x,y) = 2xy \) over the triangle:

\[
\int \int_R f(x,y)\,dA = \int_0^2 \int_0^{2-x} 2xy\,dy\,dx
\]

\[
= \int_0^2 xy^2\bigg|_{y=0}^{y=2-x} \,dx
\]

\[
= \int_0^2 x(2-x)^2 \,dx
\]

\[
= \int_0^2 4x - 4x^2 + x^3 \,dx
\]

\[
= \left[ 2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4 \right]_{x=0}^{x=2}
\]

\[
= 8 - \frac{32}{3} + 4 - 0
\]

\[
= \frac{4}{3}.
\]

**GRADING NOTE:** Since I said the quiz would only cover section 16.4 and this question isn’t designed to be solved using polar coordinates, I’ve decided to make your total score for this quiz equal to your score for question 2 plus the maximum of your scores for both questions.

2. Draw the graph of \( r = \cos(3\theta) \) and calculate the area of the region it bounds.

The graph is a ‘three leaved rose’ (see the solution to exercise 11.4 13 on pg A98 of the textbook). The total area is three times the area of one ‘petal’, and the bounds on \( \theta \) for the petal which lies along the positive \( x \)-axis are given by \( \theta \) such that \( \cos(3\theta) = 0 \), so \( 3\theta = \pm\pi/2 \), i.e. \( \theta = \pm\pi/6 \).

For a fixed value of \( \theta \) in the interval \([-\pi/6, \pi/6]\), the points inside the region
have $r$ between zero and $\cos(3\theta)$. Therefore the area is

$$\int\int_{R} 1 dA = \int_{-\pi/6}^{\pi/6} \int_{0}^{\cos(3\theta)} r \, dr \, d\theta$$

$$= \int_{-\pi/6}^{\pi/6} \frac{1}{2} r^{2} \bigg|_{r=0}^{r=\cos(3\theta)} d\theta$$

$$= \frac{1}{2} \int_{-\pi/6}^{\pi/6} \cos^{2}(3\theta) \, d\theta$$

$$= \int_{0}^{\pi/6} \cos^{2}(3\theta) \, d\theta \quad \text{(even function on symmetric interval)}$$

$$= \int_{0}^{\pi/6} \frac{1}{2} [\cos(6\theta) + 1] \, d\theta \quad \text{(double angle formula)}$$

$$= \left[ \frac{1}{12} \sin(6\theta) + \frac{1}{2} \theta \right]_{\theta=0}^{\theta=\pi/6}$$

$$= \frac{1}{12} \sin(\pi) + \frac{1}{2} \left( \frac{\pi}{6} \right) - 0$$

$$= \frac{\pi}{12}.$$

The total area is thus $3(\pi/12) = \pi/4$. 