An Extremely Short Proof of the Hairy Ball
Theorem

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Abstract. Using winding numbers, we give an extremely short proof that every continuous
field of tangent vectors on \( S^2 \) must vanish somewhere.

Consider the unit two sphere \( S^2 = \{ p \in \mathbb{R}^3 : |p| = 1 \} \) in \( \mathbb{R}^3 \). We say a function \( v : S^2 \to \mathbb{R}^3 \) is a vector field on \( S^2 \) if \( \langle v(p), p \rangle = 0 \) for each \( p \in S^2 \) and call a vector field continuous if its component functions are continuous.

**Theorem 1.** Suppose \( v \) is a continuous vector field on \( S^2 \). Then there is \( p \in S^2 \) such that \( v(p) = 0 \).

This classical theorem was originally proven by Poincaré and is sometimes called the “Hairy Ball theorem.” Theorem 1 has many interesting proofs (see, for instance, [2] and the charming book [1]) and various generalizations; for more information, see the introduction of [2]. The distinguishing attribute of the present proof is its brevity and elegance: Each of the aforementioned proofs requires computations in and between a set of stereographic coordinate charts that appropriately cover \( S^2 \). The argument here is shorter and simpler.

A regular smooth curve in the plane is a smooth map \( S^1 \to \mathbb{R}^2 \) whose derivative does not vanish anywhere. The rotation number of such a curve \( \gamma \) is \( \frac{1}{2\pi} \) times the change that the oriented angle \( \dot{\gamma} \) makes with some fixed reference direction (e.g., \( e_1 = (1, 0) \)) as the curve is traversed; in other words, it is the winding number of \( \dot{\gamma} \), thought of as a map \( S^1 \to \mathbb{R}^2 \setminus \{0\} \). The rotation number is an integer that is an invariant under regular homotopy (homotopy through regular curves).

**Proof.** Suppose for the sake of a contradiction that \( S^2 \) admits a continuous nonvanishing vector field \( v \); we may suppose \( v \) has unit length by replacing \( v \) with \( \frac{v}{|v|} \). We first note that the definition of rotation number can be extended to curves in \( S^2 \) by replacing the fixed direction \( e_1 \) by the variable direction \( v \) in the definition above.

To see this, endow \( \mathbb{R}^3 \) with a right-handed orientation so the ordered 3-tuple of standard basis vectors \( \{e_1, e_2, e_3\} \) is positively oriented and identify \( \mathbb{R}^2 \) with the subset \( \{(x, y, z) \in \mathbb{R}^3 : z = 0\} \subset \mathbb{R}^3 \). Given \( p \in S^2 \) and a unit vector \( w \in T_pS^2 \), there is a unique unit vector \( w^+ \in T_pS^2 \) such that \( \{p, w, w^+\} \) is positively oriented. For such \( p \) and \( w \), denote by \( \Phi_{p,w} \) the isometry of \( \mathbb{R}^3 \) determined by requesting that \( \Phi_{p,w} \) map the point \( p \) to \( 0 \) and send the ordered 3-tuple of tangent vectors \( \{w, w^+, p\} \subset T_p\mathbb{R}^3 \) to \( \{e_1, e_2, e_3\} \subset T_0\mathbb{R}^3 \). Clearly, \( \Phi_{p,w} \) depends continuously on \( p \) and \( w \). We define the rotation number of a curve \( \gamma \) in \( S^2 \) with respect to \( v \) to be the winding number of the continuous curve \( \Phi_{v(\gamma)}(\dot{\gamma}) \).

Consider now the family of regular smooth curves in \( S^2 \) defined as follows: \( C_{p,s} \) (for \( p \in S^2, s \in (-1, 1) \)) is the circle that is the intersection of \( S^2 \) and the plane \( \{q \in S^2 : \)}
\( \langle q, p \rangle = s \rangle \), oriented so that \( p \) is the positive normal. These curves are all regularly homotopic and so have the same rotation number with respect to \( v \), say \( n \).

Now notice that for \( s = 0 \), \( C_{p,s} \) and \( C_{-p,s} \) parametrize the same great circle but with opposite orientations. Thus, \( n = -n \) and hence \( n = 0 \). On the other hand, for \( s \) close to \( 1 \), the rotation number of \( C_{p,s} \) is close to the rotation number of a circle in the plane because \( v \) is close to \( v(p) \) on \( C_{p,s} \) by continuity. Thus, \( n \in \{-1, 1\} \). This is a contradiction.

REFERENCES


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