Abstract. We note an argument proving simultaneously Hopf’s rotation angle theorem and the $\mathcal{C}^1$ Jordan curve theorem.

The Jordan curve theorem and Hopf rotation angle theorem are fundamental results about simple, closed, plane curves (hereafter referred to as Jordan curves). The first of these asserts that a Jordan curve bounds exactly two regions: an interior and exterior. The second asserts that the net angle the tangent vector of a positively oriented $\mathcal{C}^1$ Jordan curve rotates as the curve is traversed is $2\pi$. The purpose of this note is to unite these theorems with a single proof using a weak tubular neighborhood theorem proved below. While the application to the Jordan curve theorem is standard [4, Sec. 2.1], it appears that the application of the tubular neighborhood theorem to the rotation angle theorem may be new.

Hopf’s theorem (see [1] for an exposition by Hopf), sometimes called the Umlaufsatz, is often used [2, Chap. 9] in proofs of the Gauss–Bonnet theorem. The Jordan curve theorem for general continuous curves is very subtle (see, however, [5] for an accessible proof), and we restrict our considerations to $\mathcal{C}^1$ curves.

Let $\{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$ and $S^1 \subset \mathbb{R}^2$ be the unit circle. Let $\Pi: \mathbb{R} \rightarrow S^1$ be the covering map defined by $\Pi(x) = (\cos 2\pi x, \sin 2\pi x)$. We may identify $S^1$ with $[0, 1]/\ast$, where $\ast$ is the equivalence relation generated by requiring that $0 \ast 1$. Using this identification, any function $f: S^1 \rightarrow \mathbb{R}^2$ may be regarded as a function $f: [0, 1] \rightarrow \mathbb{R}^2$, where $f(0) = f(1)$.

Given $p \in \mathbb{R}^2$ and $r > 0$, let $B_r(p) = \{q \in \mathbb{R}^2 : |q - p| \leq r\}$.

The winding number or degree $\deg(f)$ of a continuous curve $f: S^1 \rightarrow \mathbb{R}^2 \setminus \{0\}$ is an integer (see [3] for an elementary but rigorous introduction) which intuitively corresponds to $\frac{1}{2\pi}$ times the net change in oriented angle $f$ makes with a fixed reference direction as the curve is traversed. More precisely, given any such $f$, we define $\deg(f) := \tilde{f}(1) - \tilde{f}(0)$, where $\tilde{f}: S^1 \rightarrow \mathbb{R}$ is the lift of $f$ uniquely determined by requiring that

$$\Pi \circ \tilde{f} = \frac{f}{|f|} \quad \text{and} \quad \tilde{f}(0) \in [0, 1). \quad (1)$$

The winding number is a homotopy invariant: if $f$ and $g$ are homotopic (written $f \sim g$), then $\deg(f) = \deg(g)$.

A closed curve $\gamma: S^1 \rightarrow \mathbb{R}^2$ is $\mathcal{C}^1$ if its component functions $x(t)$ and $y(t)$ are continuously differentiable and regular if $\dot{\gamma}(t) \neq 0$ for all $t \in [0, 1]$. The rotation index of a regular curve $\gamma: S^1 \rightarrow \mathbb{R}^2$ is defined to be $\deg(\dot{\gamma})$. For such a curve $\gamma$, we define the
reversal \( \gamma' : S^1 \rightarrow \mathbb{R}^2 \) by \( \gamma'(t) = \gamma(1 - t) \). \( \gamma' \) is the same curve as \( \gamma \), but traversed in the opposite direction, and the rotation index of \( \gamma' \) is minus the rotation index of \( \gamma \). If \( \gamma : S^1 \rightarrow \mathbb{R}^2 \) is a closed plane curve and \( p \in \mathbb{R}^2 \setminus \gamma \) (where here, as elsewhere in this note, we abuse notation slightly by identifying \( \gamma \) with its image), the winding number of \( \gamma \) about \( p \) is defined to be \( \deg(\gamma - p) \).

**Theorem 1.** Let \( \gamma \) be a regular, \( C^1 \) Jordan curve. After possibly replacing \( \gamma \) by \( \gamma' \):

1. *(Umlaufsatz)* The rotation index of \( \gamma \) is one.
2. *(Jordan curve theorem)* \( \mathbb{R}^2 \setminus \gamma \) is a disjoint union of two path components, the interior, \( \text{int}(\gamma) \), and exterior, \( \text{ext}(\gamma) \), which are characterized by

\[
\deg(\gamma - p) = \begin{cases} 
1 & \text{if } p \in \text{int}(\gamma) \\
0 & \text{if } p \in \text{ext}(\gamma).
\end{cases}
\]

**Proof.** By compactness, the image of \( \gamma \) contains a point with smallest \( y \)-coordinate; suppose without loss of generality that \( \gamma(0) \) is such a point. After possibly replacing \( \gamma \) by its reversal \( \gamma' \), we may suppose \( \gamma(0) = ce_1 \) for some \( c > 0 \). Choose the continuous unit normal field \( \nu \) along \( \gamma \) such that \( \nu_{\gamma(0)} = -e_2 \).

For \( \epsilon \in \mathbb{R} \), consider the curve \( \gamma_\epsilon \) defined by \( \gamma_\epsilon(t) := \gamma(t) - \epsilon \nu_{\gamma(t)} \). When \( |\epsilon| \) is small, \( \gamma_\epsilon \) should be thought of a “parallel curve” to \( \gamma \). The tubular neighborhood theorem [2, Theorem 10.19] asserts that for a smooth Jordan curve \( \gamma \) and \( \epsilon > 0 \) small enough, the map from \( \gamma \times (-\epsilon, \epsilon) \) to \( \mathbb{R}^2 \) defined by \( (p, \epsilon') \mapsto \gamma_{\epsilon'}(p) \) is a diffeomorphism. For our purposes, the following less general version—proved at the end of the note—will suffice.

**Assertion.** There exists \( \epsilon > 0 \) such that:

1. For all \( \epsilon' \) such that \( 0 < |\epsilon'| \leq \epsilon \), the curves \( \gamma \) and \( \gamma_{\epsilon'} \) have disjoint images.
2. For all \( t \in [0, 1] \), if \( \gamma(t) \in B_\epsilon(\gamma(0)) \), then the \( y \)-component of \( \nu_{\gamma(t)} \) is negative.

Now fix some \( \epsilon > 0 \) such that the assertion holds and define \( p_{\text{in}} := \gamma_\epsilon(0) \), \( p_{\text{out}} := \gamma_{-\epsilon}(0) \). We claim that

1. \( \deg(\gamma - p_{\text{in}}) = 1 \) and \( \deg(\gamma - p_{\text{out}}) = 0 \).
2. \( \gamma - p_{\text{in}} \sim \gamma \).

**Figure 1.** Portions of the curves \( \gamma, \gamma'_\epsilon, \) and \( \gamma_{-\epsilon} \), where \( \epsilon' \in (0, \epsilon] \).

Intuitively (a) encodes the fact that \( \gamma \) winds once around \( p_{\text{in}} \) and zero times around \( p_{\text{out}} \), since \( p_{\text{in}} \) lies just above the lowest point of \( \gamma \), while \( p_{\text{out}} \) is entirely below \( \gamma \) (see Figure 1).

The Umlaufsatz follows immediately from combining the first part of (a) with (b). For the Jordan curve theorem, note that since \( \deg(\gamma - p) \) is a locally constant function
of \( p \in \mathbb{R}^2 \setminus \gamma \), (a) implies that \( \mathbb{R}^2 \setminus \gamma \) has at least two path components. We will show that each point in \( \mathbb{R}^2 \setminus \gamma \) can be connected by a path in \( \mathbb{R}^2 \setminus \gamma \) to a point on either \( \gamma_e \) or \( \gamma_{-e} \), and it will follow from this that \( \mathbb{R}^2 \setminus \gamma \) has at most two path components.

Fix \( p \in \mathbb{R}^2 \setminus \gamma \) and let \( q \) be a point on \( \gamma \) that is closest to \( p \). By elementary geometry, the line segment connecting \( p \) and \( q \) is orthogonal to the tangent line to \( \gamma \) at \( q \); hence \( p = q + \delta v_q \) for some \( \delta \neq 0 \). If \( \delta > 0 \), consider the point \( r := q + \epsilon v_q \), which is on \( \gamma_{-\epsilon} \). Each point on the line segment from \( p \) to \( r \) is either closer to \( p \) than \( q \) is and consequently not on \( \gamma \) by the choice of \( q \), or on the curve \( \gamma_{-\epsilon} \) for some \( 0 < \epsilon' \leq \epsilon \) and therefore not on \( \gamma \) by (i). Therefore, the line segment from \( p \) to \( r \) lies in \( \mathbb{R}^2 \setminus \gamma \). If \( \delta < 0 \), we take \( r := q - \epsilon v_q \), which is on \( \gamma_e \) and argue in the same way. We conclude that \( \mathbb{R}^2 \setminus \gamma \) has at most two path components, and this completes the proof of the Jordan curve theorem.

It remains to prove (a) and (b). We first claim that

\[
(\gamma - p_{\text{in}})^{-1} \left( \{(0, y) \in \mathbb{R}^2 : y < 0\} \right) = \{0, 1\}. \tag{2}
\]

By the choice of \( \gamma(0) \), if \( \gamma(t) - p_{\text{in}} \) were on the negative \( y \)-axis for some \( t \notin [0, 1] \), then \( \gamma(t) \) would lie on \( \gamma_e \) for some \( \epsilon' \) satisfying \( 0 < \epsilon' \leq \epsilon \), which would contradict (i). Let \( \Gamma \) be the lift of \( \gamma - p_{\text{in}} \) as in (1). Since \( \gamma - p_{\text{in}} \) is \( C^1 \) and \( \gamma(0) = e_1 \), it follows from (2) and the intermediate value theorem that \( \deg(\gamma - p_{\text{in}}) = \Gamma(1) - \Gamma(0) = 1 \). By the choice of \( \gamma(0) \), the image of \( \gamma - p_{\text{out}} \) omits the entire negative \( y \)-axis, so an argument similar to the one above shows \( \deg(\gamma - p_{\text{out}}) = 0 \). This proves (a).

For (b), define \( H : [0, 1]^2 \to \mathbb{R}^2 \setminus \{0\} \) by

\[
H(s, t) = \gamma(t) - \gamma_e(s).
\]

Notice that \( \epsilon v_\gamma \) is equal to the diagonal of \( H \) (i.e., \( \epsilon v_{\gamma(t)} = H(t, t) \)). The diagonal of \( H \) is also homotopic to the concatenation \( V \) of the left and top sides of the domain square given by

\[
V(t) = \begin{cases} 
H(0, 2t) & \text{if } t \in [0, 1/2] \\
H(2t - 1, 0) & \text{if } t \in [1/2, 1].
\end{cases}
\]

We now claim that the image of the second loop in this product omits the entire positive \( y \)-axis and hence is nullhomotopic. To see this, suppose for the sake of a contradiction that there exists \( t \in S^1 \) such that \( \gamma(0) - \gamma_e(t) \) lies on the positive \( y \)-axis. The choice of \( \gamma(0) \) implies that \( \gamma(t) - \gamma(0) \) and \( \gamma(t) - \gamma_e(t) = \epsilon v_{\gamma(t)} \) each have positive \( y \)-components and that \( |\gamma(0) - \gamma(t)| < \epsilon \). This contradicts (ii).

Consequently, since \( \gamma - p_{\text{in}} = H(0, \cdot) \), it follows that \( \gamma - p_{\text{in}} \sim v_\gamma \). Since the maps \( v_\gamma \) and \( \dot{\gamma} \) differ by constant angle \( \pi/2 \), we see that \( v_\gamma \sim \dot{\gamma} \); hence \( \gamma - p_{\text{in}} \sim \gamma \) and (b) holds.

**Proof of Assertion.** We first show there exists \( \epsilon > 0 \) such that (i) holds. Suppose for the sake of a contradiction that for each \( k \in \mathbb{N} \), there exists \( \epsilon_k \) such that \( 0 < |\epsilon_k| \leq \frac{1}{k} \) and \( t_k, t'_k \in [0, 1] \) such that \( \gamma(t_k) = \gamma_{-\epsilon_k}(t'_k) \). By compactness, we may suppose that both \( (t_k)_{k \in \mathbb{N}} \) and \( (t'_k)_{k \in \mathbb{N}} \) converge. Since \( \gamma \) is embedded, both sequences have the same limit, say \( t_\infty \). By a change of coordinates, we may suppose \( \gamma(t_\infty) = 0 \) and \( \dot{\gamma}(t_\infty) = ce_1 \), for some \( c > 0 \). Since \( \gamma \) is \( C^1 \), there exists \( \delta > 0 \) such that, for all \( t \) sufficiently close to \( t_\infty \), \( \gamma(t) \) lies on the graph of a function \( f \in C^1((-\delta, \delta)) \) satisfying \( f(0) = f'(0) = 0 \). Then for all sufficiently large \( k \in \mathbb{N} \), there exist \( x_k, x'_k \) satisfying \( \lim_{k \to \infty} x_k = \lim_{k \to \infty} x'_k = 0 \) such that

\[
\gamma(t_k) = (x_k, f(x_k)), \quad \gamma_{-\epsilon_k}(t'_k) = (x'_k, f(x'_k)) - \frac{\epsilon_k(f'(x'_k), -1)}{\sqrt{1 + (f'(x'_k))^2}}. \tag{3}
\]
From the assumption that $\gamma(t_k) = \gamma_\epsilon(t'_k)$ and (3), we conclude
\[
\frac{f(x_k) - f(x'_k)}{x_k - x'_k} = -\frac{1}{f'(x'_k)},
\]
which is clearly a contradiction, since (because $f$ is $C^1$) the left side of (4) approaches zero as $k \to \infty$, while the magnitude of the right-hand side is unbounded as $k \to \infty$.

We next show that (ii) holds as long as $\epsilon > 0$ is small enough. Since $\nu_{\gamma(0)} = -\epsilon_2$, choose (by continuity) $\delta > 0$ such that if $0 \leq t < \delta$ or $1 - \delta < t \leq 1$ then $\nu_{\gamma(t)}$ has negative $y$-component. If $\epsilon > 0$ is sufficiently small, then $B_\epsilon(\gamma(0))$ is disjoint from the image of $[\delta, 1 - \delta]$ under $\gamma$ (since this image is compact). For such a choice of $\epsilon$, if $\gamma(t) \in B_\epsilon(\gamma(0))$, then either $t < \delta$ or $t > 1 - \delta$, and therefore $\nu_{\gamma(t)}$ has negative $y$-component. ■

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REFERENCES


Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104
pjmcgrat@sas.upenn.edu