

# ON BOGOMOLOV'S BIRATIONAL ANABELIAN PROGRAM II

## ON IHARA'S QUESTION/ODA-MATSUMOTO CONJECTURE

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ABSTRACT. Let  $\Pi_K^c \rightarrow \Pi_K$  be the maximal pro- $\ell$  abelian-by-central, respectively abelian, Galois groups of a function field  $K|k$  with  $k$  algebraically closed with  $\text{char} \neq \ell$ . We show that if  $\text{td}(K|k) > 2$ , then  $K|k$  can be reconstructed from  $\Pi_K^c$  endowed with the set of the inertia groups  $T_v \subset \Pi_K$  of all the prime divisors  $v$  of  $K|k$ . As applications, one gets a group theoretical recipe to reconstruct  $K|k$  from  $\Pi_K^c$ , provided  $\text{td}(K|k) > \dim(k)+1$ , where  $\dim(k)$  is the Kronecker dimension. This extends the main known results from the basic case where  $\dim(k) = 0$ , i.e.,  $k$  is an algebraic closure of a finite field, see [B–T], [P4]. We also give an application to the *pro- $\ell$  abelian-by-central* I/OM (Ihara's question / Oda-Matsumoto conjecture), which in the cases considered here turns out to imply the classical I/OM.

### 1. INTRODUCTION

We introduce notations which will be used throughout the manuscript as follows:

- $\ell$  is a fixed rational prime number.
- $K|k$  are function fields with  $k$  algebraically closed of characteristic  $\neq \ell$ .
- $K'|K \hookrightarrow K^c|K$  are pro- $\ell$  abelian, respectively abelian-by-central, extensions.
- $\Pi_K^c := \text{Gal}(K^c|K) \rightarrow \text{Gal}(K'|K) =: \Pi_K$  are the corresponding Galois groups.

Notice that  $\Pi_K^c \rightarrow \Pi_K$  is encoded in  $\Pi_K^c$ , because  $\ker(\Pi_K^c \rightarrow \Pi_K) = [\Pi_K^c, \Pi_K^c]$ .

This manuscript concerns a program initiated by BOGOMOLOV [Bo] at the beginning of the 1990's, which has as ultimate goal to reconstruct  $K|k$  from  $\Pi_K^c$ , provided  $\text{td}(K|k) > 1$ . Notice that in contrast to Grothendieck's (birational) anabelian philosophy, where the arithmetical Galois action must be part of the picture, in Bogomolov's program there is absolutely no arithmetical action (because the base field  $k$  is algebraically closed). Note also that the condition  $\text{td}(K|k) > 1$  is necessary, because  $\text{td}(K|k) = 1$  implies that the absolute Galois group  $G_K$  is profinite free on  $|k|$  generators – as showed by DOUADY, HARBATER, POP – and therefore,  $\Pi_K^c$  depends on the cardinality of the base field  $|k|$  only.

A strategy to tackle Bogomolov's program was proposed by the author in 1999, and along the lines of that strategy, the Bogomolov program was completed for  $k$  an algebraic closure of a finite field by BOGOMOLOV–TSCHINKEL [B–T] and POP [P4].

In order to explain the results of this note, we recall a few basic facts as follows.

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First, recall that given a function field  $K|k$  as above, a **prime divisor** of  $K|k$  is any valuation  $v$  of  $K|k$  which is trivial on  $k$  and its residue field  $Kv$  satisfies  $\text{td}(K|k) = \text{td}(Kv|k) + 1$ . Equivalently, a valuation  $v$  of  $K$  is a prime divisor of  $K|k$  if and only if there exists a normal model  $X$  of  $K|k$  such that  $v$  is a Weil prime divisor of  $X$ . For any valuation  $v$  of  $K$  we denote by  $T_v \subseteq Z_v \subset \Pi_K$  the inertia/decomposition groups of some prolongation  $v$  to  $K'$  (and notice that  $T_v \subset Z_v$  depend on  $v$  only, because  $\Pi_K$  is abelian). If  $v$  is a prime divisor of  $K|k$ , by abuse of language, we call the pair  $T_v \subset Z_v$  a **divisorial group** in  $\Pi_K$ , and we denote by  $\mathfrak{In.div}(K) := \cup_v T_v \subset \Pi_K$  the set of all the **divisorial inertia elements** in  $\Pi_K$ .

Second, for a further function field  $L|l$  with  $l$  algebraically closed, let  $\text{Isom}^F(L^i, K^i)$  be the set of the isomorphisms of the pure inseparable closures up to Frobenius twists. Further, let  $\text{Isom}^c(\Pi_K, \Pi_L)$  be the set of the abelianizations  $\Phi : \Pi_K \rightarrow \Pi_L$  of the isomorphisms  $\Pi_K^c \rightarrow \Pi_L^c$  modulo multiplication by  $\ell$ -adic units. Note there exists a canonical embedding

$$\text{Isom}^F(L^i, K^i) \rightarrow \text{Isom}^c(\Pi_K, \Pi_L), \quad \phi \mapsto \Phi_\phi$$

by  $\Phi_\phi(\sigma) := \phi'^{-1} \circ \sigma \circ \phi'$  for  $\sigma \in \Pi_K$ , where  $\phi' : L^{i'} \rightarrow K^{i'}$  is any prolongation of  $\phi : L^i \rightarrow K^i$  to the pro- $\ell$  abelian closures. (Note that  $\Phi_\phi$  depends on  $\phi$  only, and not on the specific  $\phi'$ .)

Finally, recall that a natural generalization of the prime divisors of  $K|k$  are the **quasi prime divisors** of  $K|k$ , which are valuations  $v$  (not necessarily trivial on  $k$ ) satisfying: First,  $vK/vk \cong \mathbb{Z}$  and  $\text{td}(K|k) = \text{td}(Kv|kv) + 1$ , and second,  $v$  is minimal satisfying these two conditions, i.e., if  $w$  is a valuation of  $K$  satisfying the two conditions and the valuation rings satisfy  $\mathcal{O}_w \supseteq \mathcal{O}_v$ , then  $w = v$ . In particular, the prime divisors of  $K|k$  are the quasi prime divisors of  $K|k$  that are trivial on  $k$ . For a quasi prime divisor  $v$ , let  $T_v^1 \subset Z_v^1 \subset \Pi_K$  be its **minimized inertia/decomposition groups**, see Section 2, A), Decomposition Graphs, and/or TOPAZ [To], for definitions, and recall that  $T_v = T_v^1$ ,  $Z_v = Z_v^1$  if  $\text{char}(Kv) \neq \ell$ . By POP [P1] and TOPAZ [To], one can recover  $T_v^1 \subset Z_v^1$  from  $\Pi_K^c$ , thus the set of all the (minimized) quasi-divisorial inertia elements  $\mathfrak{In.q.div}(K) = \cup_v T_v^1 \subset \Pi_K$ . Moreover, the recipe to do so is invariant under isomorphisms, i.e., in the above notations, every  $\Phi \in \text{Isom}^c(\Pi_K, \Pi_L)$  has the property that  $\Phi(\mathfrak{In.q.div}(K)) = \mathfrak{In.q.div}(L)$ .

Unfortunately, for the time being, to the best of our knowledge, there is no strategy to describe  $\mathfrak{In.div}(K)$  inside  $\mathfrak{In.q.div}(K)$  in the case  $k$  is an arbitrary algebraically closed base field. The main result of this note *reduces the Bogomolov program to describing  $\mathfrak{In.div}(K)$  inside  $\mathfrak{In.q.div}(K)$  using the information encoded in  $\Pi_K^c$* , and reads as follows:

**Theorem 1.1.** *In the above notations, there exists a group theoretical recipe about pro- $\ell$  abelian-by-central groups such that the following hold:*

- 1) *The recipe reconstructs  $K|k$  from  $\Pi_K^c$  endowed with  $\mathfrak{In.div}(K) \subset \Pi_K$  in a functorial way, provided  $\text{td}(K|k) > 2$ .*
- 2) *The recipe is invariant under isomorphisms  $\Phi \in \text{Isom}^c(\Pi_K, \Pi_L)$  in the sense that if  $\Phi(\mathfrak{In.div}(K)) = \mathfrak{In.div}(L)$ , then  $\Phi = \Phi_\phi$  for a (unique)  $\phi \in \text{Isom}^F(L^i, K^i)$ .*

The proof of the above Theorem 1.1 is quite involved and relies heavily on “specialization” techniques, among other things on a result by JOSSEN [Jo] on specialization of points of abelian varieties, and previous work by the author POP [P1], [P2], [P3] and [P4]. See subsection A) of Section 2 for the quite involved strategy and structure of the proof.

Concerning applications of Theorem 1.1 above, the point is that under supplementary conditions on the base field  $k$ , one can distinguish  $\mathfrak{In}.div(K)$  inside  $\mathfrak{In}.q.div(K)$ , thus Theorem 1.1 above is applicable. We begin by recalling the Kronecker dimension  $\dim(k)$  of  $k$  is defined as follows: First, we set  $\dim(\mathbb{F}_p) = 0$  and  $\dim(\mathbb{Q}) = 0$ , and second, if  $k_0 \subset k$  is the prime field of  $k$ , we define  $\dim(k) := \text{td}(k|k_0) + \dim(k_0)$ . Hence  $\dim(k) = 0$  if and only if  $k$  is an algebraic closure of a finite field, and  $\dim(k) = 1$  if and only if  $k$  is an algebraic closure of a global field, etc. Therefore,  $\text{td}(K|k) > 1$  is equivalent to  $\text{td}(K|k) > \dim(k) + 1$  in the case  $k$  is an algebraic closure of a finite field, whereas  $\text{td}(K|k) > \dim(k) + 1$  is equivalent to  $\text{td}(K|k) > 2$  if  $k$  is an algebraic closure of a global field, etc.

In light of the above discussion and notations, one has the following generalization of the main results of BOGOMOLOV–TSCHINKEL [B–T], POP [P4]:

**Theorem 1.2.** *Let  $K|k$  be an function field with  $\text{td}(K|k) > 1$  and  $k$  algebraically closed of Kronecker dimension  $\dim(k)$ . The following hold:*

- 1) *For every nonnegative integer  $\delta$ , there exists a group theoretical recipe  $\mathfrak{dim}(\delta)$  depending on  $\delta$  such that given the profinite group  $\Pi_K^c$ , one can decide whether  $\dim(k) = \delta$  and  $\text{td}(K|k) > \dim(k)$  by applying the recipe  $\mathfrak{dim}(\delta)$  to  $\Pi_K^c$ .*
- 2) *There is a group theoretical recipe which recovers  $\mathfrak{In}.div(K) \subset \Pi_K$  from  $\Pi_K^c$ , provided  $\text{td}(K|k) > \dim(k)$ . Thus if  $\text{td}(K|k) > \dim(k) + 1$ , then by **Theorem 1.1, 1)**, one can reconstruct  $K|k$  functorially from  $\Pi_K^c$ .*
- 3) *Both recipes above are invariant under isomorphisms of profinite groups as follows: Suppose that  $\Pi_K^c$  and  $\Pi_L^c$  are isomorphic. Then  $\text{td}(K|k) = \text{td}(L|l)$ , and one has:  $\dim(k) = \delta$  and  $\text{td}(K|k) > \dim(k)$  if and only if  $\dim(l) = \delta$  and  $\text{td}(L|l) > \dim(l)$ .*
- 4) *Suppose that  $\text{td}(K|k) > \dim(k) + 1$ . Then by **Theorem 1.1, 2)**, one concludes that the canonical map below is a bijection:*

$$\text{Isom}^F(L^i, K^i) \longrightarrow \text{Isom}^c(\Pi_K, \Pi_L), \quad \phi \mapsto \Phi_\phi.$$

If  $\dim(k) = 1$ , our methods developed here work as well for the function fields  $K = k(X)$  of projective smooth surfaces  $X$  with finite (étale) fundamental group, thus for function fields of “generic” surfaces. And our methods also work provided  $\text{td}(K|k) > \dim(k) > 1$ .

An immediate consequence of Theorem 1.2 is a positive answer to a question by Ihara from the 1980’s, which in the 1990’s became a conjecture by Oda–Matsumoto, for short I/OM. The I/OM is about giving a topological/combinatorial description of the absolute Galois group of the rational numbers. See POP [P5], Introduction, for explanations concerning the I/OM. The situation we consider here is as follows: Let  $k_0$  be an arbitrary perfect field, and  $k := \bar{k}_0$  an algebraic closure. Let  $X$  be a geometrically integral  $k_0$ -variety,  $\mathcal{U}_X := \{U_i\}_i$  be a basis of open neighborhoods of the generic point  $\eta_X$ , and  $\mathcal{U}_{\bar{X}} = \{\bar{U}_i\}_i$  its base change to  $k$ . Set  $\Pi^c(\bar{U}_i) := \pi_1^c(\bar{U}_i)$  and  $\Pi(\bar{U}_i) := \pi_1^{\ell, \text{ab}}(\bar{U}_i)$ . Then letting  $K := k(\bar{X})$  be the function field of  $\bar{X}$ , it follows that  $\Pi_K^c \rightarrow \Pi_K$  is the projective limit of the system  $\Pi^c(\bar{U}_i) \rightarrow \Pi(\bar{U}_i)$ , and there exists a canonical embedding  $\text{Aut}^c(\Pi(\mathcal{U}_{\bar{X}})) \hookrightarrow \text{Aut}^c(\Pi_K)$ . Finally let  $\text{Aut}_k(K^i) \hookrightarrow \text{Aut}^F(K^i)$  be the group of all the  $k$ -automorphisms of  $K^i$ , respectively all the field automorphisms of  $K^i$  modulo Frobenius twists. Note that since  $k \subset K^i$  is the unique maximal algebraically closed subfield in  $K^i$ , every  $\phi \in \text{Aut}^F(K^i)$  maps  $k$  isomorphically onto itself, hence  $\text{Aut}^F(K^i)$  acts on  $k$ . Let  $k_K \subseteq k_0$  be the corresponding fixed field up to Frobenius twists.

**Theorem 1.3.** *In the above notations, suppose that  $\dim(X) > \dim(k_0) + 1$ . Then one has a canonical exact sequence of the form:  $1 \rightarrow \text{Aut}_k(K^i) \rightarrow \text{Aut}^c(\Pi_K) \rightarrow \text{Aut}_{k_K}^F(k) \rightarrow 1$ .*

*Thus if  $\text{Aut}_k(K^i) = 1$  and  $k_K = k_0$ , then  $v_K^c : \text{Gal}_{k_0} \rightarrow \text{Aut}^c(\Pi_K)$  is an isomorphism, hence the pro- $\ell$  abelian-by-central I/OM holds for  $\mathcal{U}_X$ , and so does the classical I/OM.*

We note that Theorem 1.3 is an immediate consequence of Theorem 1.2: Setting namely  $L|l = K|k$ , Theorem 1.2 implies that the canonical map  $\text{Aut}^F(K^i) \rightarrow \text{Aut}^c(\Pi_K)$  is an isomorphism of groups. Further, since  $k \subset K^i$  is the unique maximal algebraically closed subfield of  $K^i$ , every  $\phi \in \text{Aut}^F(K^i)$  maps  $k$  isomorphically onto itself. Conclude by noticing that one has an obvious exact sequence of groups

$$1 \rightarrow \text{Aut}_k(K^i) \rightarrow \text{Aut}^F(K^i) \rightarrow \text{Aut}_{k_K}^F(k) \rightarrow 1.$$

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## 2. PROOF OF THEOREM 1.1

### A) Formulation of a strengthening of Theorem 1.1

Let  $k$  be an algebraically closed field with  $\text{char}(k) \neq \ell$ , and  $K|k$  be a function field with  $d := \text{td}(K|k) > 1$ . We begin by recalling briefly basics about the (quasi) prime divisors of the function field  $K|k$ , see POP [P1], Section 3, for more details.

Recall that a **prime  $r$ -divisor** if  $\mathfrak{v}$  is the valuation theoretical composition  $\mathfrak{v} = v_r \circ \cdots \circ v_1$ , where  $v_1$  is a prime divisor of  $K$ , and inductively,  $v_{i+1}$  is a prime divisor of the residue function field  $K\mathfrak{v}_i|k$ , where  $\mathfrak{v}_i := v_i \circ \cdots \circ v_1$  for  $i < r$ . In particular,  $r \leq \text{td}(K|k)$ , and  $\mathfrak{v}_i = v_i \circ \cdots \circ v_1$  is a prime  $i$ -divisor for all  $i \geq 1$ . By definition, the trivial valuation will be the **rank zero** generalized prime divisor. Finally, if  $r$  is irrelevant for the context, we will speak simply about a **generalized prime divisor**.

Next recall that a valuation  $\mathfrak{v}$  is called a **quasi prime  $r$ -divisor** of  $K|k$  if  $\mathfrak{v}$  is a (valuation theoretical) composition  $\mathfrak{v} = v_r \circ \cdots \circ v_1$ , where  $v_1$  is a quasi prime divisor of  $K|k$ , and setting  $\mathfrak{v}_i := v_i \circ \cdots \circ v_1$ , inductively, one has that  $v_{i+1}$  is a quasi prime divisor of the residue function field  $K\mathfrak{v}_i|k\mathfrak{v}_i$ .

Note that the prime  $r$ -divisors of  $K|k$  are precisely the quasi prime  $r$ -divisors of  $K|k$  which are trivial on  $k$ .

- **The total (quasi) prime divisor graph**

We define the **total prime divisor graph**  $\mathcal{D}_K^{\text{tot}}$  of  $K|k$  to be the half-oriented graph as follows:

- a) The vertices of  $\mathcal{D}_K^{\text{tot}}$  are the residue fields  $K\mathfrak{v}$  of all the generalized prime divisors  $\mathfrak{v}$  of  $K|k$  viewed as distinct function fields.
- b) For  $\mathfrak{v} = v_r \circ \cdots \circ v_1$  and  $\mathfrak{w} = w_s \circ \cdots \circ w_1$ , the edges from  $K\mathfrak{v}$  to  $K\mathfrak{w}$  are as follows:
  - i) If  $\mathfrak{v} = \mathfrak{w}$ , then the trivial valuation  $\mathfrak{v}/\mathfrak{w} = \mathfrak{w}/\mathfrak{v}$  of  $K\mathfrak{v} = K\mathfrak{w}$  is the only edge from  $K\mathfrak{v} = K\mathfrak{w}$  to itself; and this is by definition a non-oriented edge.

- ii) If  $K\mathfrak{v} \neq K\mathfrak{w}$ , then the set of edges from  $K\mathfrak{v}$  to  $K\mathfrak{w}$  is non-empty iff  $s = r + 1$  and  $v_i = w_i$  for  $1 \leq i \leq r$ ; and if so, then  $w_s = \mathfrak{w}/\mathfrak{v}$  is the only edge from  $K\mathfrak{v}$  to  $K\mathfrak{w}$ , and this is by definition an oriented edge.

We define the **total quasi prime divisor graph**  $\mathcal{Q}_K^{\text{tot}}$  of  $K|k$  in a totally parallel way, but considering as vertices all the generalized quasi prime divisors (instead of the generalized prime divisors) of  $K|k$ . Notice that  $\mathcal{D}_K^{\text{tot}} \subset \mathcal{Q}_K^{\text{tot}}$  is a full subgraph.

Finally, we note that the total (quasi) prime divisor graph of  $K|k$  has the following functorial properties:

1) *Embeddings*. Let  $L|l \hookrightarrow K|k$  be an embedding of function fields which maps  $l$  isomorphically onto  $k$ . Then the canonical restriction map of valuations  $\text{Val}_K \rightarrow \text{Val}_L$ ,  $v \mapsto v|_L$ , gives rise to a surjective morphism of the total (quasi) prime divisor graphs  $\varphi_i : \mathcal{D}_K^{\text{tot}} \rightarrow \mathcal{D}_L^{\text{tot}}$  and  $\varphi_i : \mathcal{Q}_K^{\text{tot}} \rightarrow \mathcal{Q}_L^{\text{tot}}$ .

2) *Restrictions*. Given a generalized (quasi) prime divisor  $\mathfrak{v}$  of  $K|k$ , let  $\mathcal{D}_{\mathfrak{v}}^{\text{tot}} \subset \mathcal{Q}_{\mathfrak{v}}^{\text{tot}}$  be the set of all the generalized prime, respectively quasi prime, divisors  $\mathfrak{w}$  of  $K|k$  with  $\mathfrak{v} \leq \mathfrak{w}$ . Then the map  $\mathcal{Q}_{\mathfrak{v}}^{\text{tot}} \rightarrow \text{Val}_{K\mathfrak{v}}$ ,  $\mathfrak{w} \mapsto \mathfrak{w}/\mathfrak{v}$ , is an isomorphism of  $\mathcal{Q}_{\mathfrak{v}}^{\text{tot}}$  onto the total quasi prime divisor graph of  $K\mathfrak{v}|k\mathfrak{v}$ . And if  $\mathfrak{v}$  is a prime divisor, the above map also defines an isomorphism from  $\mathcal{D}_{\mathfrak{v}}^{\text{tot}}$  onto the total prime divisor graph of  $K\mathfrak{v}|k$ .

• **Decomposition graphs** [See POP [P3], Section 3, for more details.]

Let  $K|k$  be as above. For every valuation  $v$  of  $K$ , let  $1 + \mathfrak{m}_v =: U_v^1 \subset U_v := \mathcal{O}_v^\times$  be the principal  $v$ -units, respectively the  $v$ -units in  $K^\times$ . By POP [P1] and TOPAZ [To], it follows that the decomposition field  $K_v^Z$  of  $v$  is contained in  $K^{Z^1} := K[\sqrt[\ell^\infty]{U_v^1}]$ , and the inertia field  $K_v^T$  of  $v$  is contained in  $K^{T^1} := K[\sqrt[\ell^\infty]{U_v}]$ . We denote  $T_v^1 := \text{Gal}(K'|K^{T^1}) \subseteq T_v$  and  $Z_v^1 := \text{Gal}(K'|K^{Z^1}) \subseteq Z_v$  and call  $T^1 \subseteq Z^1$  the **minimized inertia/decomposition groups** of  $v$ . Recalling that  $Kv^\times = U_v/U_v^1$ , by Kummer theory one gets that

$$\Pi_{K\mathfrak{v}}^1 := Z_v^1/T^1 = \text{Hom}^{\text{cont}}(U_v/U_v^1, \mathbb{Z}_\ell(1)) = \text{Hom}^{\text{cont}}(Kv^\times, \mathbb{Z}_\ell(1)).$$

By abuse of language, we say that  $\Pi_{K\mathfrak{v}}^1$  is the **minimized residue Galois group** at  $v$ . We notice that  $K^Z = K^{Z^1}$ ,  $K^{T^1} = K^T$ ,  $\Pi_{K\mathfrak{v}}^1 = \Pi_{K\mathfrak{v}}$ , provided  $\text{char}(Kv) \neq \ell$ . On the other hand, if  $\text{char}(Kv) = \ell$ , then one must have  $\text{char}(k) = 0$ , and in this case  $T_v^1 \subseteq Z_v^1 \subseteq T_v$ , hence the residue field of  $K^{Z^1}$  contains  $(Kv)'$ , thus  $\Pi_{K\mathfrak{v}}^1 \subseteq T_v/T_v^1$  has trivial image in  $\Pi_{K\mathfrak{v}} = Z_v/T_v$ .

Anyway, for generalized prime divisors  $\mathfrak{v}$  of  $K$ , one has:  $T_{\mathfrak{v}} = T_{\mathfrak{v}}^1$ ,  $Z_{\mathfrak{v}} = Z_{\mathfrak{v}}^1$ ,  $\Pi_{K\mathfrak{v}}^1 = \Pi_{K\mathfrak{v}}$ .

Recall that if  $\mathfrak{v}$  is a quasi prime  $r$ -divisor  $\mathfrak{v}$  one has  $T_{\mathfrak{v}}^1 \cong \mathbb{Z}_\ell^r$ . Moreover, any of the equalities  $K_{\mathfrak{v}}^{Z^1} = K_{\mathfrak{v}}^Z$ ,  $K_{\mathfrak{v}}^{T^1} = K_{\mathfrak{v}}^T$ ,  $\Pi_{K\mathfrak{v}}^1 = \Pi_{K\mathfrak{v}}$ , is equivalent to  $\text{char}(K\mathfrak{v}) \neq \ell$ . Further, for generalized quasi prime divisors  $\mathfrak{v}_1$  and  $\mathfrak{v}_2$  the following are equivalent:

- i)  $Z_{\mathfrak{v}_1}^1 \cap Z_{\mathfrak{v}_2}^1 \neq 1$ .
- ii)  $T_{\mathfrak{v}_1}^1 \cap T_{\mathfrak{v}_2}^1 \neq 1$ .

And if the above equivalent conditions are satisfied, there exists a unique quasi prime divisor  $\mathfrak{v}$  of  $K|k$  with  $T_{\mathfrak{v}}^1 = T_{\mathfrak{v}_1}^1 \cap T_{\mathfrak{v}_2}^1$ . Further,  $\mathfrak{v}$  is also the unique generalized quasi prime divisor of  $K|k$  which is maximal with the property  $Z_{\mathfrak{v}_1}^1, Z_{\mathfrak{v}_2}^1 \subseteq Z_{\mathfrak{v}}^1$ . Moreover, if the above equivalent conditions are satisfied, then  $\mathfrak{v}$  is trivial on  $k$ , provided  $\min(\mathfrak{v}_1, \mathfrak{v}_2)$  is trivial on  $k$ .

In particular, for generalized (quasi) prime divisors  $\mathfrak{v}$  and  $\mathfrak{w}$  of  $K|k$  one has:  $\mathfrak{v} = \mathfrak{w}$  iff  $T_{\mathfrak{v}}^1 = T_{\mathfrak{w}}^1$  iff  $Z_{\mathfrak{v}}^1 = Z_{\mathfrak{w}}^1$ . Further,  $\mathfrak{v} < \mathfrak{w}$  iff  $T_{\mathfrak{v}}^1 \subset T_{\mathfrak{w}}^1$  strictly iff  $Z_{\mathfrak{v}}^1 \supset Z_{\mathfrak{w}}^1$  strictly. And if  $\mathfrak{v} < \mathfrak{w}$  is a (quasi)mprime  $r$ -divisor, respectively a (quasi) prime  $s$ -divisor, then  $T_{\mathfrak{w}}^1/T_{\mathfrak{v}}^1 \cong \mathbb{Z}_{\ell}^{s-r}$ .

We conclude that the partial ordering on the set of all the generalized (quasi) prime divisors  $\mathfrak{v}$  of  $K|k$  is encoded in the set of their minimized inertia/decomposition groups  $T_{\mathfrak{v}}^1 \subseteq Z_{\mathfrak{v}}^1$ . In particular, the existence of the trivial, respectively nontrivial, edge from  $K\mathfrak{v}$  to  $K\mathfrak{w}$  in  $\mathcal{D}_K^{\text{tot}}$  and/or  $\mathcal{Q}_K^{\text{tot}}$  is equivalent to  $T_{\mathfrak{v}}^1 = T_{\mathfrak{w}}^1$ , respectively to  $T_{\mathfrak{v}}^1 \subset T_{\mathfrak{w}}^1$  and  $T_{\mathfrak{w}}^1/T_{\mathfrak{v}}^1 \cong \mathbb{Z}_{\ell}$ .

Via the Galois correspondence and the functorial properties of the Hilbert decomposition theory for valuations, we attach to the total prime divisor graph  $\mathcal{D}_K^{\text{tot}}$  of  $K|k$  a graph  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  whose vertices and edges are in bijection with those of  $\mathcal{D}_K^{\text{tot}}$ , as follows:

- a) The vertices of  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  are the pro- $\ell$  groups  $\Pi_{K\mathfrak{v}}$ , viewed as distinct pro- $\ell$  groups, where  $\mathfrak{v}$  are all the quasi-prime divisors of  $K|k$ .
- b) If an edge from  $K\mathfrak{v}$  to  $K\mathfrak{w}$  exists, the corresponding edge from  $\Pi_{K\mathfrak{v}}$  to  $\Pi_{K\mathfrak{w}}$  is endowed with the pair of groups  $T_{\mathfrak{w}/\mathfrak{v}} = T_{\mathfrak{w}}/T_{\mathfrak{v}} \subseteq Z_{\mathfrak{w}}/T_{\mathfrak{v}} = Z_{\mathfrak{w}/\mathfrak{v}}$ , viewed as subgroups of  $\Pi_{K\mathfrak{v}} = Z_{\mathfrak{w}}/T_{\mathfrak{v}}$ , and notice that in this case one has that  $\Pi_{K\mathfrak{w}} = Z_{\mathfrak{w}/\mathfrak{v}}/T_{\mathfrak{w}/\mathfrak{v}}$ .

The graph  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  will be called the **total decomposition graph** of  $K|k$ , or of  $\Pi_K$ .

In a similar way, we attach to  $\mathcal{Q}_K^{\text{tot}}$  the **total quasi decomposition graph** of  $K|k$ , which we denote  $\mathcal{G}_{\mathcal{Q}_K^{\text{tot}}}$ , but using the minimized inertia/decomposition/residue groups  $T_{\mathfrak{v}}^1$ ,  $Z_{\mathfrak{v}}^1$ ,  $\Pi_{K\mathfrak{v}}^1$  instead of the inertia/decomposition/residue Galois groups (which are the same for generalized prime divisors, because  $\text{char}(k) \neq \ell$ ). Clearly,  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  is a full subgraph of  $\mathcal{G}_{\mathcal{Q}_K^{\text{tot}}}$ .

The functorial properties of the total graphs of (quasi) prime divisors translate in the following functorial properties of the total (quasi) decomposition graphs:

- 1) *Embeddings.* Let  $\iota : L|l \hookrightarrow K|k$  be an embedding of function fields which maps  $l$  isomorphically onto  $k$ . Then the canonical projection homomorphism  $\Phi_{\iota} : \Pi_K \rightarrow \Pi_L$  is an open homomorphism, and moreover, for every generalized (quasi) prime divisor  $\mathfrak{v}$  of  $K|k$  and its restriction  $\mathfrak{v}_L$  to  $L$  one has:  $\Phi_{\iota}(Z_{\mathfrak{v}}^1) \subseteq Z_{\mathfrak{v}_L}^1$  is an open subgroup, and  $\Phi_{\iota}(T_{\mathfrak{v}}^1) \subseteq T_{\mathfrak{v}_L}^1$  satisfies:  $\Phi_{\iota}(T_{\mathfrak{v}}^1) = 1$  iff  $\mathfrak{v}_L$  has divisible value group, e.g.,  $\mathfrak{v}_L$  is the trivial valuation. Therefore,  $\Phi_{\iota}$  gives rise to **morphisms** of total (quasi) decomposition graphs

$$\Phi_{\iota} : \mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{D}_L^{\text{tot}}}, \quad \Phi_{\iota} : \mathcal{G}_{\mathcal{Q}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{Q}_L^{\text{tot}}}.$$

- 2) *Restrictions.* Given a generalized (quasi) prime divisor  $\mathfrak{v}$  of  $K|k$ , let  $pr_{\mathfrak{v}} : Z_{\mathfrak{v}}^1 \rightarrow \Pi_{K\mathfrak{v}}^1$  be the canonical projection. Then for every  $\mathfrak{w} \geq \mathfrak{v}$  we have:  $T_{\mathfrak{w}}^1 \subseteq Z_{\mathfrak{w}}^1$  are mapped onto  $T_{\mathfrak{w}/\mathfrak{v}}^1 := T_{\mathfrak{w}}^1/T_{\mathfrak{v}}^1 \subseteq Z_{\mathfrak{w}}^1/T_{\mathfrak{v}}^1 =: Z_{\mathfrak{w}/\mathfrak{v}}^1$ . Therefore, the total (quasi) decomposition graph for  $K\mathfrak{v}|k\mathfrak{v}$  can be recovered from the one for  $K|k$  in a canonical way via  $pr_{\mathfrak{v}} : Z_{\mathfrak{v}}^1 \rightarrow \Pi_{K\mathfrak{v}}^1$ .

We now are in position to announce the strengthening Theorem 1.1, which is:

**Theorem 2.1.** *In the notations from the Introduction, let  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \subset \mathcal{G}_{\mathcal{Q}_K^{\text{tot}}}$  be the total decomposition, respectively quasi decomposition, graphs of  $K|k$ . The the following hold:*

- 1) *There exists a group theoretical recipe which reconstructs  $K|k$  in a functorial way from  $\mathcal{G}_{\mathcal{Q}_K^{\text{tot}}}$  endowed with  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ , provided  $\text{td}(K|k) > 2$ .*
- 2) *The group theoretical recipe above is invariant under isomorphisms as follows: Let  $L|l$  be a further function field with  $l$  algebraically closed, and  $\Phi : \mathcal{G}_{\mathcal{Q}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{Q}_L^{\text{tot}}}$  be an*

isomorphism which maps  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  isomorphically onto  $\mathcal{G}_{\mathcal{D}_L^{\text{tot}}}$ . Then there exists  $\epsilon \in \mathbb{Z}_\ell^\times$  such that  $\epsilon \cdot \Phi$  is induced by an isomorphism  $\iota : L^1|l \rightarrow K^1|k$ .

- 3) Moreover, the isomorphism  $\iota$  above is unique up to Frobenius twists and  $\iota(l) = k$ , and  $\epsilon$  is unique up to multiplication by  $p^m$  with  $m \in \mathbb{Z}$  and  $p = \text{char}(k)$ .

The proof of the above Theorem 2.1 is quite involved and will be completed in the next subsections B), C), D), E), of this section. The strategy of the proof is as follows:

- First, by POP [P3], there are group theoretical recipes which recover the basic invariants  $\widehat{U}_K$  and (the equivalence class of) the divisorial  $\widehat{U}_K$ -lattice  $\mathcal{L}_K$  for  $K|k$  from  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ . Moreover, those recipes are invariant under isomorphisms of total decomposition graphs. These facts will be reviewed and some of them deepened in the following subsection B).

- Second, using specialization techniques and the fact that the assertion of Theorem 1.1 from Introduction is known over algebraic closures of finite fields in the specific way proved in POP [P4], we define the quasi arithmetical  $\widehat{U}_K$ -lattice  $\mathcal{L}_K^0$  for  $K|k$  (which is contained in  $\mathcal{L}_K$ , but it is not a divisorial  $\widehat{U}_K$ -lattice for  $K|k$  in the sense of POP [P3]) using a group theoretical recipe which is invariant under isomorphisms  $\Phi : \mathcal{G}_{\mathcal{Q}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{Q}_L^{\text{tot}}}$  which map  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  onto  $\mathcal{G}_{\mathcal{D}_L^{\text{tot}}}$ . The technical preparation for this step is done in subsection C), where we explain the specialization techniques developed by ROQUETTE and MUMFORD, and use that to study the specialization of the Weil divisor group (by using DE JONG's alterations and the geometric description of the Weil divisor group for projective smooth varieties).

- Finally, letting  $j_K : K^\times \rightarrow \widehat{K}$  be the  $\ell$ -adic completion homomorphism, one shows that  $j_K(K^\times) \subset \mathcal{L}_K^0$ , and that  $\mathcal{L}_K^0 / (\widehat{U}_K \cdot j_K(K^\times))$  is a torsion group. That allows one to recover the rational quotients of  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ , thus of its geometric decomposition subgraphs  $\mathcal{G}_{\mathcal{D}_K}$ . One concludes by applying the main result from POP [P3], Introduction.

## B) Reviewing facts about decomposition graphs

We begin by recalling here a few basic facts from POP [P3] and proving a little bit more precise/stronger results about the (pro- $\ell$  abelian) fundamental group of quasi projective normal  $k$ -varieties, see the discussion from [P3], Appendix, section 7.3 for some of the details.

Let  $D$  be a set of prime divisors of  $K|k$ . We denote by  $T_D \subseteq \Pi_K$  the closed subgroup generated by all the  $T_v$ ,  $v \in D$ , and say that  $\Pi_{1,D} := \Pi_K/T_D$  is the **fundamental group** of the set  $D$ . In the case  $D$  equals the set of all the prime divisors  $D = D_{K|k}$  of  $K|k$ , we say that  $\Pi_{1,D_{K|k}} =: \Pi_{1,K}$  is the **(birational) fundamental group for  $K|k$** .

Recall that a set  $D$  of prime divisors of  $K|k$  is **geometric**, if there exists a normal model  $X \rightarrow k$  of  $K|k$  such that  $D = D_X$  is the set of Weil prime divisors of  $X$ . [If so, there are always quasi-projective normal models  $X$  with  $D = D_X$ .] In particular, if  $X$  is a normal model of  $K|k$  and  $\Pi_1(X)$  denotes the maximal pro- $\ell$  abelian fundamental group of  $X$ , one has canonical surjective projections  $\Pi_{1,D_X} \rightarrow \Pi_1(X)$  and  $\Pi_{1,D_X} \rightarrow \Pi_{1,K}$ . Moreover, if  $U \subset X$  is an open *smooth*  $k$ -subvariety, then  $\Pi_{1,D_U} \rightarrow \Pi_1(U)$  is an isomorphism (by the purity of the branch locus). In particular,  $\Pi_{1,D_U} = \Pi_1(U)$  is a finite  $\mathbb{Z}_\ell$ -module, and since one has the canonical surjective homomorphisms  $\Pi_{1,D_U} \rightarrow \Pi_{1,D_X} \rightarrow \Pi_{1,K}$ , it follows that  $\Pi_{1,D_X} = \Pi_{1,D}$  and  $\Pi_{1,K}$  are finite  $\mathbb{Z}_\ell$ -modules. Further, it was shown in [P3], Appendix, 7.3, see especially Fact 57, that there exist (quasi projective) normal models  $X$  such that  $\Pi_{1,D_X} \rightarrow \Pi_{1,K}$  is an isomorphism. Nevertheless, it is not clear that for every geometric set  $D$  there exists quasi projective normal models  $X$  with  $\Pi_{1,D} = \Pi_{1,D_X}$  and  $\Pi_{1,D_X} \rightarrow \Pi_1(X)$  is an isomorphism.

We will prove more precise assertions below, which will be used later on. (This was not known to the author and those whom he asked at the time [P3] was written.) We begin by recalling two fundamental facts concerning *alterations* as introduced by DE JONG and developed by GABBER, TEMKIN, and many others, see e.g. [ILO], Expose X.

Let  $D$  be a fixed geometric set of prime divisors for  $K|k$ ,  $X_0$  be any projective normal model of  $K|k$  with  $D \subseteq D_{X_0}$ , and  $S_0 \subset X_0$  a fixed closed proper subset. Usually  $S_0$  will be chosen to define  $D$  in the sense that  $D = D_{X_0 \setminus S_0}$ . By [ILO], Expose X, Theorem 2.1, there exist **prime to  $\ell$  alterations above  $S_0$** , i.e., projective generically finite separable morphisms  $Y \rightarrow X_0$  satisfying the following:

- $Y$  is a projective smooth  $k$ -variety.
- $T := f_0^{-1}(S_0)$  is a NCD (normal crossings divisor) in  $Y$ .
- $[k(Y) : K]$  is prime to  $\ell$ .

We denote by  $D_0$  the restriction of  $D_Y$  to  $K$ , and notice that  $D_{X_0} \subseteq D_0$ , because  $Y \rightarrow X_0$  is surjective. Hence finally  $D \subseteq D_{X_0} \subseteq D_0$ .

Second, let  $X_1 \rightarrow X_0$  be a dominant morphism with  $X_1$  a projective normal model of  $K|k$  such that  $D_0 \subseteq D_{X_1}$ , thus we have  $D \subseteq D_{X_0} \subseteq D_0 \subseteq D_{X_1}$ . Then by DE JONG's theory of alterations, see e.g., [ILO], Expose X, Lemma 2.2, there exists a **generically normal finite alteration** of  $X_1$ , i.e., a projective dominant  $k$ -morphism  $Z \rightarrow X_1$  satisfying the following:

- $Z$  is a projective smooth  $k$ -variety.
- The field extension  $K = k(X_1) \hookrightarrow k(Z) =: M$  is a finite and normal.
- $\text{Aut}(M|K)$  acts on  $Z$  and  $Z \rightarrow X_1$  is an  $\text{Aut}(M|K)$ -invariant.

By a standard scheme theoretical construction [recalled below], there exists a projective normal model  $X$  for  $K|k$  and a dominant  $k$ -morphism  $X \rightarrow X_1$  such that  $Z \rightarrow X_1$  factors through  $X \rightarrow X_1$ , and the resulting  $k$ -morphism  $Z \rightarrow X$  is finite. In particular, since  $Z$  is smooth, thus normal, it follows that  $Z \rightarrow X$  is the normalization of  $X$  in the field extension  $K \hookrightarrow M$ . The standard scheme theoretical construction is as follows: Let  $Z \rightarrow \text{Aut}(M|K) \backslash Z =: Z_i$  be the quotient of  $Z$  by  $\text{Aut}(M|K)$ . Then  $Z \rightarrow Z_i$  is a finite generically Galois morphism, and its function field  $M_i := k(Z_i)$  satisfies:  $M|M_i$  is Galois, and  $M_i|K$  purely inseparable. Hence there exists  $e > 0$  such that  $M_i^{(e)} := \text{Frob}^e(M_i) \subset K$ , and  $M_i^{(e)}$  is the function field of the  $e^{\text{th}}$  Frobenius twist  $Z_i^{(e)}$  of  $Z_i$ . And notice that  $Z_i^{(e)}$  is a projective normal model for the function field  $M_i^{(e)}|k$ . Further, the normalization of  $Z_i^{(e)}$  in the finite field extension  $M_i^{(e)} \hookrightarrow M_i$  is nothing but  $Z_i$ . Finally, let  $X$  be the normalization of  $Z_i^{(e)}$  in the function field extension  $M_i^{(e)} \hookrightarrow K$ . Then  $X$  is a projective normal  $k$ -variety because  $Z_i^{(e)}$  was so, and  $k(X) = K$ , thus  $X$  is a projective normal model of  $K|k$ . Further, by the transitivity of normalization, it follows that the normalization of  $X$  in  $K \hookrightarrow M_i$  equals the normalization of  $Z_i^{(e)}$  in the field extension  $M_i^{(e)} \hookrightarrow M_i$ , and that normalization is  $Z_i$ . Finally, using the transitivity of normalization again, it follows that the normalization of  $X$  in  $K \hookrightarrow M_i$  is  $Z$  itself. Finally, to prove that  $Z \rightarrow X_1$  factors through  $Z \rightarrow X$ , we proceed as follows: First, since  $X_1$  and  $X$  are both projective normal models of  $K|k$ , there is a canonical rational map  $X \dashrightarrow X_1$ . We claim that  $X \rightarrow X_1$  is actually a morphism. Indeed, let  $x \in X$  be a fixed point, and  $Z_x \subset Z$  be the preimage of  $x$  under  $Z \rightarrow X$ . Then  $\text{Aut}(M|K)$  acts transitively on  $Z_x$ , and since  $Z \rightarrow X_1$  is  $\text{Aut}(M|K)$ -invariant, it follows that the image of  $Z_x$  under  $Z \rightarrow X_1$  consists of one point, say  $x_1 \in X_1$ . Now let  $V_1 := \text{Spec } R_1 \subset X_1$

be an affine open subset containing  $x_1$ , and  $W := \text{Spec } S \subset Z$  be an  $\text{Aut}(M|K)$  invariant open subset of  $Z$  containing  $Z_x$ , and  $V := \text{Spec } R \subset X$  be the image of  $W = \text{Spec } S$  under  $Z \rightarrow X$ . Then identifying  $R_1, R$  and  $S_1$  with the corresponding  $k$ -subalgebras of finite type of  $K$ , respectively of  $M = k(Z)$ , it follows that  $W \rightarrow V_1$  and  $W \rightarrow V$  are defined by the  $k$ -embeddings  $R_1 \hookrightarrow S$ , respectively  $R \hookrightarrow S$ , defined via the inclusion  $K \hookrightarrow M$ . Now since  $S$  is the normalization of  $R$ , it follows that  $R$  is mapped isomorphically onto  $K \cap S$ . Thus since  $K \hookrightarrow M$  maps  $R_1$  into  $K \cap S$ , we conclude that  $R_1 \subset R$ . That in turns shows that  $V \dashrightarrow V_1$  is defined by the  $k$ -inclusion  $R_1 \hookrightarrow R$ , thus it is a morphism, and therefore, defined at  $x \in V$ . Conclude that  $X \dashrightarrow X_1$  is actually a  $k$ -morphism.

**Preparation/Notations 2.2.** Summarizing the discussion above, for a geometric set  $D$  of prime divisors for  $K|k$ , and  $X_0$  a projective normal model of  $K|k$  with  $D \subseteq D_{X_0}$ , we let  $S_0 \subset X_0$  be closed subset with  $D = D_{X_0 \setminus S_0}$ , and can and will consider the following:

- a) A prime to  $\ell$  alteration  $Y \rightarrow X_0$  above  $S_0$ . We denote by  $T \subset Y$  the preimage of  $S_0 \subset X_0$  under  $Y \rightarrow X_0$ , and by  $D_0$  the restriction of  $D_Y$  to  $K$ . Hence  $D \subseteq D_{X_0} \subseteq D_0$ .
- b) A morphism of projective normal models  $X \rightarrow X_0$  with  $D_0 \subseteq D_X$ . For closed subsets  $S'_0 \subset X_0$ , let  $T' \subset Y$  and  $S' \subset X$  be the corresponding preimages of  $S'_0 \subset X_0$ .
- c) A smooth projective  $k$ -variety  $Z$  together with a dominant finite  $k$ -morphism  $Z \rightarrow X$  such that  $K = k(X) \hookrightarrow k(Z) =: M$  is normal and  $\text{Aut}(M|K)$  acts on  $Z$ .

**Proposition 2.3.** *In the above notations, the following hold:*

- 1) *The canonical projection  $\Pi_{1, D_{X \setminus S'}} \rightarrow \Pi_1(X \setminus S')$  is an isomorphism. Hence one has canonical surjective projections  $\Pi_{1, D_{X_0 \setminus S'_0}} \rightarrow \Pi_{1, D_{X \setminus S'}} \rightarrow \Pi_1(X \setminus S') \rightarrow \Pi_1(X_0 \setminus S'_0)$ .*
- 2) *Suppose that  $D_0 \subseteq D_{X \setminus S'}$ . Then  $\Pi_{1, D_{X \setminus S'}} \rightarrow \Pi_1(X \setminus S') \rightarrow \Pi_{1, K}$  are isomorphisms.*

*Proof.* To 1): The existence and the surjectivity of the projections is clear. Thus it is left to show that  $\Pi_{1, D_{X \setminus S'}} \rightarrow \Pi_1(X \setminus S')$  is injective. Equivalently, one has to prove the following: Let  $\tilde{K}|K$  be an abelian  $\ell$ -power degree extension, and  $\tilde{X} \rightarrow X$  be the normalization of  $X$  in  $K \hookrightarrow \tilde{K}$ . Then  $\tilde{X} \rightarrow X$  is etale above  $S'$  if and only if none of the prime divisors  $v \in D_{X \setminus S'}$  has ramification in  $\tilde{K}|K$ . Clearly, this is equivalent to the corresponding assertion for all cyclic sub extensions of  $\tilde{K}|K$ , thus without loss of generality, we can suppose that  $\tilde{K}|K$  is cyclic, thus  $\text{Gal}(\tilde{K}|K)$  is cyclic. For points  $\tilde{x} \mapsto x$  under  $\tilde{X} \rightarrow X$  and valuations  $\tilde{v}|v$  of  $\tilde{K}|K$ , we denote by  $T_{\tilde{x}}$ , respectively  $T_{\tilde{v}}$  the corresponding inertia groups.

**Claim 1.** *Suppose that  $\tilde{G} := \text{Gal}(\tilde{K}|K) = \langle \tilde{g} \rangle$  is cyclic. Then for every  $\tilde{x} \in \tilde{X}$  there exists a prime divisor  $\tilde{w}$  of  $\tilde{K}|k$  with  $T_{\tilde{x}} \subseteq T_{\tilde{w}}$  and  $\tilde{x}$  in the closure of the center  $x_{\tilde{w}}$  of  $\tilde{w}$  on  $\tilde{X}$ .*

*Proof of Claim 1.* Compare with POP, [P2], Proof of Theorem B. Recalling the cover  $Z \rightarrow X$  with  $M := k(Z)$ , let  $\tilde{K}' := M\tilde{K}$  be the compositum of  $\tilde{K}$  and  $M = k(Z)$ , and  $\tilde{X}' \rightarrow X$  be the normalization of  $X$  in the function field extension  $K \hookrightarrow \tilde{K}'$  and the resulting canonical factorizations  $\tilde{X}' \rightarrow \tilde{X} \rightarrow X$  and  $\tilde{X}' \rightarrow Z \rightarrow X$ . Further, choosing a preimage  $\tilde{x}'$  of  $\tilde{x}$  under  $\tilde{X}' \rightarrow \tilde{X}$ , consider  $\tilde{x}' \mapsto \tilde{x} \mapsto x$  and  $\tilde{x}' \mapsto z \mapsto x$  under the above factorizations of  $\tilde{X} \rightarrow X$ . Then by the functoriality of the Hilbert decomposition/ramification theory, there are surjective canonical projections  $T_{\tilde{x}'} \rightarrow T_{\tilde{x}}$  and  $T_{\tilde{x}'} \rightarrow T_z$ . Thus given a generator  $\tilde{g}$  of  $T_{\tilde{x}}$ , there exists  $\tilde{g}' \in T_{\tilde{x}'}$  which maps to  $\tilde{g}$  under  $T_{\tilde{x}'} \rightarrow T_{\tilde{x}}$ . And if  $\tilde{w}'$  is a prime divisor of  $\tilde{K}'|k$  such that  $\tilde{g}' \in T_{\tilde{w}'}$ , then setting  $\tilde{w} := \tilde{w}'|_{\tilde{w}}$ , it follows that  $\tilde{g} \in T_{\tilde{w}}$ . Hence letting  $K' := M^{\tilde{g}}$

be the fixed field of  $\tilde{g}$  in  $M$ , and replacing  $\tilde{K}|K$  by  $\tilde{K}'|K'$ , we can suppose that from the beginning we have  $M \subset \tilde{K}$ , and  $\tilde{K}|K$  is cyclic with Galois group  $\tilde{G} = \langle \tilde{g} \rangle$ , and  $M|K$  is a cyclic subextension, say with Galois group  $G = \langle g \rangle$ , where  $g = \tilde{g}|_M$ . And notice that  $\tilde{G} = T_{\tilde{x}}$  and  $G = T_z$ . Thus  $G$  acts on the local ring  $\mathcal{O}_z$  of  $z$ , and  $\tilde{G}$  acts on the local ring  $\mathcal{O}_{\tilde{x}}$  of  $\tilde{x}$ .

**Case 1.**  $K = M$ . Then  $X = Z$  is a projective smooth model for  $K|k$ , thus  $\mathcal{O}_x = \mathcal{O}_z$  is a regular ring. Let  $\mathcal{O}_x^n$  be the normalization of  $\mathcal{O}_x$  in  $K \hookrightarrow \tilde{K}$ , and  $\text{Spec } \mathcal{O}_x^n \rightarrow \text{Spec } \mathcal{O}_x$  the restriction of  $\tilde{X} \rightarrow X$  to  $\text{Spec } \mathcal{O}_x^n$ . Since  $\text{Spec } \mathcal{O}_x$  is regular, by the purity of the branch locus it follows that  $T_{\tilde{x}}$  is generated by inertia groups of the form  $T_{\tilde{w}}$ , with  $\tilde{w}$  prime divisors of  $\tilde{K}|k$  having the center on  $\text{Spec } \mathcal{O}_x^n \subset \tilde{X}$ . Since  $T_{\tilde{x}}$  is cyclic, it follows that there exists  $\tilde{w}$  with  $T_{\tilde{x}} \subseteq T_{\tilde{w}}$ , etc.

**Case 2.**  $K \subset M$  strictly. Then letting  $(\mathcal{O}, \mathfrak{m})$  be the local ring  $(\mathcal{O}_z, \mathfrak{m}_z)$  at  $z$ , one has  $T_z = G$ . Then proceeding as in the in the proof of Theorem B, explanations after Fact 2.2, (the proof of) Lemma 2.4 of loc.cit. is applicable, and by Step 3 of that proof, it follows by Lemma 2.6 of loc.cit. that there exists a local ring  $(\mathcal{O}', \mathfrak{m}')$  which has the properties:

- $G$  acts on  $(\mathcal{O}', \mathfrak{m}')$ , and  $(\mathcal{O}', \mathfrak{m}')$  dominates  $(\mathcal{O}_z, \mathfrak{m}_z)$ .
- There exist local parameters  $(t'_1, \dots, t'_d)$  of  $\mathcal{O}'$ , and a primitive character  $\chi$  of  $G$  such that  $\sigma(t'_i) = t'_i$  for  $1 \leq i < d$  and  $\sigma(t'_d) = \chi(\sigma)t'_d$ .

Since  $\mathcal{O}'$  dominates  $\mathcal{O}_z$ , it follows that  $\mathcal{O}^T := \mathcal{O}'^G$  dominates  $\mathcal{O}_z^G = \mathcal{O}_x$ , and  $\mathcal{O}^T$  is a regular ring by Step 4 of loc.cit. Further  $G = T_z$  is a quotient of the ramification group  $T_{\mathcal{O}^T}$  of  $\mathcal{O}^T$  in  $K \hookrightarrow \tilde{K}$ , and since  $K \hookrightarrow \tilde{K}$  is cyclic of  $\ell$ -power order, it follows that  $T_{\mathcal{O}^T} = \text{Gal}(\tilde{K}|K) = T_{\tilde{x}}$ . Thus  $\mathcal{O}^T$  has a unique prolongation  $\tilde{\mathcal{O}}^T$  to  $\tilde{K}$ . And since  $\mathcal{O}^T$  dominates  $\mathcal{O}_x$ , it follows that  $\tilde{\mathcal{O}}^T$  dominates  $\mathcal{O}_{\tilde{x}}$  (which is the unique prolongation of  $\mathcal{O}_x$  to  $\tilde{K}$ ). On concludes in the same way as at Case 1 above.

**Claim 2.** *Let  $\tilde{K}|K$  be an abelian  $\ell$ -power Galois extension as above. Then a prime divisor  $w$  of  $K|k$  ramifies in  $\tilde{K}|K$  iff  $w$  has a prolongation  $w_L$  to  $L$  which ramifies in  $\tilde{L}|L$ .*

*Proof of Claim 2.* Let  $\hat{K}|K$  be a (minimal) finite normal extension with  $L, \tilde{K} \subseteq \hat{K}$ , and  $\text{Gal}(\hat{K}|K) =: \hat{G} \rightarrow \tilde{G} := \text{Gal}(\tilde{K}|K)$ ,  $\hat{g} \mapsto g$ , be the corresponding projection of Galois groups. Setting  $\tilde{L} = L\tilde{K}$  inside  $\hat{K}$ , it follows that  $\text{Gal}(\tilde{L}|L) =: H \rightarrow \tilde{G}$  is an isomorphism, and  $\hat{H} := \text{Gal}(\hat{K}|L) \rightarrow H$  is surjective, because  $\tilde{K}$  and  $L$  are linearly disjoint over  $K$ . Thus there exists a Sylow  $\ell$ -group  $\hat{H}_\ell$  of  $\hat{H}$  with  $\hat{H}_\ell \rightarrow H$  surjective. Since  $(\hat{G} : \hat{H}) = [L : K]$  is prime to  $\ell$ , it follows that  $\hat{H}_\ell$  is a Sylow  $\ell$ -group of  $\hat{G}$  as well. Let  $\hat{w}|\hat{w}|w$  denote the prolongations of  $w$  to  $\tilde{K}$ , respectively  $\hat{K}$ , and  $T_{\hat{w}} \rightarrow T_{\tilde{w}}$  be the corresponding surjective projections of the inertia groups. For every  $\sigma \in T_{\tilde{w}}$ , let  $\hat{\sigma} \in T_{\hat{w}}$  be a preimage of order a power of  $\ell$ . Then  $\hat{\sigma}$  is contained in a Sylow  $\ell$ -group of  $\hat{G}$ , hence there exists a conjugate  $\hat{\sigma}^{\hat{\tau}}$  which lies in  $\hat{H}_\ell$ . Thus replacing  $\hat{w}|\hat{w}|w$  by  $\hat{w}^{\hat{\tau}}|\hat{w}^{\hat{\tau}}|w$ , and  $\sigma, \hat{\sigma}$  by  $\sigma^{\hat{\tau}}, \hat{\sigma}^{\hat{\tau}}$ , respectively  $\hat{\sigma}^{\hat{\tau}}$ , without loss of generality, we can suppose that  $\hat{\sigma} \in T_{\hat{w}} \subseteq \hat{H}_\ell$  is a preimage of  $\sigma \in T_{\tilde{w}}$ . Thus setting  $\tilde{w}_L := \hat{w}|_{\tilde{L}}$  and  $w_L := \hat{w}|_L$ , it follows that  $\tilde{w}_L|w_L$  are prolongations of  $\tilde{w}|w$  to  $\tilde{L}$ , respectively  $L$ . And taking into account that  $\hat{H} \rightarrow \tilde{G}$  factors as  $\hat{H} \rightarrow H \rightarrow \tilde{G}$ , it follows that the image  $\sigma_L \in H$  of  $\hat{\sigma} \in \hat{H}$  under  $\hat{H} \rightarrow H$  lies in  $T_{\tilde{w}_L} \subset H$ . This concludes the proof of Claim 2.

Coming back to assertion 1) of Proposition 2.3, we prove the following more precise result:

**Lemma 2.4.** *Let  $\tilde{K}|K$  be a finite abelian pro- $\ell$  field extension, and set  $\tilde{L} := L\tilde{K}$ . Then for the corresponding normalization  $\tilde{X} \rightarrow X$  and  $\tilde{Y} \rightarrow Y$ , the following assertions are equivalent:*

- i)  $\tilde{X} \rightarrow X$  is étale above  $S'$ .
- ii) All  $v \in D_{X \setminus S'} \cap D_0$  are unramified in  $\tilde{K}|K$ .
- iii) All  $w \in D_{Y \setminus T'}$  are unramified in  $\tilde{L}|L$ .
- iv)  $\tilde{Y} \rightarrow Y$  is étale above  $T'$ .

*Proof of Lemma 2.4.* First, i) implies ii) in an obvious way. For the implication ii)  $\Rightarrow$  iii), let  $v_L \in D_{Y \setminus T'}$  have restriction  $v := (v_L)|_K$  to  $K$ . Then  $v \in D_0 \cap D_{X \setminus S'}$ , as observed above. On the other hand, by Claim 2) above one has that  $v$  does not ramify in  $\tilde{K}|K$  iff all its prolongations to  $L$  do not ramify in  $\tilde{L}|L$ . Hence  $v_L$  does not ramify in  $\tilde{L}|L$ . Next, since  $Y$  is smooth, thus regular, assertions iii), iv) are equivalent by the purity of the branch locus. Thus it is left to prove that iv) implies i). By contradiction, suppose that  $\tilde{X} \rightarrow X$  is branched at some point  $x \in X \setminus S'$ . Then by the discussion before Claim 1 combined with Claim 1, it follows that there exists a prime divisor  $w$  of  $K|k$  which is ramified in  $\tilde{K}|K$  and  $x$  lies in the closure of the center  $x_w$  of  $w$ . Thus since  $x \in X \setminus S'$  and  $S' \subset X$  is closed, it follows that  $x_w \in X \setminus S'$ . By Claim 2 there exists a prolongation  $w_L$  of  $w$  to  $L$  which is ramified in  $\tilde{L}|L$ , and let  $y_w$  be the center of  $w_L$  on  $Y$ . Then by the compatibility of center of valuations with (separated) morphism, it follows that  $w_L$  and its restriction  $w$  to  $K$  have the same center  $x_0$  on  $X_0$ , and that center satisfies  $y_w \mapsto x_0$ ,  $x_w \mapsto x_0$ . Now let  $S'_0 \subset X_0$  be the closed subset such that  $S' \subset X$  is the preimage of  $S'_0$ , and let  $T' \subset Y$  be the preimage of  $S'_0$ . Then  $x_0 \in X_0 \setminus S'_0$  (because  $x \in X \setminus S'$ ), and  $y_w \in Y \setminus T'$ . Since  $w_L$  is branched in  $\tilde{Y} \rightarrow Y$  and has center  $y_L \in Y \setminus T'$ , it follows that  $\tilde{Y} \rightarrow Y$  is not étale above  $T'$ , contradiction! This concludes the proof of Lemma 2.4, thus of assertion 1) of Proposition 2.3.

To 2): Let  $S'_0 \subset X_0$  be such that  $S' \subset X$  is the preimage of  $S'_0$ , and  $T' \subset Y$  be the preimage of  $S'_0$  in  $Y$ . In the notations from the proof of assertion 2) above, suppose that all  $v \in D_{X \setminus S'}$  are unramified in  $\tilde{K}|K$ . We claim that all prime divisors  $w$  of  $K|k$  are unramified in  $\tilde{K}|K$ . Indeed, by Claim 2 above and the discussion there after, it follows that a prime divisor  $w$  of  $K|k$  is ramified in  $\tilde{K}|K$  iff  $w$  has a prolongation  $w_L$  to  $L$  which is ramified in  $\tilde{L}|L$  iff there exists  $v_L \in D_Y$  which is ramified in  $\tilde{L}|L$  iff the restriction  $v := (v_L)|_K$  is ramified in  $\tilde{K}|K$ . In other words,  $\Pi_{1, D_Y} = \Pi_{1, L}$  if and only if  $\Pi_{1, D_{X \setminus S'}} = \Pi_{1, K}$ , etc.  $\square$

Coming to the second part of this subsection, let  $X$  be a normal model of  $K|k$ , hence  $D_X$  is a geometric set of prime divisors for  $K|k$ . [Recall that there always exists a quasi projective normal model  $\tilde{X}$  for  $K|k$  such that  $D_X = D_{\tilde{X}}$ .] We notice that by Krull's Hauptidealsatz, it follows that  $\mathbb{G}_m(X) := \Gamma(X, \mathcal{O}_X)^\times$  depends on  $D_X$  only, and not on  $X$ , hence the group of principal divisors  $\mathcal{H}_K(X) := K^\times / \mathbb{G}_m(X)$  on  $X$  depends only on  $D_X$ , and not on  $X$ . Since  $\text{Div}(X)$  depends on  $D_X$  only and not on  $X$ , it follows that  $\mathfrak{C}\mathfrak{I}(X)$  depends on  $D_X$  only, thus the canonical exact sequence

$$0 \rightarrow \mathcal{H}_K(X) \xrightarrow{\text{div}} \text{Div}(X) \xrightarrow{\text{pr}} \mathfrak{C}\mathfrak{I}(X) \rightarrow 0$$

depends on  $D_X$  only, and not on the specific  $X$ . Next we consider the resulting exact sequence of  $\ell$ -adically complete groups, which is obtained by tensoring the above sequence

with  $\mathbb{Z}/\ell^e$  and taking projective limits as  $e \rightarrow \infty$ . First, for every  $n = \ell^e$  one gets

$$0 \rightarrow {}_n\mathfrak{C}\mathfrak{l}(X) \rightarrow \mathcal{H}_K(X)/n \rightarrow \text{Div}(X)/n \rightarrow \mathfrak{C}\mathfrak{l}(X)/n \rightarrow 0,$$

where  ${}_n\mathfrak{C}\mathfrak{l}(X)$  is the  $n$ -torsion of  $\mathfrak{C}\mathfrak{l}(X)$ . Since  $1 \rightarrow \mathbb{G}_m(X)/k^\times \rightarrow K^\times/k^\times \rightarrow \mathcal{H}_K(X) \rightarrow 1$  is an exact sequence of free abelian groups, so is  $1 \rightarrow \mathbb{G}_m(X)/n \rightarrow K^\times/n \rightarrow \mathcal{H}_K(X)/n \rightarrow 1$ . Hence if  $U(X)/n \subset K^\times/n$  is the preimage of  ${}_n\mathfrak{C}\mathfrak{l}(X)$  under  $K^\times/n \rightarrow \mathcal{H}_K(X)/n$ , it follows that the resulting sequence  $1 \rightarrow \mathbb{G}_m(X)/n \rightarrow U(X)/n \rightarrow {}_n\mathfrak{C}\mathfrak{l}(X) \rightarrow 0$  is exact, thus the long exact sequence above give rise canonically to a long exact sequence:

$$(*) \quad 1 \rightarrow U(X)/n \rightarrow K^\times/n \rightarrow \text{Div}(X)/n \rightarrow \mathfrak{C}\mathfrak{l}(X)/n \rightarrow 0.$$

Taking limits over  $e \rightarrow \infty$ , we get the long exact sequence below, which depends on the geometric set  $D_X$  only, and not on the specific (quasi projective) normal model  $X$  for  $K|k$ :

$$(\dagger) \quad 1 \rightarrow \widehat{U}(X) \hookrightarrow \widehat{K} \xrightarrow{\text{div}} \widehat{\text{Div}}(X) \xrightarrow{\text{pr}} \widehat{\mathfrak{C}\mathfrak{l}}(D_X) \rightarrow 0.$$

Recall that a geometric set  $D = D_X$  is called **complete regular like**, if for every geometric set  $D_{\tilde{X}} \supseteq D_X$  one has  $\widehat{U}(X) = \widehat{U}(\tilde{X})$  and  $\widehat{\mathfrak{C}\mathfrak{l}}(\tilde{X}) \cong \widehat{\mathfrak{C}\mathfrak{l}}(X) \oplus \mathbb{Z}_\ell^r$ , where  $r := |D_{\tilde{X}} \setminus D_X|$ . For a complete regular like geometric set  $D = D_X$ , let  $\mathfrak{C}\mathfrak{l}'(X) \subseteq \mathfrak{C}\mathfrak{l}''(X)$  be the maximal divisible, respectively  $\ell$ -divisible subgroups of  $\mathfrak{C}\mathfrak{l}(X)$ . Then by structure of  $\mathfrak{C}\mathfrak{l}(X)$ , see e.g. [P3], Appendix, 7.3, it follow that  $\mathfrak{C}\mathfrak{l}'(X) \subseteq \mathfrak{C}\mathfrak{l}''(X)$  is a (finite) torsion group of order prime to  $\ell$ . Thus if  $\text{Div}^0(D_X) \subseteq \text{Div}'(D_X)$  are the preimages of  $\mathfrak{C}\mathfrak{l}'(X) \subseteq \mathfrak{C}\mathfrak{l}''(X)$  in  $\text{Div}(X)$ , it follows that  $\text{Div}^0(D_X)_{(\ell)} = \text{Div}'(D_X)_{(\ell)}$  inside  $\text{Div}(X)_{(\ell)}$ .<sup>1</sup> Moreover, both  $\widehat{U}_K := \widehat{U}(X)$  and  $\text{Div}'(D_X)_{(\ell)}$  are birational invariants of  $K|k$ . We let  $\mathcal{L}_K \subset \widehat{K}$  be the preimage of  $\text{Div}'(X)_{(\ell)} \subset \widehat{\text{Div}}(X)$ , and call it the **canonical divisorial  $\widehat{U}_K$ -lattice** for  $K|k$ . Notice that  $\widehat{\text{div}}(\mathcal{L}_K) = \text{Div}'(X)_{(\ell)}$ .

Next we recall the Galois theoretical counterpart of the above exact sequence  $(\dagger)$ . Let  $D = D_X$  be the geometric set of prime divisors of  $K|k$  defined by  $X$ , and recall the exact sequence  $1 \rightarrow T_D \rightarrow \Pi_K \rightarrow \Pi_{1,D} \rightarrow 1$ , where  $T_D \subseteq \Pi_K$  is the subgroup generated by all the  $T_v$ ,  $v \in D$ . Then after fixing an identification of the  $\ell$ -adic Tate module  $\mathbb{T}_{\mathbb{G}_m, \ell}$  with  $\mathbb{Z}_\ell$  (note that there is no Galois action involved, because all base fields are algebraically closed), one has the Kummer isomorphism  $\widehat{K} = \text{Hom}_{\text{cont}}(\Pi_K, \mathbb{Z}_\ell)$ , i.e.,  $\widehat{K}$  is the  $\ell$ -adic dual of  $\Pi_K$ , and the inclusion  $T_v \hookrightarrow \Pi_K$  gives rise canonically to the restriction map  $\widehat{K} \rightarrow \text{Hom}(T_v, \mathbb{Z}_\ell)$ . For every  $v \in D$ , let  $\tau_v \in T_v$  be the canonical generator of the inertia group  $T_v$  at  $v$ , i.e., the unique generator  $\tau_v \in T_v$  such that some (thus every) uniformizing parameter  $t_v \in K \subset \widehat{K}$  satisfies  $t_v(\tau_v) = 1$ . Then  $\text{Hom}(T_v, \mathbb{Z}_\ell) = \mathbb{Z}_\ell \cdot t_v$  can be canonically identified with  $\mathbb{Z}_\ell$ , and the resulting  $j^v : \widehat{K} = \text{Hom}(\Pi_K, \mathbb{Z}_\ell) \rightarrow \text{Hom}(T_v, \mathbb{Z}_\ell) = \mathbb{Z}_\ell$  is nothing but the  $\ell$ -adic completion of the valuation  $v : K^\times \rightarrow \mathbb{Z}$ . We further consider  $\text{Div}_D := \bigoplus_v \mathbb{Z}_\ell$  and  $j = \bigoplus_v j^v : \widehat{K} \rightarrow \text{Div}_D$  and its  $\ell$ -adic completion  $\hat{j} : \widehat{K} \rightarrow \widehat{\text{Div}}_D$ . Finally, let  $\mathfrak{F}_D$  be the free abelian pro- $\ell$  group on  $(\tau_v)_{v \in D}$ . Via the canonical map  $\mathfrak{F}_D \rightarrow T_D \subseteq \Pi_K$ , one gets a long exact sequence  $1 \rightarrow \mathfrak{R}_D \rightarrow \mathfrak{F}_D \rightarrow \Pi_K \rightarrow \Pi_{1,K} \rightarrow 1$  of pro- $\ell$  abelian groups, where  $\mathfrak{R}_D \subset \mathfrak{F}_D$  is the relation module for the system of inertia generators  $(\tau_v)_v$ , and the first four terms of the exact sequence have no torsion. Since the  $\ell$ -adic dual of  $\mathfrak{F}_D$  is canonically isomorphic to  $\widehat{\text{Div}}_D$ , by taking  $\ell$ -adic duals, on gets an exact sequence:

$$(\dagger)_{\text{Gal}} \quad 1 \rightarrow \widehat{U}_D \hookrightarrow \widehat{K} \xrightarrow{\hat{j}} \widehat{\text{Div}}_D \xrightarrow{\text{can}} \widehat{\mathfrak{C}\mathfrak{l}}_D \rightarrow 0,$$

<sup>1</sup>Recall that for an abelian group  $A$ , we denote  $A_{(\ell)} := A \otimes \mathbb{Z}_{(\ell)}$ .

in which each of the terms is the  $\ell$ -adic dual of the corresponding pro- $\ell$  group,  $\hat{j}$  is as constructed above, and the other morphisms are canonical.

Now recalling that  $D = D_X$  for some quasi projective normal model of  $K|k$ , it is obvious that the above exact sequences  $(\dagger)$  and  $(\dagger)_{\text{Gal}}$  are canonically isomorphic, and in particular one has  $\widehat{U}_D = \widehat{U}(X)$  inside  $\widehat{K}$ , and  $\widehat{\mathfrak{C}}\mathfrak{I}_D \cong \widehat{\mathfrak{C}}\mathfrak{I}(X)$  canonically. On the other hand, the exact sequence  $(\dagger)_{\text{Gal}}$  was constructed from  $\Pi_K$  endowed with the canonical system  $(\tau_v)_v$  of the inertia groups  $(T_v)_{v \in D}$  only. Unfortunately, we do not have a group theoretical recipe yet to identify canonical system of inertia generators  $(\tau_v)_v$  —say, up to simultaneous raising to some power  $\epsilon \in \mathbb{Z}_\ell^\times$ . Nevertheless, if  $(\tau'_v)_v$  is an arbitrary system of inertia generators, there exist unique  $\epsilon_v \in \mathbb{Z}_\ell^\times$  such that  $\tau'_v = \tau_v^{\epsilon_v}$  for all  $v \in D$ , thus  $t_v(\tau'_v) = \epsilon_v$ . Hence  $j'^v := \epsilon_v j^v$  satisfies:  $\ker(j^v) = \ker(j'^v)$  and  $\text{im}(j^v) = \text{im}(j'^v)$ . We conclude that  $\ker(\hat{j}) = \bigcap_v \ker j^v$  and  $\text{im}(\hat{j}) \subset \widehat{K}$  depend on  $\Pi_K$  endowed with  $(T_v)_{v \in D}$  only, and not on the system of inertia generators  $(\tau_v)_v$ . Thus using the fact that  $(\dagger)$  and  $(\dagger)_{\text{Gal}}$  are canonically isomorphic, we get that  $\widehat{U}(X) = \widehat{U}_D$  and  $\widehat{\mathfrak{C}}\mathfrak{I}(X) \cong \widehat{\mathfrak{C}}\mathfrak{I}_D$  depend on  $\Pi_K$  endowed with  $(T_v)_{v \in D}$  only, and not on the canonical inertia generators  $(\tau_v)_v$ . We thus conclude that the fact that a geometric set of prime divisors  $D$  is complete regular like (as define above) is encoded in  $\Pi_K$  endowed with  $(T_v)_{v \in D}$ . And we also notice that if  $D$  is complete regular like,  $\widehat{U}_D$  equals actually the  $\ell$ -adic dual  $\widehat{U}_K$  of  $\Pi_{1,K}$ . Recall that a **geometric decomposition graph**  $\mathcal{G}_{\mathcal{D}_K}$  for  $K|k$  is called complete regular like, if for every vertex  $\mathfrak{v}$  of  $\mathcal{G}_{\mathcal{D}_K}$  one has that the set  $D_{\mathfrak{v}}$  of 1-vertices of the residual geometric decomposition graph  $\mathcal{G}_{\mathcal{D}_{K^{\mathfrak{v}}}}$  is complete regular like. Further, recall that by POP [P3], Remark 11 and Propositions 22 and 23, the following hold:

- 1) The geometric decomposition graphs  $\mathcal{G}_{\mathcal{D}_K}$  as well as the **complete regular like** decomposition graphs for  $K|k$  can be recovered by group theoretical recipes from  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ .
- 2) Let  $\mathcal{G}_{\mathcal{D}_K}$  be a *complete regular like* decomposition graph,  $D = D_X$  be the geometric set of 1-vertices of  $\mathcal{G}_{\mathcal{D}_K}$ , and consider the isomorphic canonical exact sequences  $(\dagger)$ ,  $(\dagger)_{\text{Gal}}$  from above. Then there are group theoretical recipes by which one recovers from  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  endowed with  $\mathcal{G}_{\mathcal{D}_K}$  the following:
  - a)  $\widehat{U}(X) = \widehat{U}_K = \widehat{U}_D$  inside  $\widehat{K}$ , and the isomorphy type of  $\widehat{\mathfrak{C}}\mathfrak{I}(X) \cong \widehat{\mathfrak{C}}\mathfrak{I}_D$ .
  - b) The set of divisorial  $\widehat{U}_D$ -lattices of the form  $\epsilon \cdot \mathcal{L}_K \subset \widehat{K}$  with  $\epsilon \in \mathbb{Z}_\ell^\times$ .
  - c) Systems of inertia generators  $(\tau_v^{\epsilon_v})_v$  with  $\epsilon_v \in \epsilon \cdot \mathbb{Z}_{(\ell)}^\times$ ,  $v \in D$ ,  $\epsilon \in \mathbb{Z}_\ell^\times$ , satisfying:

$$\sum_v n_v v \in \text{Div}'(X)_{(\ell)} \quad \Leftrightarrow \quad \sum_v \epsilon_v n_v v \in \epsilon \cdot \text{Div}'(X)_{(\ell)}$$

- d) The subgroups  $\hat{j}(\epsilon \cdot \mathcal{L}_K) = \epsilon \cdot \text{Div}'(X)_{(\ell)} \subseteq \epsilon \cdot \text{Div}(X)_{(\ell)} \subset \widehat{\text{Div}}_D$  with  $\epsilon \in \mathbb{Z}_\ell^\times$ .

- 3) Hence one can recover from  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  endowed with  $\mathcal{G}_{\mathcal{D}_K}$  exact sequences of the form

$$1 \rightarrow \widehat{U}_K \rightarrow \epsilon \cdot \mathcal{L}_K \xrightarrow{j'} \epsilon \cdot \text{Div}(X)_{(\ell)} \xrightarrow{\text{can}'_D} \widehat{\mathfrak{C}}\mathfrak{I}(X) \rightarrow 0,$$

where  $\epsilon \in \mathbb{Z}_\ell^\times$ , and  $j' := \bigoplus_v \epsilon_v j^v$  for some  $\epsilon_v \in \epsilon \cdot \mathbb{Z}_{(\ell)}^\times$  satisfying condition c) above.

- 4) Replacing  $D = D_X$  by any geometric subset  $D' = D_{X'} \subseteq D_X = D$ , one gets by restriction the corresponding exact sequence for  $D'$ , namely:

$$1 \rightarrow \widehat{U}_{D'} \rightarrow \epsilon \cdot \mathcal{L}_{D'} \xrightarrow{j'_{D'}} \epsilon \cdot \text{Div}(D')_{(\ell)} \xrightarrow{\text{can}'_{D'}} \widehat{\mathfrak{C}}\mathfrak{I}(D') \rightarrow 0.$$

- 5) The recipes to recover the above information from  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  are invariant under isomorphisms of total decomposition graphs as follows, see loc.cit. Proposition 5.4, as

follows: Let  $L|l$  be a further function field with  $l$  an algebraically closed field of characteristic  $\neq \ell$ , and  $\Phi : \mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{D}_L^{\text{tot}}}$  an isomorphism of total decomposition graphs. Then  $\mathcal{G}_{\mathcal{D}_K}$  is a complete regular like decomposition graph for  $K|k$  if and only if its image  $\mathcal{G}_{\mathcal{D}_L}$  under  $\Phi$  is a complete regular like decomposition graphs for  $L|l$ . Let  $\mathcal{G}_{\mathcal{D}_K} \subset \mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  be a decomposition graph having  $D_K$  as set of 1-vertices, and  $\mathcal{G}_{\mathcal{D}_L} \subset \mathcal{G}_{\mathcal{D}_L^{\text{tot}}}$  be its image via  $\Phi$ , having  $D_L$  as set of 1-vertices. Further, recall that the Kummer isomorphism defined by  $\Phi$  is the  $\ell$ -adic dual  $\hat{\phi} : \widehat{L} \rightarrow \widehat{K}$  of  $\Phi$ . Then there exists  $\epsilon$  and  $(\epsilon_v)_v$  as at point 2) above such that the diagram below is commutative:

$$\begin{array}{ccccccc} 0 & \rightarrow & \widehat{U}_L & \longrightarrow & \mathcal{L}_L & \xrightarrow{J_L} & \text{Div}(D_L)_{(\ell)} & \xrightarrow{\text{can}_L} & \widehat{\mathcal{C}}\mathfrak{l}(D_L) & \rightarrow & 0 \\ & & \downarrow \hat{\phi} & & \downarrow \hat{\phi} & & \downarrow (\epsilon_v)_v & & \downarrow \text{can} & & \\ 0 & \rightarrow & \widehat{U}_K & \longrightarrow & \epsilon \cdot \mathcal{L}_K & \xrightarrow{J_K} & \epsilon \cdot \text{Div}(D_K)_{(\ell)} & \xrightarrow{\text{can}_K} & \widehat{\mathcal{C}}\mathfrak{l}(D_K) & \rightarrow & 0 \end{array}$$

- 6) Finally, replacing  $D_K$  by any co-finite subset  $D'_K \subseteq D_K$ , and  $D_L$  by the image  $D'_L \subseteq D_L$  of  $D'_K$  under  $\Phi$ , the long exact sequence from 4) above together with the commutative diagram from 5) give rise to a commutative diagram for  $D'_K$  and  $D'_L$ :

$$\begin{array}{ccccccc} 0 & \rightarrow & \widehat{U}_{D'_L} & \longrightarrow & \mathcal{L}_L & \xrightarrow{J'_L} & \text{Div}(D'_L)_{(\ell)} & \xrightarrow{\text{can}'_L} & \widehat{\mathcal{C}}\mathfrak{l}(D'_L) & \rightarrow & 0 \\ & & \downarrow \hat{\phi} & & \downarrow \hat{\phi} & & \downarrow (\epsilon_v)_v & & \downarrow \text{can}' & & \\ 0 & \rightarrow & \widehat{U}_{D'_K} & \longrightarrow & \epsilon \cdot \mathcal{L}_K & \xrightarrow{J'_K} & \epsilon \cdot \text{Div}(D'_K)_{(\ell)} & \xrightarrow{\text{can}'_K} & \widehat{\mathcal{C}}\mathfrak{l}(D'_K) & \rightarrow & 0 \end{array}$$

### C) Specialization techniques

We begin by recalling briefly the specialization/reduction results concerning projective integral/normal/smooth  $k$ -varieties and (finite)  $k$ -morphisms between such varieties, see MUMFORD [Mu], II, §§ 7–8, ROQUETTE [Ro], and GROTHENDIECK-MURRE [G–M]. We first recall the general facts and then come back with specifics in our situation. We consider the following context:  $k$  is an algebraically closed field, and  $\text{Val}_k$  is the space of all the valuations  $v$  of  $k$ . For  $v \in \text{Val}_k$  we denote by  $\mathcal{O}_v \subset k$  its valuation ring, and let  $\mathcal{O}_v \rightarrow kv$ ,  $a \mapsto \bar{a}$ , be its residue field. One has the reduction map  $\mathcal{O}_v[T_1, \dots, T_N] \rightarrow kv[T_1, \dots, T_N] =: \mathcal{R}_v$ ,  $f \mapsto \bar{f}$ , defined by mapping each coefficient of  $f$  to its residue in  $kv$ . In particular, the reduction map above gives rise to a reduction of the ideals  $\mathfrak{a} \subset k[T_1, \dots, T_n]$  to ideals  $\mathfrak{a}_v \subset \mathcal{R}_v$  defined by  $\mathfrak{a} \mapsto \mathfrak{A} := \mathfrak{a} \cap \mathcal{O}_v[T_1, \dots, T_n]$  followed by  $\mathfrak{A} \rightarrow \mathfrak{a}_v$ ,  $f \mapsto \bar{f}$ . Finally, we recall that  $\text{Val}_k$  carries the Zariski topology  $\tau^{\text{Zar}}$  which has as a basis the subsets of the form

$$\mathcal{U}_A = \{v \in \text{Val}_k \mid v(a) = 0 \ \forall a \in A\}, \quad A \subset k^\times \text{ finite subsets.}$$

We notice the trivial valuation  $v_0$  on  $k$  belongs to every Zariski open non-empty set, thus  $\tau^{\text{Zar}}$  is a prefilter on  $\text{Val}_k$ . We will denote by  $\mathfrak{D}$  ultrafilters on  $\text{Val}_k$  with  $\tau^{\text{Zar}} \subset \mathfrak{D}$ .

**Remarks 2.5.** In the above notations, we consider/remark the following:

- 1) Let  ${}^*k := \prod_v kv/\mathfrak{D}$  be the ultraproduct of  $(kv)_v$ , and consider the canonical embedding of  $k$  into  ${}^*k$ , defined by  $a \mapsto (a_v)/\mathfrak{D}$ , where  $a_v = \bar{a}$  if  $a$  is a  $v$ -unit, and  $a_v = 0$  else.
- 2) Let  $\mathfrak{a} := (f_1, \dots, f_r) \subset R := k[T_1, \dots, T_N]$  be an ideal. The fact that  $\mathfrak{a}$  is a prime ideal, respectively that  $V(\mathfrak{a}) = \text{Spec } R/\mathfrak{a} \subseteq \mathbb{A}_k^N$  is normal/smooth is an open condition involving the coefficients if  $f_1, \dots, f_r$  as parameters. Therefore, there exists a Zariski open subset  $\mathcal{U}_{\mathfrak{a}} \subset \text{Val}_k$  such that for all  $v \in \mathcal{U}_{\mathfrak{a}}$  the following hold:

- a)  $f_1, \dots, f_r \in \mathfrak{A}$ , and  $\mathfrak{a}_v = (\bar{f}_1, \dots, \bar{f}_r)$ .
- b) If  $\mathfrak{a}$  is prime, then so is  $\mathfrak{a}_v = (\bar{f}_1, \dots, \bar{f}_r)$ .
- c) If  $V(\mathfrak{a}) \hookrightarrow \mathbb{A}_k^N$  is a normal/smooth  $k$ -subvariety, then so is  $V(\mathfrak{a}_v) \hookrightarrow \mathbb{A}_{k_v}^N$ .

**Definition/Remark 2.6.** In the context of Remark 2.5, we introduce notations as follows:

- 1)  $\mathfrak{a}_* := \prod_v \mathfrak{a}_v / \mathfrak{D} \subset \prod_v \mathcal{R}_v =: \mathcal{R}_*$  is the  $\mathfrak{D}$ -ultraproduct of the ideals  $(\mathfrak{a}_v)_v$ . One has a canonical embedding  $\mathfrak{a} \hookrightarrow \mathfrak{a}_*$  defined by  $f \mapsto f_*$ , where  $f_*$  is the image of  $f$  under  $R \hookrightarrow \mathcal{R}_*$ . In particular,  $\mathfrak{a} = \mathfrak{a}_* \cap R$  inside  $\mathcal{R}_*$ .
- 2)  $\mathfrak{a}k_* = (f_{1*}, \dots, f_{r*}) \subset Rk_* = k_*[T_1, \dots, T_N]$  is called the finite part of  $\mathfrak{a}_*$ . Notice that  $V(\mathfrak{a}k_*) \hookrightarrow \mathbb{A}_{k_*}^N$  is nothing but the base change of  $V(\mathfrak{a}) \hookrightarrow \mathbb{A}_k^N$  under  $k \hookrightarrow k_*$ .
- 3) Finally, if  $\mathfrak{a}$  is a prime ideal, then so  $\mathfrak{a}_*$ , and the canonical inclusion of  $k$ -algebras  $R/\mathfrak{a} \hookrightarrow \mathcal{R}_*/\mathfrak{a}_*$  gives rise to inclusions of their quotient fields  $\kappa(\mathfrak{a}) \hookrightarrow \kappa(\mathfrak{a}_*)$ , and one has: The algebraic closure of  $\bar{\kappa}(\mathfrak{a})$  of  $\kappa(\mathfrak{a})$  and  $\kappa(\mathfrak{a}_*)$  are linearly disjoint over  $\kappa(\mathfrak{a})$ .

Let  $V$  be a projective reduced  $k$ -scheme, and  $V = \text{Proj } k[T_0, \dots, T_N]/\mathfrak{p} \hookrightarrow \mathbb{P}_k^N$  be a fixed projective embedding, where  $\mathfrak{p} = (f_1, \dots, f_r) \subset k[T_0, \dots, T_N]$  is a homogeneous prime ideal. Then  $\mathfrak{P} := \mathfrak{p} \cap \mathcal{O}_v[T_0, \dots, T_N]$  is a homogeneous prime ideal of  $\mathcal{O}_v[T_0, \dots, T_N]$  and therefore,  $\mathcal{V} := \text{Proj } \mathcal{O}_v[T_0, \dots, T_N]/\mathfrak{P} \hookrightarrow \mathbb{P}_{\mathcal{O}_v}^N$  is a projective  $\mathcal{O}_v$ -scheme, etc.

**Fact 2.7.** In the above notations, the following hold:

- 1)  $\mathcal{V} \hookrightarrow \mathbb{P}_{\mathcal{O}_v}^N$  is the scheme theoretical closure of  $V$  under  $V \hookrightarrow \mathbb{P}_k^N \hookrightarrow \mathbb{P}_{\mathcal{O}_v}^N$ , and  $V \hookrightarrow \mathbb{P}_k^N$  is the generic fiber of  $\mathcal{V} \hookrightarrow \mathbb{P}_{\mathcal{O}_v}^N$ . Further, the special fiber  $\mathcal{V}_v \hookrightarrow \mathbb{P}_{k_v}^N$  is reduced.
- 2) If  $V = \cup_i V_i$  with  $V_i$  closed subsets, then  $\mathcal{V} = \cup_i \mathcal{V}_i$ , and  $\mathcal{V}_v = \cup_i \mathcal{V}_{i,v}$ . Further, if  $V$  is connected, so is  $\mathcal{V}_v$ , and if  $V$  is integral, then  $\mathcal{V}_v$  is of pure dimension equal to  $\dim(V)$ .
- 3) If  $V$  is integral and normal, then the local ring at any generic point  $\eta_i$  of  $\mathcal{V}_v$  is a valuation ring  $\mathcal{O}_{v\eta_i}$  of  $k(V)$  dominating  $\mathcal{O}_v$ .

**Definition 2.8.**

- 1) The valuation  $v_{\eta_i}$  with valuation ring  $\mathcal{O}_{v\eta_i}$  introduced at Fact 2.7, c), above is called a Deuring constant reduction of  $k(V)$  at  $v$ .
- 2) The  $v$ -reduction/specialization map for closed subsets of  $V$  is defined by

$$\text{sp}_v : \{S \subseteq V \mid S \text{ closed}\} \rightarrow \{\mathcal{S}'_v \subseteq \mathcal{V}_v \mid \mathcal{S}'_v \text{ closed}\}, \quad S \mapsto \mathcal{S}'_v.$$

- 3) In particular, if  $P \subset V$  is a prime Weil divisor, then  $\mathcal{P} \subset \mathcal{V}$  is a relative Weil divisor of  $\mathcal{V}$ . Finally,  $\text{sp}_v$  gives rise to a Weil divisor reduction/specialization homomorphism

$$\text{sp}_v : \text{Div}(V) \rightarrow \text{Div}(\mathcal{V}_v), \quad P \mapsto \sum_i \mathcal{P}_{v,i}$$

where  $\mathcal{P}_{v,i}$  are the irreducible components of the special fiber  $\mathcal{P}_v$  of  $\mathcal{P}$ .

**Fact 2.9.** There exists a Zariski open nonempty set  $\mathcal{U} \subset \text{Val}_k$  such that each  $v \in \mathcal{U}$  satisfies:

- a) If  $V$  is integral/normal/smooth, then so are  $\mathcal{V}$  and  $\mathcal{V}_v$ . In particular, if  $V$  is integral and normal, then  $\mathcal{V}_v$  is integral and normal, and  $v_{\eta_v}$  is called the canonical (Deuring) constant reduction of  $k(V)$  at  $v$ , and we denote it by  $v_{k(V)}$ .
- b) If  $S_1, \dots, S_r \subseteq V$  are distinct closed subsets of  $V$ , then  $\text{sp}_v(S_1), \dots, \text{sp}_v(S_r)$  are distinct closed subsets of  $\mathcal{V}_v$ . And if  $\mathcal{P} = \cup_i P_i$  is a NCD (normal crossings divisor) in  $V$ , then so is  $\mathcal{P}_v = \cup_i \text{sp}_v(P_i)$  in  $\mathcal{V}_v$ .

- c)  $\text{sp}_v : \text{Div}(V) \rightarrow \text{Div}(\mathcal{V}_v)$  is compatible with principal divisors, and therefore gives rise to a reduction/specialization homomorphism  $\text{sp}_v : \mathfrak{Cl}(V) \rightarrow \mathfrak{Cl}(\mathcal{V}_v)$ .
- d) If  $H \subset \mathbb{P}_k^N$  is a general hyperplane, then  $\mathcal{H} \subset \mathbb{P}_{\mathcal{O}_v}^N$  is a general relative hyperplane and  $\mathcal{H}_v \subset \mathbb{P}_{kv}^N$  is a general hyperplane, and  $(\mathcal{H} \cap \mathcal{V})_v = \mathcal{H}_v \cap \mathcal{V}_v$  inside  $\mathbb{P}_{kv}^N$ .
- e) Let  $\mathbb{G}_m(V) \subset k(V)$  be the group of invertible global section on  $V$ , and  $\mathbb{G}_m(\mathcal{V}_v)$  be correspondingly defined. Then  $\mathbb{G}_m(V)/k^\times \rightarrow \mathbb{G}_m(\mathcal{V}_v)/kv^\times$  is an isomorphism.

In the above context and notations, let  $W \rightarrow V$  be a dominant morphism of projective integral normal  $k$ -varieties, which is generically finite, and  $K := k(V) \hookrightarrow k(W) =: L$  be the corresponding finite field extension. Let  $\tilde{L}|L$  be a finite Galois extension with  $[\tilde{L} : L]$  relatively prime to  $[L : K]$ , and  $\tilde{K}|K$  be the maximal Galois subextension of  $K \hookrightarrow \tilde{L}$  having degree  $[\tilde{K} : K]$  relatively prime to  $[L : K]$ . Then the canonical projection of Galois groups  $H := \text{Gal}(\tilde{L}|L) \rightarrow \text{gal}(\tilde{K}|K) =: G$  is surjective. Further, let  $\tilde{V} \rightarrow V$  and  $\tilde{W} \rightarrow W$  be the normalizations of  $V$  in  $K \hookrightarrow \tilde{K}$ , respectively of  $W$  in  $L \hookrightarrow \tilde{L}$ . Then by the universal property of normalization,  $\tilde{W} \rightarrow W \rightarrow V$  factors as  $\tilde{W} \rightarrow \tilde{V} \rightarrow V$ .

Finally, let  $V \hookrightarrow \mathbb{P}_k^N$ , etc., be projective  $k$ -embeddings, and consider the corresponding projective  $\mathcal{O}_v$ -schemes  $V \hookrightarrow \mathcal{V}$ , etc., as defined above. Then by general scheme theoretical non-sense,  $W \rightarrow V$ , etc., give rise to dominant rational  $\mathcal{O}_v$ -maps  $\mathcal{W} \dashrightarrow \mathcal{V}$ , etc.

**Fact 2.10.** There exists a Zariski open nonempty set  $\mathcal{U} \subset \text{Val}_k$  such that each  $v \in \mathcal{U}$  satisfies:

- a) The rational maps above are actually morphisms of  $\mathcal{O}_v$ -schemes, and one has commutative diagrams of morphisms of projective  $\mathcal{O}_v$ -schemes

$$\begin{array}{ccc}
 \tilde{W} & \rightarrow & W \\
 \downarrow & & \downarrow \\
 \tilde{V} & \rightarrow & V
 \end{array}
 \quad
 \begin{array}{ccc}
 \tilde{\mathcal{W}} & \rightarrow & \mathcal{W} \\
 \downarrow & & \downarrow \\
 \tilde{\mathcal{V}} & \rightarrow & \mathcal{V}
 \end{array}
 \quad
 \begin{array}{ccc}
 \tilde{\mathcal{W}}_v & \rightarrow & \mathcal{W}_v \\
 \downarrow & & \downarrow \\
 \tilde{\mathcal{V}}_v & \rightarrow & \mathcal{V}_v
 \end{array}$$

where the LHS diagrams is the generic fiber of the middle one, respectively RHS diagrams is the special fiber of the middle one.

- b) Moreover, the degrees of the morphism which correspond to each other in the above diagrams are equal. Thus the canonical constant reductions  $v_L, v_K, v_{\tilde{L}}$  are the unique prolongations of  $v_K$  to  $L, \tilde{K},$  and  $\tilde{L}$ , respectively.
- c) The residue field extension  $\tilde{K}v_{\tilde{K}}|Kv_K$  is the maximal Galois subextension of  $\tilde{L}v_{\tilde{L}}|Lv_L$  which has degree relatively prime to  $[L : K] = [Lv_L|Kv_K]$ .

**Fact 2.11.** Let  $W$  be a projective smooth  $k$ -variety, and  $T' \subset W$  be either empty, or the support of a normal crossings divisor in  $W$ , and set  $\mathcal{W}' := W \setminus T'$ . For  $v \in \text{Val}_k$  define correspondingly  $\mathcal{T}' \hookrightarrow \mathcal{W}, \mathcal{T}'_v \hookrightarrow \mathcal{W}_v$ , thus  $\mathcal{W}' = \mathcal{W} \setminus \mathcal{T}', \mathcal{W}'_v = \mathcal{W}_v \setminus \mathcal{T}'_v$ , and let  $D_{\mathcal{W}'}$  and  $D_{\mathcal{W}'_v}$  be the corresponding geometric sets of Weil prime divisors. Then there exists a Zariski open nonempty set  $\mathcal{U} \subset \text{Val}_k$  such that for  $v \in \mathcal{U}$  the special fiber  $\mathcal{W}_v$  is smooth,  $\mathcal{T}'_v$  either empty or a NCD, and the following hold: First, by the purity of the branch locus, and second, by GROTHENDICK–MURRE [G–M] applied to  $\mathcal{W}'_v$ , the canonical maps of pro- $\ell$  abelian fundamental groups below are isomorphisms:

$$\Pi_{1, D_{\mathcal{W}'}} \rightarrow \Pi_1(W') \rightarrow \Pi_1(\mathcal{W}') \leftarrow \Pi_1(\mathcal{W}'_v) \leftarrow \Pi_{1, D_{\mathcal{W}'_v}}.$$

We now come back to the context of Theorem 2.1. Recall the notations and the context from Preparation/Notations 2.2, and the  $k$ -morphisms  $X \rightarrow X_0 \leftarrow Y, Z \rightarrow X$ , further the

closed subset  $S_0 \subset X_0$  and the algebraic set of prime divisors  $D = D_{X_0 \setminus S_0}$ , and finally, the preimages  $S \rightarrow S_0 \leftarrow T$  of  $S_0$  under  $X \rightarrow X_0 \leftarrow Y$ . Finally, by Proposition 2.3, one has canonical identifications, respectively a surjective canonical projection as below:

$$\Pi_{1, D_{X \setminus S}} = \Pi_1(X \setminus S) \rightarrow \Pi_1(X) = \Pi_{1, D_X} = \Pi_{1, K}.$$

After choosing projective embeddings for each of the  $k$ -varieties  $X_0, X, Y, Z$ , we get for every  $v \in \text{Val}_k$  the corresponding  $\mathcal{O}_v$ -schemes  $X_0 \hookrightarrow \mathcal{X}_0 \leftarrow \mathcal{X}_{0v}$ , etc., and for every of the  $k$ -morphisms above, we get corresponding dominant rational  $\mathcal{O}_v$ -maps, etc. Then applying Facts 2.7–2.11, one has the following:

**Fact/Notations 2.12.** There exists a Zariski open subset  $\mathcal{U} \subset \text{Val}_k$  of valuations  $v$  satisfying the following: The special fiber of each of the above  $k$ -varieties is irreducible and normal/smooth if the generic fiber was so. Further, all the dominant rational  $\mathcal{O}_v$ -maps under discussion above are actually  $\mathcal{O}_v$ -morphisms such that the degrees of the generic fiber  $k$ -morphisms and the corresponding special fiber  $kv$ -morphisms are equal. In particular, the canonical constant reductions  $v_L$  and  $v_M$  are the unique prolongations of  $v_K$  to  $L$ , respectively  $M$ . Finally, let  $S'_0 \subset X_0$  be either empty or equal to  $S_0$ , hence its preimages  $S' \subset X$  and  $T' \subset Y$  are either empty or equal to  $S$ , respectively  $T$ . We set  $X'_0 := X_0 \setminus S'_0$ ,  $X' := X \setminus S'$ ,  $Y' := Y \setminus T'$ , and for  $v \in \mathcal{U}$  consider the corresponding  $X_0 \rightarrow \mathcal{X}_0 \leftarrow \mathcal{X}_{0v}$ ,  $S'_0 \rightarrow \mathcal{S}'_0 \leftarrow \mathcal{S}'_{0v}$  and the resulting  $X'_0 \rightarrow \mathcal{X}'_0 \leftarrow \mathcal{X}'_{0v}$ , etc. Then the resulting  $kv$ -morphisms satisfy the following:

- a)  $\mathcal{S}_v \subset \mathcal{X}_v$ ,  $\mathcal{T}_v \subset \mathcal{Y}_v$  are the preimages of  $\mathcal{S}_{0v}$  under  $\mathcal{X}_v \rightarrow \mathcal{X}_{0v} \leftarrow \mathcal{Y}_v$ , and the restriction  $\mathcal{D}_0$  of  $D_{\mathcal{Y}_v}$  to  $Kv_K$  satisfies  $D_{\mathcal{X}_{0v}} \subseteq \mathcal{D}_0 \subseteq D_{\mathcal{X}_v}$ .
- b) Further, setting  $\mathcal{D} := D_{\mathcal{X}_{0v} \setminus \mathcal{S}_{0v}}$ , one has that  $\mathcal{D} \subseteq \text{sp}_v(D)$ , and further:
  - $\mathcal{Y}_v \rightarrow \mathcal{X}_{0v}$  is a prime to  $\ell$  alteration above  $\mathcal{S}_{0v}$ , hence  $\mathcal{T}_v$  is NCD in  $\mathcal{Y}_v$ .
  - $\mathcal{X}_v \rightarrow \mathcal{X}_{0v}$  are projective normal models of  $Kv_K|kv$ , and  $\mathcal{D} \subseteq \mathcal{D}_0 \subseteq D_{\mathcal{X}_v}$ .
  - $\mathcal{Z}_v \rightarrow \mathcal{X}_v$  is a finite generically normal alteration with  $\text{Aut}(Mv_M|Kv_K) = \text{Aut}(M|K)$ .
- c) Applying Proposition 2.3 to the fibers, one has:
  - The canonical maps  $\Pi_{1, D_{X'}} \rightarrow \Pi_1(X')$  and  $\Pi_{1, D_{\mathcal{X}'_v}} \rightarrow \Pi_1(\mathcal{X}'_v)$  are isomorphisms.
  - $\Pi_{1, D_X} \rightarrow \Pi_1(X) \rightarrow \Pi_{1, K}$  and  $\Pi_{1, D_{\mathcal{X}_v}} \rightarrow \Pi_1(\mathcal{X}_v) \rightarrow \Pi_{1, Kv_K}$  are isomorphisms.
- d) By the functoriality of fundamental group and Fact 2.11 one has:
  - $X' \hookrightarrow \mathcal{X}' \leftarrow \mathcal{X}'_v$  give rise to surjective projections  $\Pi_1(X') \rightarrow \Pi_1(\mathcal{X}') \leftarrow \Pi_1(\mathcal{X}'_v)$ .
  - $Y' \hookrightarrow \mathcal{Y}' \leftarrow \mathcal{Y}'_v$  give rise to canonical isomorphisms  $\Pi_1(Y') \rightarrow \Pi_1(\mathcal{Y}') \leftarrow \Pi_1(\mathcal{Y}'_v)$ .

**Proposition 2.13.** *The canonical maps  $\Pi_1(X') \rightarrow \Pi_1(\mathcal{X}') \leftarrow \Pi_1(\mathcal{X}'_v)$  are isomorphisms.*

*Proof.* We consider the canonical commutative diagram of  $\mathcal{O}_v$ -morphism, and the corresponding maps of fundamental groups:

$$\begin{array}{ccc}
Y' & \hookrightarrow & \mathcal{Y}' & \leftarrow & \mathcal{Y}'_v \\
(*) & \downarrow & \downarrow & & \downarrow \\
X' & \hookrightarrow & \mathcal{X}' & \leftarrow & \mathcal{X}'_v
\end{array}
\quad
\begin{array}{ccc}
\Pi_1(Y') & \rightarrow & \Pi_1(\mathcal{Y}') & \leftarrow & \Pi_1(\mathcal{Y}'_v) \\
(\dagger) & \downarrow & \downarrow & & \downarrow \\
\Pi_1(X') & \rightarrow & \Pi_1(\mathcal{X}') & \leftarrow & \Pi_1(\mathcal{X}'_v)
\end{array}$$

The homomorphisms in these diagrams satisfy: First, since the vertical morphism in the diagram (\*) are finite, having degree equal to  $[L : K]$ , the vertical homomorphisms in the diagram (\dagger) have open images of index dividing  $[L : K]$ . Thus since  $[L : K]$  is prime to  $\ell$ , it follows that the vertical maps in the diagram (\dagger) are actually surjective. Further, the morphisms in the first row of the diagram (\dagger) are isomorphisms, and those of the second row

are surjective by Fact/Notations 2.12, d). We have to prove that the homomorphisms in the second row of the diagram (†) are isomorphisms too.

**Claim 1.**  $\Pi_1(X') \rightarrow \Pi_1(\mathcal{X}')$  is injective.

Indeed, let  $\tilde{X} \rightarrow X'$  any finite abelian  $\ell$ -power degree etale cover of  $X'$ ,  $\tilde{K} = k(\tilde{X})$  its function field, and  $\tilde{\mathcal{X}} \rightarrow \mathcal{X}'$  the normalization of  $\mathcal{X}'$  in  $K \hookrightarrow \tilde{K}$ . We claim that  $\tilde{\mathcal{X}} \rightarrow \mathcal{X}'$  is etale. Since being etale is an open condition, it is sufficient to prove that the cover of the special fiber  $\tilde{\mathcal{X}}_v \rightarrow \mathcal{X}'_v$  is etale. (Notice that we do not know yet whether  $\tilde{\mathcal{X}}_v$  is reduced.) Let  $\tilde{L} := L\tilde{K}$ , and  $\tilde{Y} \rightarrow Y'$  be the normalization of  $Y'$  in  $L \hookrightarrow \tilde{L}$ . Since  $\Pi_1(Y') \rightarrow \Pi_1(X')$  is surjective, and  $\tilde{X} \rightarrow X'$  is etale, it follows that  $\text{Gal}(\tilde{L}|L)$  is a quotient of  $\Pi_1(Y')$ . Thus the normalization  $\tilde{\mathcal{Y}} \rightarrow \mathcal{Y}'$  of  $\mathcal{Y}'$  in  $L \hookrightarrow \tilde{L}$  is etale. Further, since the morphisms in the first row of the diagram (†) are isomorphisms, it follows that the corresponding  $\tilde{\mathcal{Y}} \rightarrow \mathcal{Y}'$ ,  $\tilde{\mathcal{Y}}_v \rightarrow \mathcal{Y}'_v$  are etale covers of integral  $\mathcal{O}_v$ -schemes satisfying  $\deg(\tilde{Y} \rightarrow Y) = \deg(\tilde{\mathcal{Y}}_v \rightarrow \mathcal{Y}_v)$ . In particular, the canonical constant reduction  $v_L$  has a unique prolongation to  $v_{\tilde{L}}$  to  $\tilde{L}$ , and  $v_{\tilde{L}}|v_L$  is totally inert. Since  $v_L|v_K$  is totally inert by the fact that  $v \in \mathcal{U}$ , it follows that  $v_{\tilde{L}}|v_K$  is totally inert as well. Thus  $v_K$  has a unique prolongation  $v_{\tilde{K}}$  to  $\tilde{K}$  as well, and  $v_{\tilde{K}}|v_K$  is totally inert, i.e.,  $\text{Gal}(\tilde{K}|K)$  equals the decomposition group above  $v_K$ , and the inertia group above  $v_K$  is trivial. Equivalently,  $\tilde{\mathcal{X}}_v$  is integral, and  $Kv_K = \kappa(\mathcal{X}'_v) \hookrightarrow \kappa(\tilde{\mathcal{X}}_v) = \tilde{K}v_{\tilde{K}}$  is Galois extension with  $\text{Gal}(\tilde{K}v_{\tilde{K}}|Kv_K) = \text{Gal}(\tilde{K}|K)$ . Hence we have the following:  $\tilde{K}v_{\tilde{K}}|Kv_K$  is an abelian  $\ell$ -power degree extension such that  $\tilde{L}v_{\tilde{L}}$  is the compositum  $\tilde{L}v_{\tilde{L}} = Lv_L\tilde{K}v_{\tilde{K}}$ , and  $\tilde{\mathcal{Y}}_v \rightarrow \mathcal{Y}'_v$  is etale. By Proposition 2.3, it follows that that  $\tilde{\mathcal{X}}_v \rightarrow \mathcal{X}'_v$  is etale as well. Thus  $\tilde{\mathcal{X}} \rightarrow \mathcal{X}'$  is etale, and Claim 1 is proved. Conclude that  $\Pi_1(X') \rightarrow \Pi_1(\mathcal{X}')$  is an isomorphism.

**Claim 2.**  $\Pi_1(\mathcal{X}'_v) \rightarrow \Pi_1(\mathcal{X}')$  is injective.

In order to prove the claim, via the canonical isomorphisms  $\Pi_1(Y') \rightarrow \Pi_1(\mathcal{Y}') \leftarrow \Pi_1(\mathcal{Y}'_v)$ , we can identify these groups with a finite  $\mathbb{Z}_\ell$ -module  $\Pi$ , thus the canonical surjective projections  $\Pi \rightarrow \Pi_1(\mathcal{X}'_v) \rightarrow \Pi_1(\mathcal{X}')$  are defined as quotients of  $\Pi$  by  $\Delta := \ker(\Pi \rightarrow \Pi_1(\mathcal{X}'))$  and  $\Delta_v := \ker(\Pi \rightarrow \Pi_1(\mathcal{X}'_v))$ , which are finite  $\mathbb{Z}_\ell$ -submodules of  $\Pi$ . The following conditions are obviously equivalent:

- i)  $\Pi_1(\mathcal{X}'_v) \rightarrow \Pi_1(\mathcal{X}')$  is an isomorphism
- ii)  $\Delta_v = \Delta$
- iii)  $\ell\Delta + \Delta_v = \Delta$

where iii)  $\Rightarrow$  ii) follows by Nakayama's Lemma, because  $\Delta \subseteq \Pi_1(Y)$  is a finite  $\mathbb{Z}_\ell$ -module. On the other hand, since  $\Delta$  is a finite  $\mathbb{Z}_\ell$ -module, there exist only finitely many subgroups  $\Sigma$  such that  $\ell\Delta \subseteq \Sigma \subseteq \Delta$ . Thus for all sufficiently large  $\ell^e$  [precisely, for  $\Pi \rightarrow \bar{\Pi} := \Pi/\ell^e$ , one must have that  $\Delta/\ell \rightarrow \bar{\Delta}/\ell$  is injective], the above conditions are equivalent as well to:

- iv)  $\bar{\Delta}_v = \bar{\Delta}$ .
- v)  $\Pi_1(\mathcal{X}'_v)/\ell^e \rightarrow \Pi_1(\mathcal{X}')/\ell^e$  is an isomorphism.

Thus we conclude that it is sufficient to show that given  $e > 0$ , there exists a Zariski open non-empty subset  $\mathcal{U}_e \subset \mathcal{U}$  of valuations  $v$  such that condition v) above is satisfied. By contradiction, suppose that this is not the case, hence for every Zariski open subset  $\mathcal{U}' \subset \mathcal{U}$ , there exists some  $v \in \mathcal{U}'$  such that condition v) does not hold at  $v$ . We notice that  $\Pi/\ell^e$  is finite, thus it has only finitely many quotients. Therefore, there exist a quotient  $\Pi/\ell^e \rightarrow G$  such that the set  $\Sigma_G := \{v \in \mathcal{U} \mid G = \Pi_1(\mathcal{X}'_v)/\ell^e \rightarrow \Pi_1(\mathcal{X}')/\ell^e \text{ is not an isomorphism}\}$  is

in  $\mathfrak{D}$ . For  $v \in \Sigma$ , and the etale cover  $\tilde{\mathcal{X}}_v \rightarrow \mathcal{X}'_v$ , it follows that the base change  $\tilde{\mathcal{X}}_v \times_{\mathcal{X}'_v} \mathcal{Y}'_v \rightarrow \mathcal{Y}'_v$  of  $\tilde{\mathcal{X}}_v \rightarrow \mathcal{X}'_v$  to  $\mathcal{Y}'_v$  is etale and has Galois group  $G$ . Since the morphisms in the first row of the diagram  $(\dagger)$  are isomorphisms, there exists a unique etale cover  $\tilde{\mathcal{Y}} \rightarrow \mathcal{Y}'$  with  $\text{Gal}(\tilde{\mathcal{Y}}|\mathcal{Y}') = G$  and special fiber  $\tilde{\mathcal{Y}}_v = \tilde{\mathcal{X}}_v \times_{\mathcal{X}'_v} \mathcal{Y}'_v$ . Further, if  $\tilde{Y} \rightarrow Y$  is the generic fiber of  $\tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ , recalling the discussion/notations from Definition/Remark 2.6, one has the following: Let  $\kappa(\mathcal{X}'_*) := \prod_v \kappa(\mathcal{X}'_v)/\mathfrak{D}$ , and  $\kappa(\tilde{\mathcal{X}}_*)$ ,  $\kappa(\mathcal{Y}'_*)$  and  $\kappa(\tilde{\mathcal{Y}}_*)$  be correspondingly defined. Then by general principles of ultraproducts of fields one has that  $\kappa(\tilde{\mathcal{Y}}_*) = \kappa(\tilde{\mathcal{X}}'_*)L$ ,  $\kappa(\tilde{\mathcal{Y}}_*) = \kappa(\tilde{\mathcal{X}}'_*)\tilde{L}$ , and  $\kappa(\mathcal{X}'_*) \rightarrow \kappa(\tilde{\mathcal{X}}_*)$ ,  $\kappa(\mathcal{Y}'_*) \rightarrow \kappa(\tilde{\mathcal{Y}}_*)$  are Galois field extensions with Galois group  $G$ . On the other hand, by Definition/Remark 2.6, c), the fields  $\bar{K}$  and  $\kappa(\mathcal{X}'_*)$  are linearly disjoint over  $K$ . Therefore, there exists a unique finite abelian  $\ell$ -power degree extension  $\tilde{K}|K$  such that  $\kappa(\tilde{\mathcal{X}}_*) = \kappa(\mathcal{X}'_*)\tilde{K}$ . But then the linear disjointness of  $\bar{K}$  and  $\kappa(\mathcal{X}'_*)$  over  $K$  implies that  $\tilde{L} = L\tilde{K}$ . And since  $\tilde{Y} \rightarrow Y'$  is etale with Galois group  $G$ , by Lemma 2.4, it follows that the normalization  $\tilde{X} \rightarrow X'$  of  $X'$  in  $K \hookrightarrow \tilde{K}$  is etale (with Galois group  $G$ ). Hence the corresponding  $\tilde{\mathcal{X}} \rightarrow \mathcal{X}'$  is an etale cover with Galois group  $G$ , etc., contradiction! This concludes the proof of Proposition 2.13.  $\square$

Finally, concerning notations and general facts, we mention the following:

- e) For  $v \in \mathcal{U}$ , let  $H \subset \mathbb{P}_{kv}^N$  be a general hyperplane. Then  $\mathcal{X}_H := \mathcal{X}_v \cap H$  is a projective normal  $kv$ -variety, and a Weil prime divisor of  $\mathcal{X}_v$ , whose valuation we denote by  $v_H$ . And  $\mathcal{S}_H := \mathcal{S}_v \cap H$  is a closed subset of  $\mathcal{X}_H$ , and we let  $D_{\mathcal{X}_H \setminus \mathcal{S}_H}$  be the corresponding geometric set of Weil prime divisors of the function field  $\kappa(\mathcal{X}_H)|kv$  of  $\mathcal{X}_H$ .
- f) For a valuation  $\mathfrak{v}$  of  $K$ , we let  $U_{\mathfrak{v}} := \mathcal{O}_{\mathfrak{v}}^{\times} \subset K^{\times}$  be the  $\mathfrak{v}$ -units, and  $\hat{j}_{\mathfrak{v}} : \hat{U}_{\mathfrak{v}} \rightarrow \widehat{K\mathfrak{v}}$  be the  $\ell$ -adic completion of the  $\mathfrak{v}$ -reduction homomorphism  $j_{\mathfrak{v}} : U_{\mathfrak{v}} \rightarrow K\mathfrak{v}^{\times}$ .

**Proposition 2.14.** *Let  $\mathcal{U}$  be the Zariski open non-empty set from Fact/Notations 2.12. In the above notations, for  $v \in \mathcal{U}_D$  we set  $\mathfrak{v} := v_H \circ v_K$ , thus  $K\mathfrak{v} = (Kv_K)v_H$  is a function field over  $kv$ , and consider the geometric sets of prime divisors:  $D_{X \setminus S}$  of  $K|k$ ,  $D_{\mathcal{X}_v \setminus \mathcal{S}_v}$  of the function field  $Kv_K|kv$ , respectively  $D_{\mathcal{X}_H \setminus \mathcal{S}_H}$  of the function field  $K\mathfrak{v}|kv$ . Then one has:*

- 1)  $\hat{U}_K \subseteq \hat{U}_{D_{X \setminus S}} \subseteq \hat{U}_{v_K}$ , and  $\hat{j}_{v_K}$  maps  $\hat{U}_K \subseteq \hat{U}_{D_{X \setminus S}}$  isomorphically onto  $\hat{U}_{Kv_K} \subseteq \hat{U}_{D_{\mathcal{X}_v \setminus \mathcal{S}_v}}$ .
- 2)  $\mathfrak{v} := v_H \circ v_K$  is a quasi prime divisor of  $K|k$  satisfying  $\mathfrak{v}|_k = v$  and  $K\mathfrak{v} = \kappa(\mathcal{X}_v)$ .
- 3)  $\hat{U}_K \subseteq \hat{U}_{D_{X \setminus S}} \subseteq \hat{U}_{\mathfrak{v}}$ , and  $\hat{j}_{\mathfrak{v}}$  maps  $\hat{U}_K \subseteq \hat{U}_{D_{X \setminus S}}$  isomorphically onto  $\hat{U}_{K\mathfrak{v}} \subseteq \hat{U}_{D_{\mathcal{X}_H \setminus \mathcal{S}_H}}$ .

*Proof.* To 1): The  $\ell$ -adic duals of the isomorphisms  $\Pi_1(X) \leftarrow \Pi_1(\mathcal{X}_v)$ ,  $\Pi_1(X \setminus S) \leftarrow \Pi_1(\mathcal{X}_v \setminus \mathcal{S}_v)$  are the isomorphism  $\hat{U}_K \rightarrow \hat{U}_{Kv_K}$ , respectively  $\hat{U}(X \setminus S) \rightarrow \hat{U}(\mathcal{X}_v \setminus \mathcal{S}_v)$ . On the other hand, by Kummer theory it follows that the last two isomorphisms are defined by the  $\ell$ -adic completion of the residue homomorphism  $j_{v_K} : U_{v_K} \rightarrow Kv_K^{\times}$ . This completes the proof of assertion 1).

The proof of assertion 2) is clear, because  $w_H$  is trivial on  $kv$ , thus the restriction of  $\mathfrak{v}$  to  $k$  equals the one of  $v_K$ , which is  $v$  by the definition of  $v_K$ .

Finally, assertion 3) is proved actually in POP [P3], at the end of the proof of assertion 1) of loc.cit., Proposition 23, and we omit that argument here.  $\square$

We next make a short preparation for the second application of the specialization techniques. Let  $W \rightarrow V$  be a finite morphism of projective normal  $k$ -varieties with function fields  $K := k(V)$  and  $M := k(W)$ . Consider the inclusion  $\mathcal{I} : \text{Div}(V) \rightarrow \text{Div}(W)$  and the

norm  $\mathcal{N} : \text{Div}(W) \rightarrow \text{Div}(V)$  maps, which map principal divisors to principal divisors and  $\mathcal{N} \circ \mathcal{I} = [M : K] \cdot \text{id}_{\text{Div}(V)}$  on  $\text{Div}(V)$ . Thus one has a commutative diagram of the form:

$$\begin{array}{ccccccc} 1 & \rightarrow & \mathcal{H}(M) & \hookrightarrow & \text{Div}(W) & \longrightarrow & \mathfrak{C}\mathfrak{l}(W) \rightarrow 0 \\ & & \uparrow \downarrow & & \uparrow \downarrow & & \uparrow \downarrow \\ 1 & \rightarrow & \mathcal{H}(K) & \hookrightarrow & \text{Div}(V) & \longrightarrow & \mathfrak{C}\mathfrak{l}(V) \rightarrow 0 \end{array}$$

where the upper arrows are defined by  $\mathcal{I}$ , the down arrows are defined by  $\mathcal{N}$ , and the composition  $\downarrow \circ \uparrow$  is the multiplication by  $[M : K]$ . Thus  $\mathcal{I}$  and  $\mathcal{N} \circ \mathcal{I}$  map elements of infinite order to such, and if  $[x] \in \mathfrak{C}\mathfrak{l}(X)$  has finite order  $o_{[x]}$ , the order of  $\mathcal{I}([x])$  divides  $o_{[x]}$ , and the order of  $\mathcal{N} \circ \mathcal{I}([x])$  equals  $o_{[x]}/n$ , where  $n$  is the g.c.d. of  $o_{[x]}$  and  $[M : K]$ .

**Fact 2.15.** There exists a Zariski open nonempty set  $\mathcal{U} \subset \text{Val}_k$  such that each  $v \in \mathcal{U}$  satisfies:

- 1) The inclusion/norm morphisms  $\mathcal{I} : \text{Div}(V) \rightarrow \text{Div}(W)$ ,  $\mathcal{N} : \text{Div}(W) \rightarrow \text{Div}(V)$  are compatible with the specialization maps  $\text{sp}_{W,v}$  and  $\text{sp}_{V,v}$ , and with the inclusion/norm morphisms  $\mathcal{I}_v : \text{Div}(\mathcal{V}_v) \rightarrow \text{Div}(\mathcal{W}_v)$  and  $\mathcal{N}_v : \text{Div}(\mathcal{W}_v) \rightarrow \text{Div}(\mathcal{V}_v)$ , i.e., one has:

$$\text{sp}_{W,v} \circ \mathcal{I} = \text{sp}_{V,v} \circ \mathcal{I}_v, \quad \mathcal{N}_v \circ \text{sp}_{W,v} = \mathcal{N} \circ \text{sp}_{V,v}$$

- 2) The specialization maps  $\text{sp}_v$  are compatible with principal divisors, hence define specialization morphisms  $\text{sp}_v : \mathfrak{C}\mathfrak{l}(V) \rightarrow \mathfrak{C}\mathfrak{l}(\mathcal{V}_v)$ ,  $\text{sp}_v : \mathfrak{C}\mathfrak{l}(W) \rightarrow \mathfrak{C}\mathfrak{l}(\mathcal{W}_v)$  which are compatible with the inclusion/norm morphisms, thus one has commutative diagrams:

$$\begin{array}{ccccccc} \mathcal{H}(M) & \xrightarrow{\text{sp}_v} & \mathcal{H}(Mv_M) & & \text{Div}(W) & \xrightarrow{\text{sp}_v} & \text{Div}(\mathcal{W}_v) & & \mathfrak{C}\mathfrak{l}(W) & \xrightarrow{\text{sp}_v} & \mathfrak{C}\mathfrak{l}(\mathcal{W}_v) \\ \uparrow \downarrow & & \uparrow \downarrow \\ \mathcal{H}(K) & \xrightarrow{\text{sp}_v} & \mathcal{H}(Kv_K) & & \text{Div}(V) & \xrightarrow{\text{sp}_v} & \text{Div}(\mathcal{V}_v) & & \mathfrak{C}\mathfrak{l}(V) & \xrightarrow{\text{sp}_v} & \mathfrak{C}\mathfrak{l}(\mathcal{V}_v) \end{array}$$

in which the vertical maps are the defined by the inclusion/norm homomorphisms, and the horizontal maps are the corresponding specialization homomorphism (as introduced in Fact 2.10).

- 3) Recalling that  $\mathfrak{C}\mathfrak{l}^0(\bullet) \subseteq \mathfrak{C}\mathfrak{l}'(\bullet)$  are the maximal divisible, respectively  $\ell$ -divisible, subgroups of  $\mathfrak{C}\mathfrak{l}(\bullet)$ , one has:  $\text{sp}_v$  maps  $\mathfrak{C}\mathfrak{l}^0(V) \subseteq \mathfrak{C}\mathfrak{l}'(V)$  into  $\mathfrak{C}\mathfrak{l}^0(\mathcal{V}_v) \subseteq \mathfrak{C}\mathfrak{l}'(\mathcal{V}_v)$ , respectively, and correspondingly for  $W$ .
- 4) Finally, for  $[x] \in \mathfrak{C}\mathfrak{l}(V)$  and  $[y] := \mathcal{I}([x])$ , one has:
  - a) The order of  $\text{sp}_v[x]$  is infinite if and only if the order of  $\text{sp}_v[y]$  is infinite.
  - b) If  $\text{sp}_v[y]$  has finite order, then  $\text{sp}_v[x]$  has finite order, and their orders satisfy:

$$o_{[M:K] \cdot \text{sp}_v[x]} | o_{\text{sp}_v[y]} | o_{\text{sp}_v[x]}.$$

We conclude this preparation by mentioning the following fundamental fact about specialization of points of abelian  $k$ -varieties, which follows from JOSSEN [Jo], Theorem 1 (which generalizes an earlier result by PINK [Pk] to arbitrary finitely generated base fields).

**Fact 2.16.** Let  $A$  be an abelian variety over  $k$ , and  $x \in A(k)$ . Then there exists  $\mathcal{U}_A \subset \text{Val}_k$  Zariski open such for  $v \in \mathcal{U}_A$ , there exists an abelian  $\mathcal{O}_v$ -scheme  $\mathcal{A}$  with generic fiber  $A$  and special fiber  $\mathcal{A}_v$ , and the specialization  $x_v \in \mathcal{A}_v(kv)$  of  $x \in A(k)$  satisfying:

- 1) If  $x$  has finite order  $o_x$ , then the order  $o_{x_v}$  of  $x_v$  is finite, and  $o_x = o_{x_v}$ .
- 2) If  $o_x$  is infinite, then for every  $\ell^e > 0$  there exists  $\Sigma_x \subset \mathcal{U}_A$  Zariski dense satisfying:
  - i)  $kv$  is an algebraic closure of a finite field with  $\text{char}(kv) \neq \ell$ .

ii)  $o_{x_v}$  is divisible by  $\ell^e$ .

*Proof.* Let  $R \subset k$  be a finitely generated  $\mathbb{Z}$ -algebra, say  $R = \mathbb{Z}[a_1, a_1^{-1}, \dots, a_n, a_n^{-1}]$ ,  $a_i \in k^\times$ , over which  $A$  and  $x \in A(k)$  are defined, i.e., there exists an  $R$ -abelian scheme  $\mathcal{A}_R$  and an  $R$ -point  $x_R \in \mathcal{A}_R(R)$ , such that  $A$  and  $x \in A(k)$  are the base changes of  $\mathcal{A}_R$  and  $x_R \in \mathcal{A}_R(R)$  under the canonical inclusion  $R \hookrightarrow k$ . We denote by  $\mathcal{A}_s$  and  $x_s \in \mathcal{A}_s(\kappa(s))$  the fibers of  $\mathcal{A}_R$ , respectively  $x_R$ , at  $s \in \text{Spec } R$ , and notice that if the order  $o_x$  of  $x$  is finite, then  $o_x = o_{x_R}$ . Thus the set of all the points  $s \in \text{Spec } R$  such that  $o_x = o_{x_R} = o_{x_s}$  is a Zariski open subset in  $\text{Spec } R$ . Thus replacing  $R$  by a  $R[a^{-1}]$  for a properly chosen  $a \in R$ ,  $a \neq 0$ , we can suppose without loss of generality that  $o_x = o_{x_R} = o_{x_s}$  for all  $s \in \text{Spec } R$ .

Let  $\mathcal{U}_A$  be the set of all the valuations  $v$  of  $k$  with  $R \subset \mathcal{O}_v$ . Notice that  $v \in \mathcal{U}_A$  if and only if  $v$  has a center  $s \in \text{Spec } R$  via the inclusion  $R \hookrightarrow k$  if and only if  $a_i$  is a  $v$ -unit for  $i = 1, \dots, n$ . In particular, the base change  $\mathcal{A}_{\mathcal{O}_v} := \mathcal{A}_R \times_R \mathcal{O}_v$  is an abelian  $\mathcal{O}_v$ -scheme with generic fiber  $A$ , and the base change  $x_{\mathcal{O}_v}$  of  $x_R$  under  $R \hookrightarrow \mathcal{O}_v$  is an  $\mathcal{O}_v$ -point of  $\mathcal{A}_{\mathcal{O}_v}$  whose generic fiber is  $x$ , and whose special fiber  $x_v \in \mathcal{A}_v(kv)$  is the base change of  $x_s$  under the canonical embedding  $\kappa(s) \hookrightarrow kv$ . Thus the assertions 1) and 2) of Fact 2.16 follow from the discussion above, whereas the more difficult assertion 3) follows from JOSSEN [Jo], Theorem 1 applied to the abelian  $R$ -scheme  $\mathcal{A}_R$  over  $\text{Spec } R$  (recalling that  $R$  is a  $\mathbb{Z}$ -algebra of finite type). Namely, the set  $\Sigma_{x_R}$  of all  $s \in \text{Spec } R$  with  $o_{x_s}$  divisible by  $\ell^e$  is Zariski dense. Hence the set  $\Sigma_x$  of all the  $v$  which have center in  $\Sigma_{x_R}$  is Zariski dense in  $\text{Val}_k$ , etc.

This completes the proof of Fact 2.16.  $\square$

We now define the quasi arithmetical  $\widehat{U}$ -lattice  $\mathcal{L}_K^0 \subseteq \mathcal{L}_K \subset \widehat{K}$  mentioned in subsection A). Recall that a  $\mathbb{Z}_\ell$ -submodule  $\Delta \subset \widehat{K}$  is said to have **finite co-rank**, if there exists some geometric set of prime divisors  $D$  such that  $\Delta \subseteq \widehat{U}_D$ . Since the family of geometric sets for  $K|k$  is closed under intersecion (and union), it follows that the set of all the finite corank  $\mathbb{Z}_\ell$ -submodules  $\Delta \subset \widehat{K}$  is filtered w.r.t. inclusion, and that their union  $\widehat{K}_{\text{fin}}$  is given by:

$$\widehat{K}_{\text{fin}} = \bigcup_D \widehat{U}_D \subset \widehat{K}, \quad D \text{ geometric.}$$

Clearly,  $\widehat{K}_{\text{fin}}$  is a birational invariant of  $K|k$ , and if  $\mathcal{L}'_K \subset \widehat{K}$  is any divisorial  $\widehat{U}_K$ -lattice, then  $\mathcal{L}'_K \subset \widehat{K}_{\text{fin}}$  and  $\widehat{K}_{\text{fin}} = \mathcal{L}'_K \otimes \mathbb{Z}_\ell$ . Further, if  $L|l$  is another function field over an algebraically closed field  $l$ , then every embedding  $L|l \hookrightarrow K|k$  induces an embedding  $\widehat{L}_{\text{fin}} \hookrightarrow \widehat{K}_{\text{fin}}$ .

**Remarks 2.17.** In the context/notations from subsection B) above and Proposition 2.14, let  $J_K : K^\times \rightarrow \widehat{K}$ ,  $J_{K\mathfrak{v}} : K\mathfrak{v}^\times \rightarrow \widehat{K\mathfrak{v}}$  be the  $\ell$ -adic completion homomorphisms. Let  $\mathcal{L}'_K \subset \widehat{K}$ ,  $\mathcal{L}'_{K\mathfrak{v}} \subset \widehat{K\mathfrak{v}}$  be fixed divisorial lattices, and  $\mathcal{L}_K \subset \widehat{K}$ ,  $\mathcal{L}_{K\mathfrak{v}} \subset \widehat{K\mathfrak{v}}$  be the canonical ones. Then there exist unique  $\epsilon, \epsilon_{\mathfrak{v}} \in \mathbb{Z}_\ell^\times / \mathbb{Z}_\ell^\times$  such that  $\mathcal{L}'_K = \epsilon \cdot \mathcal{L}_K$  and  $\mathcal{L}'_{K\mathfrak{v}} = \epsilon_{\mathfrak{v}} \cdot \mathcal{L}_{K\mathfrak{v}}$ .

1) For every  $\eta \in \mathbb{Z}_\ell^\times$  the following conditions are equivalent:<sup>2</sup>

- i)  $\eta \cdot J_K(K^\times)_{(\ell)} \subset \mathcal{L}'_K$ .
- i)'  $\eta \cdot J_K(K^\times)_{(\ell)} \cap \mathcal{L}'_K$  is non-trivial.
- ii)  $\eta \cdot \mathcal{L}_K = \mathcal{L}'_K$
- ii)'  $\eta \cdot \mathcal{L}_K \cap \mathcal{L}'_K$  has infinite  $\mathbb{Z}_\ell$ -rank.

Actually,  $\eta := \epsilon$  is the unique  $\eta \in \mathbb{Z}_\ell^\times / \mathbb{Z}_\ell^\times$  satisfying the above equivalent conditions.

<sup>2</sup>Recall that for every abelian group  $A$  we denote  $A_{(\ell)} := A \otimes \mathbb{Z}_\ell$ .

- 2) Correspondingly, the same is true about  $\mathcal{L}'_{K\mathfrak{v}}$ , and  $\eta_v := \epsilon_v$  is the unique  $\eta_v \in \mathbb{Z}_\ell^\times / \mathbb{Z}^\times_{(\ell)}$  such that  $J_{K\mathfrak{v}}(K\mathfrak{v}^\times) \subseteq \epsilon_v \cdot \mathcal{L}'_{K\mathfrak{v}}$ , etc.
- 3) Since  $J_{\mathfrak{v}}(U_{\mathfrak{v}} \cap J_K(K^\times)) = J_{K\mathfrak{v}}(K\mathfrak{v}^\times)$ , it follows that  $\epsilon/\epsilon_v \in \mathbb{Z}_\ell^\times / \mathbb{Z}^\times_{(\ell)}$  is unique such that  $J_{\mathfrak{v}}(\mathcal{L}'_K)$  and  $\epsilon/\epsilon_v \cdot \mathcal{L}'_{K\mathfrak{v}}$  are equal modulo  $J_{\mathfrak{v}}(\widehat{U}_K) \cdot \widehat{U}_{K\mathfrak{v}} \subset \widehat{K\mathfrak{v}}_{\text{fin}}$ .
- 4) Conclude that for every  $\theta_v \in \mathbb{Z}_\ell^\times / \mathbb{Z}^\times_{(\ell)}$  the following conditions are equivalent:
  - j)  $\theta_v \cdot J_{K\mathfrak{v}}(K\mathfrak{v}^\times)_{(\ell)} \subseteq \widehat{J}_{\mathfrak{v}}(\mathcal{L}'_K)$ .
  - jj)  $\theta_v \cdot J_{K\mathfrak{v}}(K\mathfrak{v}^\times)_{(\ell)} \cap \widehat{J}_{\mathfrak{v}}(\mathcal{L}'_K)$  is non-trivial.
 Moreover,  $\theta_v := \epsilon/\epsilon_v$  is the unique  $\theta_v \in \mathbb{Z}_\ell^\times / \mathbb{Z}^\times_{(\ell)}$  satisfying the conditions j), jj).
- 5) For every finite corank  $\mathbb{Z}_\ell$ -submodule  $\Delta \subset \widehat{K}_{\text{fin}}$  with  $\widehat{U}_K \subset \Delta$ , one has: There exists a Zariski open non-empty subset  $\mathcal{U}_\Delta \subset \text{Val}_k$  such that for every  $v \in \mathcal{U}_\Delta$  there exist “many” quasi prime divisors  $\mathfrak{v}$  with  $\mathfrak{v}|k = v$  and the following are satisfied:
  - a)  $\Delta \subset \widehat{U}_{\mathfrak{v}}$ , and  $J_{\mathfrak{v}}$  is injective on  $\Delta$  and maps  $\Delta$  into  $\widehat{K\mathfrak{v}}_{\text{fin}}$ .
  - b) Thus for  $\epsilon, \epsilon_v$  from 1), 2) above, one has:  $\epsilon \cdot \Delta \subset \mathcal{L}'_K$  iff  $\epsilon_v \cdot J_{\mathfrak{v}}(\Delta) \subset \mathcal{L}'_{K\mathfrak{v}}$ .

**Definition 2.18.** Let  $\mathcal{L}_K$  be the canonical divisorial  $\widehat{U}_K$ -lattice for  $K|k$ . For every submodule of finite corank  $\Delta \subset \mathcal{L}_K$ , let  $\mathcal{D}_\Delta$  be the set of all the quasi prime divisors  $\mathfrak{v}$  satisfying conditions 5a), 5b) above and having  $k\mathfrak{v}$  an algebraic closure of a finite field. We set

$$\Delta(\mathfrak{v}) := \widehat{U}_K \cdot \{x \in \Delta \mid \widehat{J}_{\mathfrak{v}}(x) \in J_{K\mathfrak{v}}(K\mathfrak{v}^\times)_{(\ell)}\}, \quad \Delta^0 := \bigcap_{\mathfrak{v} \in \mathcal{D}_\Delta} \Delta(\mathfrak{v}),$$

and call  $\mathcal{L}_K^0 := \bigcup_{\Delta} \Delta^0$  the quasi arithmetical  $\widehat{U}_K$ -lattice of  $\mathcal{G}_{\mathcal{D}_K}$ .

**Proposition 2.19.** *In the above notations, suppose that  $\text{td}(K|k) > 2$ . Let  $L|l$  be a further function field with  $l$  algebraically closed, and  $\Phi : \Pi_K \rightarrow \Pi_L$  be an isomorphism which maps  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \subset \mathcal{G}_{\mathcal{Q}_K^{\text{tot}}}$  isomorphically onto  $\mathcal{G}_{\mathcal{D}_L^{\text{tot}}} \subset \mathcal{G}_{\mathcal{Q}_L^{\text{tot}}}$ . Further let  $\widehat{U}_K \subset \mathcal{L}_K$  and  $\widehat{U}_L \subset \mathcal{L}_L$  be the corresponding canonical divisorial lattices, and  $\mathcal{L}_K^0 \subset \mathcal{L}_K$  and  $\mathcal{L}_L^0 \subset \mathcal{L}_L$  be as defined above. Then there exist a unique  $\epsilon \in \mathbb{Z}_\ell^\times / \mathbb{Z}^\times_{(\ell)}$  such the following hold:*

- 1) *The Kummer isomorphism  $\phi : \widehat{L} \rightarrow \widehat{K}$  of  $\Phi$  maps  $\mathcal{L}_L^0$  isomorphically onto  $\epsilon \cdot \mathcal{L}_K^0$ .*
- 2)  *$\widehat{U}_K \cdot J_K(K^\times) \subseteq \mathcal{L}_K^0$  and  $\mathcal{L}_K^0 / (\widehat{U}_K \cdot J_K(K^\times))$  is a torsion  $\mathbb{Z}_{(\ell)}$ -module.*

*Proof.* To 1): Since  $\text{td}(K|k) > 2$ , it follows that for all quasi prime divisors  $\mathfrak{v}$  of  $K|k$ , one has  $\text{td}(K\mathfrak{v}|k\mathfrak{v}) > 1$ . Further, the total decomposition graph  $\mathcal{G}_{\mathcal{D}_{K\mathfrak{v}}^{\text{tot}}}$  of  $K\mathfrak{v}|k\mathfrak{v}$  can be recovered from  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  via the canonical projection  $Z_{\mathfrak{v}} \rightarrow \Pi_{K\mathfrak{v}}$ . Moreover, if  $\mathfrak{w}$  is the quasi prime divisor of  $L|l$  corresponding to  $\mathfrak{v}$ , then the isomorphism  $\Phi : \mathcal{G}_{\mathcal{Q}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{Q}_L^{\text{tot}}}$  give rise to a residual isomorphism of total quasi decomposition graphs of the residue fields  $\Phi_{\mathfrak{v}} : \mathcal{G}_{\mathcal{Q}_{K\mathfrak{v}}^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{Q}_{L\mathfrak{w}}^{\text{tot}}}$ . Now suppose that  $k\mathfrak{v}$  is an algebraic closure of a finite field of characteristic  $\neq \ell$ . Then by Theorems 2.2 and 2.1 of POP [P4], it follows that there exists an  $\ell$ -adic unit  $\epsilon_v \in \mathbb{Z}_\ell^\times$  such that  $\epsilon_v \cdot \Phi_{\mathfrak{v}}$  is defined by an isomorphism of the pure inseparable closures  $\iota_{\mathfrak{v}} : L\mathfrak{w}^i \rightarrow K\mathfrak{v}^i$  of  $L\mathfrak{w}$  and  $K\mathfrak{v}$  respectively. In particular, the Kummer isomorphism  $\epsilon_v^{-1} \cdot \widehat{\phi}_{\mathfrak{v}} : \widehat{L\mathfrak{w}} \rightarrow \widehat{K\mathfrak{v}}$  of  $\epsilon_v \cdot \Phi$  is nothing but the  $\ell$ -adic completion  $\widehat{\iota}_{\mathfrak{v}}$  of  $\iota_{\mathfrak{v}}$ , i.e.,  $\widehat{\phi}_{\mathfrak{v}} = \epsilon_v \cdot \widehat{\iota}_{\mathfrak{v}}$ .

Since by hypothesis one has that  $\Phi$  maps  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  isomorphically onto  $\mathcal{G}_{\mathcal{D}_L^{\text{tot}}}$ , it follows by the discussion above in Subsection B), 5) and 6), that replacing  $\Phi$  by some multiple  $\epsilon^{-1} \cdot \Phi$  with  $\epsilon \in \mathbb{Z}_\ell^\times$ , *mutatis mutandis*, we can suppose that the Kummer isomorphism  $\widehat{\phi} : \widehat{L} \rightarrow \widehat{K}$  of  $\Phi$  maps  $\mathcal{L}_L$  isomorphically onto  $\mathcal{L}_K$ . In particular, since  $\mathcal{L}_L$  is the unique divisorial lattice for

$L|l$  which intersects  $J_L(L^\times)_{(\ell)}$  non-trivially, it follows that  $\mathcal{L}_K$  is the unique divisorial lattice for  $K|k$  which intersects  $\hat{\phi}(J_L(L^\times)_{(\ell)})$  non-trivially. Now we claim that the above  $\epsilon_{\mathfrak{v}}$  lies actually in  $\mathbb{Z}_{(\ell)}$ . Indeed, by Remarks 2.17 above we have:  $\mathcal{L}_L$  is the unique divisorial lattice for  $L|l$  such that  $J_{\mathfrak{w}}(\mathcal{L}_L)$  intersects  $J_{L\mathfrak{w}}(L\mathfrak{w}^\times)$  non-trivially, and  $\mathcal{L}_K$  is the unique divisorial lattice for  $K|k$  such that  $J_{\mathfrak{v}}(\mathcal{L}_K)$  intersects  $J_{K\mathfrak{v}}(K\mathfrak{v}^\times)_{(\ell)}$  non-trivially. On the other hand, since  $\hat{\phi}_{\mathfrak{v}} \circ \hat{j}_{\mathfrak{w}} = \hat{j}_{\mathfrak{v}} \circ \hat{\phi}$ , and  $\hat{\phi}(\mathcal{L}_L) = \mathcal{L}_K$ , and  $\hat{\phi}_{\mathfrak{v}}(J_{L\mathfrak{w}}(L\mathfrak{w}^\times)_{(\ell)}) = \epsilon_{\mathfrak{v}} \cdot J_{K\mathfrak{v}}(K\mathfrak{v}^\times)_{(\ell)}$ , one has:

- i)  $J_{K\mathfrak{v}}(K\mathfrak{v}^\times)_{(\ell)} \subseteq \hat{j}_{\mathfrak{v}}(\mathcal{L}_K)$
- ii)  $\epsilon_{\mathfrak{v}} \cdot J_{K\mathfrak{v}}(K\mathfrak{v}^\times)_{(\ell)} = \hat{\phi}_{\mathfrak{v}}(J_{L\mathfrak{w}}(L\mathfrak{w}^\times)_{(\ell)}) \subseteq \hat{\phi}_{\mathfrak{v}}(\hat{j}_{\mathfrak{w}}(\mathcal{L}_L)) = \hat{j}_{\mathfrak{v}}(\mathcal{L}_K)$

In particular, by Remarks 2.17, 4), j), jj), above, it follows that  $\epsilon_{\mathfrak{v}} \in \mathbb{Z}_{(\ell)}$ , as claimed. Thus without loss of generality, we can and will suppose that  $\epsilon_{\mathfrak{v}} = 1$  for all  $\mathfrak{v}$ . Finally, let  $\Xi \subset \mathcal{L}_L$  be a finite corank submodule, and  $\Delta = \hat{\phi}(\Xi)$  be its image under  $\hat{\phi}$ . Then in the notations from Definition 2.18, it follows that if  $\mathfrak{v}$  and  $\mathfrak{w}$  are quasi-prime divisors of  $K|k$ , respectively  $L|l$  which correspond to each other under  $\Phi$ , then  $\mathfrak{v} \in \mathcal{D}_\Delta$  iff  $\mathfrak{w} \in \mathcal{D}_\Xi$ . Hence  $\hat{\phi}$  maps  $\Xi(\mathfrak{w})$  isomorphically onto  $\Delta(\mathfrak{v})$ , thus  $\hat{\phi}$  maps  $\Xi^0 := \bigcap_{\mathfrak{w} \in \mathcal{D}_\Xi} \Xi(\mathfrak{w})$  isomorphically onto  $\Delta^0 := \bigcap_{\mathfrak{v} \in \mathcal{D}_\Delta} \Delta(\mathfrak{v})$ . We conclude that  $\hat{\phi}$  maps  $\mathcal{L}_L^0 := \bigcup_{\Xi} \Xi^0$  isomorphically onto  $\mathcal{L}_K^0 := \bigcup_{\Delta} \Delta^0$ , as claimed.

To 2): First, the inclusion  $\widehat{U}_K \cdot J_K(K^\times) \subseteq \mathcal{L}_K^0$  is clear, because  $\hat{j}_{\mathfrak{v}}(J_K(K^\times)) \subseteq J_{K\mathfrak{v}}(K\mathfrak{v}^\times)$  for all  $\mathfrak{v}$ , and therefore  $\hat{j}_{\mathfrak{v}}(J_K(K^\times)_{(\ell)}) \subseteq J_{K\mathfrak{v}}(K\mathfrak{v}^\times)_{(\ell)}$  as well. The torsion assertion is much more involved. Let  $\mathcal{G}_{D_K}$  be a complete regular like decomposition graph, and let  $D_K$  be the 1-vertices of  $\mathcal{G}_{D_K}$ . Then  $D_K$  is complete regular like, and recalling the facts from subsection B), consider the corresponding canonical exact sequence:

$$1 \rightarrow \widehat{U}_K \rightarrow \mathcal{L}_K \xrightarrow{J} \text{Div}(D_K)_{(\ell)} \xrightarrow{\text{can}} \widehat{\mathcal{C}}\mathfrak{l}(D_K) \rightarrow 0.$$

The fact that  $\mathcal{L}_K^0 / (\widehat{U}_K \cdot J_K(K^\times))$  is a torsion  $\mathbb{Z}_{(\ell)}$ -module is equivalent to the fact that for every  $\mathfrak{x} \in \mathcal{L}_K^0 \subset \mathcal{L}_K$ , there exists some positive integer  $n > 0$  such that  $\text{div}(\mathfrak{x}^n) \in \text{div}(K^\times)$ , thus principal. Let  $\mathfrak{x} \in \mathcal{L}_K^0$  be a fixed element throughout.

- *By contradiction*, suppose that  $\text{div}(\mathfrak{x}^n)$  is not a principal divisor for all  $n > 0$ .

Let  $D' \subseteq D_K$  be any geometric set of prime divisors for  $K|k$  such that  $\mathfrak{x} \in \widehat{U}_{D'}$ . Then  $\mathfrak{x} \in \widehat{U}_D$  for every geometric set  $D \subseteq D'$ . Hence considering a projective normal variety  $X_0$  with  $D_K \subset D_{X_0}$ , we can choose a closed subset  $S_0 \subset X_0$  such that  $D := D_{X_0 \setminus S_0} \subseteq D'$ , and  $X_0 \setminus S_0$  is smooth. Recalling the context from Fact/Notations 2.12, we consider  $X \rightarrow X_0$ , and the preimage  $S \subset X$  of  $S_0$  under  $X \rightarrow X_0$ , one has:  $D_X$  is a complete regular like set of prime divisors for  $K|k$  (because it contains  $D_K$  which is so). Further, Since  $X_0 \setminus S_0$  is smooth, one has  $\Pi_{1, X_0 \setminus S_0} = \Pi_1(X_0 \setminus S_0)$ , and by Proposition 2.3, it follows that all the canonical projections below are isomorphisms:

$$\Pi_{1, D} = \Pi_{1, D_{X_0 \setminus S_0}} \rightarrow \Pi_{1, D_{X \setminus S}} \rightarrow \Pi_1(X \setminus S) \rightarrow \Pi_1(X_0 \setminus S_0).$$

Therefore, the corresponding  $\ell$ -adic duals are isomorphic as well, hence  $\widehat{U}_D = \widehat{U}_{X \setminus S}$  inside  $\widehat{K}$ . Further, since  $\mathcal{L}_K$  is a birational invariant of  $K|k$ , considering the canonical exact sequence

$$1 \rightarrow \widehat{U}_K \rightarrow \mathcal{L}_K \xrightarrow{J} \text{Div}(X)_{(\ell)} \xrightarrow{\text{can}} \widehat{\mathcal{C}}\mathfrak{l}(X) \rightarrow 0,$$

one has that  $\mathbf{x} \in \mathcal{L}_K^0 \subset \mathcal{L}_K$  satisfies:  $\mathbf{x} \in \widehat{U}_D = \widehat{U}_{X \setminus S}$ , and therefore  $\mathbf{x} \in \mathcal{L}_K^0 \cap \widehat{U}_{X \setminus S}$  inside  $\widehat{K}$ . Further,  $\text{div}(\mathbf{x}^n)$  is not a principal divisor for any positive integer  $n > 0$ .

Next recall that  $\mathfrak{Cl}^0(X) \subset \mathfrak{Cl}'(X)$  are the maximal divisible, respectively  $\ell$ -divisible, subgroups of  $\mathfrak{Cl}(X)$ , and that  $\text{Div}^0(X) \subseteq \text{Div}'(X)$  are their preimages in  $\text{Div}(X)$ , respectively. Then by subsection B), 2), one has that  $\text{div}(\mathcal{L}_K) = \text{Div}'(X)_{(\ell)}$ . Therefore,  $\text{div}(\mathbf{x}) \in \text{Div}'(X)_{(\ell)}$ , thus its divisor class  $[\mathbf{x}]$  satisfies  $[\mathbf{x}] \in \mathfrak{Cl}'(X)$ . And since  $\text{div}(\mathbf{x}^n)$  is not principal for all  $n > 0$ , it follows that  $[\mathbf{x}] \in \mathfrak{Cl}'(X)$  has infinite order, thus the same is true for every multiple  $m[\mathbf{x}]$ ,  $m \neq 0$ . On the other hand, by the structure of  $\mathfrak{Cl}(X)$ , see e.g. POP [P3], Appendix, it follows that  $\mathfrak{Cl}'(X)/\mathfrak{Cl}^0(X)$  is a prime-to- $\ell$  torsion group. Thus we conclude that some multiple  $m[\mathbf{x}]$  lies in  $\mathfrak{Cl}^0(X)$  and has infinite order. *Mutatis mutandis*, we can replace  $\mathbf{x} \in \mathcal{L}_K^0$  by its power  $\mathbf{x}^m \in \mathcal{L}_K^0$ , thus without loss we can suppose that  $[\mathbf{x}] \in \mathfrak{Cl}^0(X)$  has infinite order.

Now let  $\mathcal{U} \subset \text{Val}_k$  be the Zariski open subset introduced at Fact/Definitions factdefinitions, and consider  $\mathcal{U}_D^{\max} := \{v \in \mathcal{U}_D \mid kv = \overline{\mathbb{F}}_q \text{ for some prime } q \neq \ell\}$ . Then by general valuation theoretical non-sense, it follows that  $\mathcal{U}_D^{\max}$  is Zariski dense in  $\text{Val}_k$ .

Step 1. We claim that there exists a Zariski dense subset  $\Sigma_x$  of  $\text{Val}_k$  with  $\Sigma_x \subset \mathcal{U}_D^{\max}$  such that  $\text{sp}_v[\mathbf{x}] \in \mathfrak{Cl}(\mathcal{X}_v)$  has order divisible by  $\ell$ . Indeed, in order to do so, recall the finite generically normal alteration  $Z \rightarrow X$  of  $X$  from Fact/Notations 2.12. Since  $Z$  is a projective smooth  $k$ -variety, the connected component  $\text{Pic}_Z^0$  of  $\text{Pic}_Z$  is an abelian  $k$ -variety. Further, the image  $[\mathbf{y}] = \mathcal{I}([\mathbf{x}])$  of  $[\mathbf{x}]$  under  $\mathcal{I} : \mathfrak{Cl}(X) \rightarrow \mathfrak{Cl}(Z)$  satisfies: First,  $[\mathbf{y}]$  lies in the divisible part  $\mathfrak{Cl}^0(Z)$  of  $\mathfrak{Cl}(Z)$ , because  $[\mathbf{x}]$  lies in the divisible part  $\mathfrak{Cl}^0(X)$  of  $\mathfrak{Cl}^0(X)$ , and second,  $[\mathbf{y}]$  has infinite order by Fact 2.15. On the other hand, the divisible part  $\mathfrak{Cl}^0(Z)$  is nothing but the  $k$ -rational points of the abelian variety  $\text{Pic}_Z^0$ . Thus by Fact 2.16 above, it follows that for any given positive integer  $e > 0$ , there exists  $\Sigma_x \subset \mathcal{U}_D^{\max}$  which is dense in  $\text{Val}_k$  such that all  $v \in \Sigma_x$  satisfy: The  $v$ -specialization  $\text{sp}_v([\mathbf{y}]) \in \text{Pic}_{Z_v}^0(kv) = \mathfrak{Cl}^0(Z_v)$  has order divisible by  $\ell^e$ . On the other hand, by Fact 2.15, 4), one has that the order of  $\text{sp}_v([\mathbf{x}])$  is divisible by the order of  $\text{sp}_v([\mathbf{y}])$ , thus by  $\ell^e$ . We thus conclude that  $o_{\text{sp}_v([\mathbf{x}])}$  is divisible by  $\ell$ , as claimed.

Step 2. Recalling that  $\widehat{U}_{D_{X \setminus S}} = \widehat{U}_D$  and  $\mathbf{x} \in \mathcal{L}_K^0 \cap \widehat{U}_D$ , set  $\Delta := \widehat{U}_{D_{X \setminus S}}$  and keep in mind the definition of  $\mathcal{D}_\Delta$ . We claim that for every  $v \in \mathcal{U}_D^{\max}$  there exist some  $\mathfrak{v} \in \mathcal{D}_\Delta$  such that  $v = \mathfrak{v}|_K$  (or in other words,  $\mathcal{U}_D^{\max}$  is contained in the restriction of  $\mathcal{D}_\Delta$  to  $K$ ). Indeed, for every  $v \in \mathcal{U}_D^{\max}$ , consider the general hyperplane sections  $\mathcal{X}_H := H \cap \mathcal{X}_v$  and  $\mathcal{S}_H := H \cap \mathcal{S}_v$ , thus  $\mathcal{S}_H \subset \mathcal{X}_H$  is a closed subset. Then  $\mathcal{X}_H$  is a projective normal variety over  $kv$ , and  $\mathcal{X}_H \subset \mathcal{X}_v$  is a Weil prime divisor, say with valuation  $v_H$ . Then setting  $D_H := D_{\mathcal{X}_H \setminus \mathcal{S}_H}$ , and  $\mathfrak{v} := v_H \circ v_K$ , by Proposition 2.14, 2), 3), it follows that  $\mathfrak{v} \in \mathcal{D}_\Delta$ .

Step 3. Consider  $\mathfrak{v} = v_H \circ v_K$  with  $v \in \mathcal{U}_D^{\max}$  and  $v_H$  as above. Then by Proposition 2.14 it follows that  $\widehat{U}_K \subseteq \widehat{U}_{D_{X \setminus S}} \subseteq \widehat{U}_{v_K}$ , and  $\hat{j}_{v_K}$  maps  $\widehat{U}_K \subseteq \widehat{U}_{D_{X \setminus S}}$  bijectively onto  $\widehat{U}_{Kv_K} \subseteq \widehat{U}_{D_{\mathcal{X}_v \setminus \mathcal{S}_v}}$ , and  $\widehat{U}_K \subseteq \widehat{U}_{D_{X \setminus S}} \subseteq \widehat{U}_{\mathfrak{v}}$ , and  $\hat{j}_{\mathfrak{v}}$  maps  $\widehat{U}_K \subseteq \widehat{U}_{D_{X \setminus S}}$  bijectively onto  $\widehat{U}_{K\mathfrak{v}} \subseteq \widehat{U}_{D_{\mathcal{X}_H \setminus \mathcal{S}_H}}$ . And setting  $\mathbf{x}_H := \hat{j}_{\mathfrak{v}}(\mathbf{x}) = \hat{j}_{v_H}(\hat{j}_{v_K}(\mathbf{x}))$ , it follows that  $\mathbf{x}_H \in \widehat{U}_{K\mathfrak{v}} \cdot J_{K\mathfrak{v}}(K\mathfrak{v}^\times)$  by the fact that  $\mathbf{x} \in \Delta^0 \subseteq \Delta(\mathfrak{v})$ . Thus in particular,  $\mathbf{x}_H$  has a trivial divisor class  $[\mathbf{x}_H] = 0$ .

On the other hand, if  $v \in \Sigma_x$ , then the divisor class of  $[\mathbf{x}_v]$  in  $\mathfrak{Cl}(\mathcal{X}_v)$  has finite order  $o_{v,\mathbf{x}}$  divisible by  $\ell$ . Let  $\text{div}_v(\mathbf{x}_v) = \sum_i n_i P_i$  with distinct Weil prime divisors  $P_i$  of  $\mathcal{X}_v$ . Since  $\mathcal{X}_H := H \cap \mathcal{X}_v$  is a general hyperplane section, it follows that  $Q_i := H \cap P_i$  are distinct prime

Weil divisors of  $\mathcal{X}_H$ , and the divisor class of  $[\mathbf{x}_H] := [\sum_i n_i Q_i]$  in  $\mathfrak{Cl}(\mathcal{X}_H)$  has order equal to the order of  $[\mathbf{x}]$ . This contradicts the fact proved above that  $[\mathbf{x}_H] = 0$ .  $\square$

#### D) Recovering the rational quotients

We recall briefly the notion of an abstract quotient of a decomposition graph, see POP [P3], Sections 4, and 5 for details.

First, let  $\mathcal{G}_\alpha$  be the pair consisting of a pro- $\ell$  abelian group  $G_\alpha$  endowed with a system of pro-cyclic subgroups  $(T_{v_\alpha})_{v_\alpha}$  which makes  $G_\alpha$  curve like of genus  $g = 0$ , i.e., there exists a system of generators  $(\tau_{v_\alpha})_\alpha$  of the  $(T_{v_\alpha})_\alpha$  such that the following are satisfied:

- i)  $\prod_\alpha \tau_{v_\alpha} = 1$ , and this is the only pro-relation satisfied by  $(\tau_{v_\alpha})_\alpha$ .
- ii)  $G_\alpha$  is topologically generated by  $(\tau_{v_\alpha})_\alpha$ .

We fix a distinguished system of generators  $\mathfrak{T} := (\tau_{v_\alpha})_{v_\alpha}$  for  $(T_{v_\alpha})_{v_\alpha}$ , and notice that if  $\mathfrak{T}' := (\tau'_{v_\alpha})_{v_\alpha}$  is another distinguished systems of generators, then there exists a unique  $\ell$ -adic unit  $\epsilon \in \mathbb{Z}_\ell^\times$  such that  $\mathfrak{T}' = \mathfrak{T}^\epsilon$ . We let  $\widehat{\mathcal{L}}_{\mathcal{G}_\alpha} := \text{Hom}_{\text{cont}}(G_\alpha, \mathbb{Z}_\ell)$  be the  $\ell$ -adic dual of  $G_\alpha$ , and  $\mathcal{L}_{\mathfrak{T}} \subset \widehat{\mathcal{L}}_{\mathcal{G}_\alpha}$  be the  $\mathbb{Z}_{(\ell)}$ -submodule of all the functionals  $\varphi : G_\alpha \rightarrow \mathbb{Z}_\ell$  satisfying  $\varphi(\tau_{v_\alpha}) \in \mathbb{Z}_{(\ell)}$  for all  $v_\alpha$ . Notice that if  $\mathfrak{T}' = \mathfrak{T}^\epsilon$  as above, then  $\epsilon \cdot \mathcal{L}_{\mathfrak{T}'} = \mathcal{L}_{\mathfrak{T}}$ , thus the  $\ell$ -adic equivalence class of the  $\mathbb{Z}_{(\ell)}$ -modules of the form  $\mathcal{L}_{\mathfrak{T}}$  depends on  $\mathcal{G}_\alpha$  only, and not on the distinguished system of generators. We will denote such lattices by  $\mathcal{L}_{\mathcal{G}_\alpha}$  and call them the divisorial lattices of  $\mathcal{G}_\alpha$ .

A morphism  $\Phi_\alpha : \mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_\alpha$  is every open group homomorphism  $\Phi_\alpha : \Pi_K \rightarrow G_\alpha$  which for every prime divisor  $v$  of  $K|k$  satisfies –inductively on  $\text{td}(K|k)$ – the following:

- a) There exists  $v_\alpha$  such that  $\Phi_\alpha(T_v) \subseteq T_{v_\alpha}$ .
- b) If  $\Phi_\alpha(T_v) = 1$ , then the induced homomorphism  $\Phi_{\alpha,v} : \Pi_{Kv} \rightarrow G$  has open image and defines a morphism of the residual (total) decomposition graph  $\Phi_{\alpha,v} : \mathcal{G}_{\mathcal{D}_{Kv}^{\text{tot}}} \rightarrow \mathcal{G}_\alpha$ .

Notice that taking  $\ell$ -adic duals,  $\Phi_\alpha$  gives rise to a Kummer homomorphism  $\widehat{\phi}_\alpha : \widehat{\mathcal{L}}_{\mathcal{G}_\alpha} \rightarrow \widehat{K}$ .

A morphism  $\Phi_\alpha : \mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_\alpha$  is called divisorial, if for each complete regular like decomposition graph  $\mathcal{D}_K$  for  $K|k$  one has  $\widehat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha}) \subset \epsilon \cdot \mathcal{L}_{\mathcal{D}_K}$  for some  $\ell$ -adic unit  $\epsilon \in \mathbb{Z}_\ell^\times$ . (Notice that it is enough to have the inclusion  $\widehat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha}) \subset \epsilon \cdot \mathcal{L}_{\mathcal{D}_K}$  for a single complete regular like decomposition graph  $\mathcal{D}_K$  for  $K|k$  in order to have it for all.)

**Definition 2.20.** We say that  $\Phi_\alpha : \mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_\alpha$  is an abstract rational quotient of  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ , if  $\Phi_\alpha$  is surjective and divisorial, and for every complete regular like decomposition graph  $\mathcal{D}_K$  for  $K|k$  the following conditions are satisfied:

- i) For all vertices  $\mathfrak{v}$  of  $\mathcal{D}_K$ , denoting by  $U_{\mathfrak{v}} \subset K^\times$  the  $\mathfrak{v}$ -units and by  $j_{\mathfrak{v}} : \widehat{U}_{\mathfrak{v}} \rightarrow \widehat{K}\mathfrak{v}$  the  $\ell$ -adic completion of the  $\mathfrak{v}$ -reduction homomorphism  $U_{\mathfrak{v}} \rightarrow K\mathfrak{v}^\times$ , one has: If  $j_{\mathfrak{v}}$  is non-trivial on  $\widehat{\mathcal{L}}_\alpha \cap \widehat{U}_{\mathfrak{v}}$ , then  $\widehat{\mathcal{L}}_\alpha \subset \widehat{U}_{\mathfrak{v}}$  and  $j_{\mathfrak{v}}$  is injective on  $\widehat{\mathcal{L}}_\alpha$ .
- ii) For every finite  $\mathbb{Z}_\ell$ -module  $\Delta \subset \mathcal{L}_{\mathcal{D}_K} \otimes \mathbb{Z}_\ell$  with  $\widehat{U}_{\mathcal{D}_K} \subseteq \Delta$ , there exist 1-vertices  $v$  of  $\mathcal{D}_K$  such that  $\Delta \subset \widehat{U}_v$  and the  $\ell$ -adic completion  $j^v : \widehat{K} \rightarrow \mathbb{Z}_\ell$  of  $v : K^\times \rightarrow \mathbb{Z}$  and  $j_v : \widehat{U}_v \rightarrow \widehat{K}v$  satisfy:  $\Delta \cap \ker(j_v) = \Delta \cap \widehat{\mathcal{L}}_\alpha$  and  $j^v(\widehat{\mathcal{L}}_\alpha) \neq 0$ .

We recall that by POP [P3], Proposition 40, one has: If  $x \in K$  is a generic element, i.e.,  $\kappa_x = k(x)$  is relatively algebraically closed in  $K$ , then the canonical projection of Galois groups  $\Phi_{\kappa_x} : \Pi_K \rightarrow \Pi_{\kappa_x}$  defines an abstract rational quotient  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\kappa_x}$  in the sense above.

We also notice that the canonical divisorial lattice for  $\mathcal{G}_{\kappa_x}$  is nothing but  $\mathcal{L}_{\kappa_x} := J_{\kappa_x}(\kappa_x^\times) \subset \widehat{\kappa}_x$ , and the Kummer homomorphism  $\hat{\phi}_{\kappa_x} : \widehat{\kappa}_x \rightarrow \widehat{K}$  maps  $\mathcal{L}_{\kappa_x}$  isomorphically onto  $J_K(\kappa_x^\times)$ .

Using Proposition 2.19 above, we are now in the position to recover the geometric rational quotients of  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  in the case  $\text{td}(K|k) > 2$  as follows:

**Proposition 2.21.** *Let  $k$  is an arbitrary algebraically closed field, and  $\text{td}(K|k) > 2$ . Then in the notations from above and of Proposition 2.19 the following hold:*

- 1) Let  $\Phi_\alpha : \mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_\alpha$  be an abstract rational quotient. Then the following are equivalent:
  - i)  $\Phi_\alpha$  is geometric, i.e., there exists a generic element  $x \in K$  and an isomorphism of decomposition graphs  $\Phi_{\alpha, \kappa_x} : \mathcal{G}_\alpha \rightarrow \mathcal{G}_{\kappa_x}$  such that  $\Phi_{\kappa_x} = \Phi_{\alpha, \kappa_x} \circ \Phi_\alpha$ .
  - ii) There exists  $\epsilon \in \mathbb{Z}_\ell^\times$  such that  $\hat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha}) \subset \epsilon \cdot \mathcal{L}_K^0$ .

2) Let  $L|l$  be a function field over an algebraically closed field  $l$ , and  $\Phi : \mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{H}_{\mathcal{D}_L^{\text{tot}}}$  be an abstract isomorphism of decomposition graphs. Then  $\Phi$  is compatible with geometric rational quotients in the sense that if  $\Phi_{\kappa_y} : \mathcal{H}_{\mathcal{D}_L^{\text{tot}}} \rightarrow \mathcal{G}_{\kappa_y}$  is a geometric rational quotient of  $\mathcal{H}_{\mathcal{D}_L^{\text{tot}}}$ , then  $\Phi_\alpha := \Phi_{\kappa_y} \circ \Phi$  is a geometric rational quotient of  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ .

*Proof.* To 1): The implication i)  $\Rightarrow$  ii) is clear by the characterization of the geometric rational quotients and Proposition 2.19, 2), precisely the assertion that  $J_K(K^\times) \subset \mathcal{L}_K^0$ . For the implication ii)  $\Rightarrow$  i) we proceed as follows: Let  $\mathcal{L}_{\mathcal{G}_\alpha}$  be a divisorial lattice for  $\mathcal{G}_\alpha$  satisfying  $\mathcal{L}_\alpha := \hat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha}) \subset \epsilon \cdot \mathcal{L}_K^0$ . Then replacing  $\mathcal{L}_{\mathcal{G}_\alpha}$  by its  $\epsilon^{-1}$  multiple, without loss we can and will suppose that  $\mathcal{L}_\alpha = \hat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha}) \subset \mathcal{L}_K^0$ , thus  $\mathcal{L}_\alpha \subset \mathcal{L}_K^0 \subseteq \mathcal{L}_K$ . Recall that by POP [P3], Fact 32, (1),  $\mathcal{L}_\alpha$  is  $\mathbb{Z}_{(\ell)}$ -saturated in  $\mathcal{L}_K$ , i.e.,  $\mathcal{L}_K/\mathcal{L}_\alpha$  is  $\mathbb{Z}_{(\ell)}$ -torsion free. Further, if  $u \in K \setminus k$ , and  $\kappa_u \subset K$  is the corresponding relatively algebraically closed subfield, then  $J_K(\kappa_u^\times)_{(\ell)}$  is  $\mathbb{Z}_{(\ell)}$ -saturated in  $\mathcal{L}_K$  as well, i.e.,  $\mathcal{L}_K/J_K(\kappa_u^\times)_{(\ell)}$  is  $\mathbb{Z}_{(\ell)}$ -torsion free. Finally, by Proposition 2.19, 2), for every  $u \in \mathcal{L}_K^0$ , there exists a multiple  $u^n$  such that  $u^n \in \widehat{U}_K \cdot J_K(K^\times)$ .

Let  $u \in \mathcal{L}_\alpha \subset \mathcal{L}_K^0$  be a non-trivial element, and  $n_u > 0$  be the minimal positive integer with  $u^{n_u} \in \widehat{U}_K \cdot J_K(K^\times)$ , say  $u^{n_u} = \theta \cdot J_K(u)$  for some  $u \in K^\times$  and  $\theta \in \widehat{U}_{\mathcal{D}_K}$ , and notice that  $J_K(u)$  and  $\theta$  with the property above are unique, because  $\widehat{U}_{\mathcal{G}} \cap \mathcal{L}_\alpha$  is trivial, by POP [P3], Fact 32. Further,  $n_u$  is relatively prime to  $\ell$ , because  $\mathcal{L}_K/J_K(\kappa_u^\times)_{(\ell)}$  has no non-trivial  $\mathbb{Z}_{(\ell)}$ -torsion. As in the proof of Proposition 5.3 from POP [P4], one concludes that actually  $J_K(\kappa_u^\times)_{(\ell)} = \mathcal{L}_\alpha$ . Note that in the proofs of the Claims 1, 2, 3, of loc.cit, and the conclusion of the proof of Proposition 5.3 of loc.cit., it was nowhere used that  $k$  is an algebraic closure of a finite field.

This concludes the proof of assertion 1).

To 2): Let  $\Phi_{\kappa_y} : \mathcal{H}_{\mathcal{D}_L^{\text{tot}}} \rightarrow \mathcal{G}_{\kappa_y}$  be a geometric rational quotient of  $\mathcal{H}_{\mathcal{D}_L^{\text{tot}}}$ . Then as in the proof of assertion 2) of Proposition 5.3 of POP [P4], it follows that

$$\Phi_\alpha = \Phi_{\kappa_y} \circ \Phi : \mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\kappa_y} =: \mathcal{G}_\alpha$$

is an abstract rational quotient of  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  with Kummer homomorphism  $\hat{\phi}_\alpha = \hat{\phi} \circ \hat{\phi}_{\kappa_y} : \widehat{\kappa}_y \rightarrow \widehat{K}$ . We show that  $\Phi_\alpha$  satisfies hypothesis ii) from Proposition 2.21, 1). Indeed,  $\mathcal{G}_\alpha := \mathcal{G}_{\kappa_y}$  has  $\mathcal{L}_{\mathcal{G}_\alpha} := \mathcal{L}_{\kappa_y} = J_{\kappa_y}(\kappa_y^\times)$  as canonical divisorial lattice, and since  $\Phi_{\kappa_y} : \mathcal{H}_{\mathcal{D}_L^{\text{tot}}} \rightarrow \mathcal{G}_{\kappa_y}$  is defined by the  $k$ -embedding  $\kappa_y \hookrightarrow L$ , one has by definitions that  $\hat{\phi}_{\kappa_y}(\mathcal{L}_{\kappa_y}) = J_L(\kappa_y^\times)_{(\ell)} \subseteq J_L(L^\times)_{(\ell)}$ . Thus finally,  $\hat{\phi}_{\kappa_y}(\mathcal{L}_{\kappa_y}) \subseteq J_L(L^\times) \subseteq \mathcal{L}_L^0$  by Proposition 2.19, 2) applied to  $L|l$ . On the other

hand, by Proposition 2.19, 1), one has that  $\hat{\phi}(\mathcal{L}_L^0) = \epsilon \cdot \mathcal{L}_K^0$  for some  $\epsilon \in \mathbb{Z}_\ell^\times$ , and therefore:

$$\hat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha}) = \hat{\phi}(\hat{\phi}_{\kappa_y}(\mathcal{L}_{\kappa_y})) \subset \hat{\phi}(\mathcal{L}_L^0) = \epsilon \cdot \mathcal{L}_K^0.$$

Hence by applying assertion 1) we have:  $\Phi_\alpha$  is a geometric quotient of  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ , etc.  $\square$

### E) Concluding the proof of Theorem 2.1

First, by Proposition 2.21 above it follows that in the context of Theorem 2.1 one can recover the geometric rational quotients of  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  from  $\mathcal{G}_{\mathcal{Q}_K^{\text{tot}}}$  endowed with  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ , and that the recipe to do so is invariant under isomorphisms  $\Phi : \mathcal{G}_{\mathcal{Q}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{Q}_L^{\text{tot}}}$  which map  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  onto  $\mathcal{G}_{\mathcal{D}_L^{\text{tot}}}$ . Conclude by applying POP [P3], Main Result, Introduction.

## 3. PROOF OF THEOREM 1.2

### A) Recovering the nature of $k$

In this sub-section we prove the first assertion of Theorem 1.2. Precisely, for each non-negative integer  $\delta \geq 0$ , we will give inductively on  $\delta$  a group theoretical recipe  $\mathfrak{dim}(\delta)$  in terms of  $\mathcal{G}_{\mathcal{Q}_K^{\text{tot}}}$ , thus in terms of  $\Pi_K^c$ , such that  $\mathfrak{dim}(\delta)$  is true if and only if

$$(*) \quad \text{td}(K|k) > \dim(k) \quad \text{and} \quad \dim(k) = \delta.$$

To simplify language, recovering the above information about  $k$  will be called “recovering the nature of  $k$ .” Note that  $\text{char}(k)$  is not part of “nature of  $k$ ” in the above sense. Moreover,  $\mathfrak{dim}(\delta)$  will be invariant under isomorphisms as follows: If  $L|l$  is a further function field over an algebraically closed field  $l$ , and  $\mathcal{G}_{\mathcal{Q}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{Q}_L^{\text{tot}}}$  is an isomorphism, then  $\mathfrak{dim}(\delta)$  holds for  $K|k$  if and only if  $\mathfrak{dim}(\delta)$  holds for  $L|l$ .

Before giving the recipes  $\mathfrak{dim}(\delta)$ , let us recall the following few facts about generalized quasi prime divisors, in particular, the facts from POP [P4], [P6], Section 4, and TOPAZ [To2].

First, letting  $\mathcal{Q}(K|k)$  be the set of all the quasi prime divisors of  $K|k$ , let  $\mathcal{T}^1(K) \subset \Pi_K$  the topological closure of  $\cup_{\mathfrak{v} \in \mathcal{Q}(K|k)} T_{\mathfrak{v}}^1$  in  $\Pi_K$ . Further, for  $l \subseteq k$  an algebraically closed subfield, let  $\mathcal{Q}_l(K|k)$  be the set of all the quasi prime divisors  $\mathfrak{v}$  with  $\mathfrak{v}|_l = \mathfrak{w}|_l$ , and  $\mathcal{T}_l^1(K) \subset \Pi_K$  be the topological closure of  $\cup_{\mathfrak{v} \in \mathcal{Q}_l(K|k)} T_{\mathfrak{v}}^1$ . Then by [P6], Theorem A, one has:  $\mathcal{T}_l^1(K) \subseteq \mathcal{T}^1(K)$  consist of minimized inertia elements, and  $\mathcal{T}_l(K)$  consists of minimized inertia elements at valuations  $v$  with  $v|_l = \mathfrak{w}|_l$ .

Second, for every generalized quasi prime  $r$ -divisor  $\mathfrak{w}$ , recall the canonical exact sequence

$$1 \rightarrow T_{\mathfrak{w}}^1 \rightarrow Z_{\mathfrak{w}}^1 \rightarrow Z_{\mathfrak{w}}^1/T_{\mathfrak{w}}^1 =: \Pi_{K\mathfrak{w}}^1 \rightarrow 1,$$

and recalling that for  $\mathfrak{w} \leq \mathfrak{v}$ , one has  $T_{\mathfrak{w}}^1 \subset T_{\mathfrak{v}}^1$ ,  $Z_{\mathfrak{w}}^1 \subset Z_{\mathfrak{v}}^1$ , we endow  $\Pi_{K\mathfrak{w}}^1 = Z_{\mathfrak{w}}^1/T_{\mathfrak{w}}^1$  with its minimized inertia/decomposition groups  $T_{\mathfrak{v}/\mathfrak{w}}^1 := T_{\mathfrak{v}}^1/T_{\mathfrak{w}}^1 \subset Z_{\mathfrak{v}}^1/T_{\mathfrak{w}}^1 =: Z_{\mathfrak{v}/\mathfrak{w}}^1 \subset \Pi_{K\mathfrak{w}}^1$ , which in turn, give rise in the usual way to the total quasi decomposition graph  $\mathcal{G}_{\mathcal{Q}_{K\mathfrak{w}}^{\text{tot}}}$  for  $K\mathfrak{w}|k\mathfrak{w}$ .<sup>3</sup>

Further, let  $\mathcal{T}_l(K\mathfrak{w}) \subseteq \mathcal{T}(K\mathfrak{w}) \subset \Pi_{K\mathfrak{w}}^1$  be the images of  $\mathcal{T}_l^1(K) \cap Z_{\mathfrak{w}}^1 \subseteq \mathcal{T}^1(K) \cap Z_{\mathfrak{w}}^1$  under the canonical projection  $Z_{\mathfrak{w}}^1 \rightarrow \Pi_{K\mathfrak{w}}^1$ . Then  $\mathcal{T}^1(\bullet)$  behaves functorially, in the sense that for  $\mathfrak{w} \leq \mathfrak{v}$ , one has: The image of  $\mathcal{T}^1(K\mathfrak{w}) \cap Z_{\mathfrak{v}/\mathfrak{w}}^1$  under  $Z_{\mathfrak{v}/\mathfrak{w}}^1 \rightarrow \Pi_{K\mathfrak{v}}^1$  equals  $\mathcal{T}^1(K\mathfrak{v})$ .

<sup>3</sup> Note that if  $\text{char}(K\mathfrak{w}) = \ell$ , the groups  $T_{\mathfrak{v}/\mathfrak{w}}^1 \subset Z_{\mathfrak{v}/\mathfrak{w}}^1 \subset \Pi_{K\mathfrak{w}}^1$  are not Galois groups over  $K\mathfrak{w}$ .

Finally, recall that a pro- $\ell$  abelian group  $G$  endowed with a system of procyclic subgroups  $(T_\alpha)_\alpha$  is called **complete curve like**, if there exists a system of generators  $(\tau_\alpha)_\alpha$  with  $\tau_\alpha \in T_\alpha$  such that letting  $T \subseteq G$  be the closed subgroup of  $G$  generated by  $(\tau_\alpha)_\alpha$ , the following hold:

- i)  $\prod_\alpha \tau_\alpha = 1$  and this is the only profinite relation satisfied by  $(\tau_\alpha)_\alpha$ .<sup>4</sup>
- ii) The quotient  $G/T$  is a finite  $\mathbb{Z}_\ell$ -module.

A case of special interest is that of quasi prime  $r$ -divisors  $\mathfrak{w}$ , with  $r = d - 1$ , where  $d = \text{td}(K|k)$ , thus  $\text{td}(K\mathfrak{w}|k\mathfrak{w}) = 1$ . Then  $\mathfrak{w}K/\mathfrak{w}k \cong \mathbb{Z}^{d-1}$  and the following hold: Let  $(t_2, \dots, t_d)$  be a system of elements of  $K$  such that  $(\mathfrak{w}t_2, \dots, \mathfrak{w}t_d)$  define a basis of  $\mathfrak{w}K/\mathfrak{w}k$ , and  $t_1 \in \mathcal{O}_\mathfrak{w}^\times$  be such that its residue  $\bar{t}_1 \in K\mathfrak{w}$  is a separable transcendence basis of  $K\mathfrak{w}|k\mathfrak{w}$ . Then  $(t_1, \dots, t_d)$  is a separable transcendence basis of  $K|k$ , and letting  $k_1 \subset K$  be the relative algebraic closure of  $k(t_2, \dots, t_d)$  in  $K$ , it follows that  $\text{td}(K|k_1) = 1$ , and  $k_1\mathfrak{w} = k\mathfrak{w}$ . Hence we are in the situation of Section 4 from [P6], Theorem 4.1, thus we have:

**Fact 3.1.** In the above notations, the following hold:

- I) for every non-trivial element  $\sigma \in \mathcal{T}^1(K\mathfrak{w})$  there exists a unique quasi prime divisor  $\mathfrak{v}_\sigma > \mathfrak{w}$  such that  $\sigma \in T_{\mathfrak{v}_\sigma}^1$ , thus the image of  $\sigma$  in  $\Pi_{K\mathfrak{w}}^1$  lies in  $T_{\mathfrak{v}/\mathfrak{w}}^1$ .
- II) Let  $(T_\alpha)_\alpha$  be a maximal system of distinct maximal cyclic subgroups of  $\Pi_{K\mathfrak{w}}^1$  satisfying one of the following conditions:
  - i)  $T_\alpha \subset \mathcal{T}^1(K\mathfrak{w})$  for each  $\alpha$ .
  - ii)  $T_\alpha \subset \mathcal{T}_l^1(K\mathfrak{w})$  for each  $\alpha$ .

Then  $\Pi_{K\mathfrak{w}}^1$  endowed with  $(T_\alpha)_\alpha$  is complete curve like if and only if:

- a)  $k\mathfrak{w}$  is an algebraic closure of a finite field, provided i) is satisfied.
- b)  $l\mathfrak{w} = k\mathfrak{w}$ , provided ii) is satisfied.

Moreover, if either i), or ii), is satisfied, then  $(T_\alpha)_\alpha$  is actually the set of the minimized inertia groups  $T_\alpha = T_{v_\alpha}^1$  at all the prime divisors  $v_\alpha := \mathfrak{v}_\alpha/\mathfrak{w}$  of  $K\mathfrak{w}|k\mathfrak{w}$ .

We are now prepared to give the recipes  $\mathfrak{dim}(\delta)$ .

- The recipe  $\mathfrak{dim}(0)$ :

We say that  $K|k$  satisfies  $\mathfrak{dim}(0)$  if and only if for every quasi prime divisor  $\mathfrak{w}$  of  $K|k$  with  $\text{td}(K\mathfrak{w}|k\mathfrak{w}) = \text{td}(K|k) - 1$ , one has:  $\Pi_{K\mathfrak{w}}^1$  endowed with any maximal system of maximal pro-cyclic subgroups  $(T_\alpha)_\alpha$  satisfying condition i) above is complete curve like.

- Given  $\delta > 0$ , and the recipes  $\mathfrak{dim}(0), \dots, \mathfrak{dim}(\delta - 1)$ , we define  $\mathfrak{dim}(\delta)$  as follows:

First, if  $\text{td}(K|k) \leq \delta$ , we say that  $\mathfrak{dim}(\delta)$  does not hold for  $K|k$ . For the remaining discussion, we suppose that  $\text{td}(K|k) > \delta$ , and proceed as follows:

For  $\delta' < \delta$ , let  $\mathcal{D}_{\delta'}(K)$  be the set of all the minimized quasi divisorial subgroups  $T_{\mathfrak{v}}^1 \subset Z_{\mathfrak{v}}^1$  of  $\Pi_K$  such that  $\Pi_{K\mathfrak{v}}^1$  endowed with the family  $(T_{\mathfrak{v}'/\mathfrak{v}}^1)_{\mathfrak{v}' < \mathfrak{v}}$ , does not satisfy  $\mathfrak{dim}(\delta')$  for any  $\delta' < \delta$ . We set  $\mathfrak{In}_{\delta}(K) := \cup_{\mathfrak{v} \in \mathcal{D}_{\delta}(K)} T_{\mathfrak{v}}^1 \subset \Pi_K$ , and notice that  $\mathfrak{In}_{\delta}(K)$  consists of minimized inertia elements (by its mere definition), and it is closed under taking powers. Hence by POP [P6], Introduction, Theorem A, the topological closure  $\overline{\mathfrak{In}}_{\delta}(K) \subset \Pi_K$  consists of minimized inertia elements in  $\Pi_K$ , and it is closed under taking powers, because  $\mathfrak{In}_{\delta}(K) \subset \Pi_K$  was so. For  $d := \text{td}(K|k)$  and every quasi  $(d - 1)$ -divisorial subgroup  $T_{\mathfrak{v}}^1 \subset Z_{\mathfrak{v}}^1$  of  $\Pi_K$ , we set

<sup>4</sup>This implies by definition that  $\tau_\alpha \rightarrow 1$  in  $G$ , thus every open subgroup of  $G$  contains almost all  $T_\alpha$ .

$\mathfrak{In}_{\mathfrak{v}} := \overline{\mathfrak{In}_{\delta}}(K) \cap Z_{\mathfrak{v}}^1$ , and note that  $\mathfrak{In}_{\mathfrak{v}}$  consists of minimized inertia elements of  $\Pi_K$  which are contained in  $Z_{\mathfrak{v}}^1$ , it is topologically closed, and closed under taking powers. Thus the image  $\pi_{\mathfrak{v}}(\overline{\mathfrak{In}_{\mathfrak{v}}}) \subset \Pi_{K\mathfrak{v}}^1$  under the canonical projection  $\pi_{\mathfrak{v}} : Z_{\mathfrak{v}}^1 \rightarrow Z_{\mathfrak{v}}^1/T_{\mathfrak{v}}^1 = \Pi_{K\mathfrak{v}}^1$  consists of minimized inertia elements, is topologically closed, and closed under taking powers. Finally, by Fact 3.1, I), above, it follows that for every  $\sigma \in \mathfrak{In}_{\mathfrak{v}}^1$  which has a nontrivial image under  $Z_{\mathfrak{v}}^1 \rightarrow \Pi_{K\mathfrak{v}}^1$ , there exists a unique minimal valuation  $\mathfrak{v}_{\sigma}$  such that  $\sigma$  is a minimized inertia element at  $\mathfrak{v}_{\sigma}$ , and in particular,  $\mathfrak{v}_{\sigma} > \mathfrak{v}$  by loc.cit. Hence  $T_{\mathfrak{v}}^1 \subset T_{\mathfrak{v}_{\sigma}}^1$ , and moreover, since  $\text{td}(K\mathfrak{v}|k\mathfrak{v}) = 1$ , it follows that  $\mathfrak{v}_{\sigma}$  is a quasi prime  $d$ -divisor of  $K|k$ . Thus we have:

**Fact 3.2.** Let  $\mathfrak{v}_i, i \in I_{\mathfrak{v}}$ , be the set of distinct quasi prime divisors of  $K|k$  which satisfy:  $\mathfrak{v}_i > \mathfrak{v}$  and  $T_{\mathfrak{v}_i}^1$  contains some  $\sigma \in \mathfrak{In}_{\mathfrak{v}}^1$  with  $\sigma \notin T_{\mathfrak{v}}^1$ . Then the following hold:

- i)  $\mathfrak{v}_i$  are quasi prime  $d$ -divisors of  $K|k$ .
- ii) Setting  $T_{\mathfrak{v}_i}^1 := T_{\mathfrak{v}_i/\mathfrak{v}}^1 \subset \Pi_{K\mathfrak{v}_i}^1$ , one has  $T_{\mathfrak{v}_i}^1 \cap T_{\mathfrak{v}_i'}^1 = \{1\}$  for  $\mathfrak{v}_i \neq \mathfrak{v}_i'$ .
- iii)  $\mathfrak{In}_{\mathfrak{v}}^1 \subset \cup_i T_{\mathfrak{v}_i}^1$ , thus the image of  $\mathfrak{In}_{\mathfrak{v}}^1$  under  $Z_{\mathfrak{v}}^1 \rightarrow \Pi_{K\mathfrak{v}}^1$  equals  $\cup_i T_{\mathfrak{v}_i}^1$ .

**Proposition 3.3.** *In the above notations suppose that  $d := \text{td}(K|k) > \delta$ . The one has:*

- 1) *Suppose  $\mathcal{D}_{\delta}(K)$  is empty. Then  $\dim(k) < \delta$  and  $\dim(k)$  is the maximal  $\delta_k$  such that there exists some quasi prime divisor  $\mathfrak{v}$  whose  $\Pi_{K\mathfrak{v}}^1$  endowed with  $(T_{\mathfrak{v}_i}^1)_{\mathfrak{v}_i}$  satisfies  $\mathfrak{dim}(\delta_k)$ . Further, a quasi prime divisor  $\mathfrak{v}$  of  $K|k$  is a prime divisor if and only if  $\Pi_{K\mathfrak{v}}^1$  endowed with its  $(T_{\mathfrak{v}_i}^1)_{\mathfrak{v}_i}$  satisfies  $\mathfrak{dim}(\delta_k)$ .*
- 2) *Suppose  $\mathcal{D}_{\delta}(K)$  is non-empty. Then  $\dim(k) \geq \delta$ , and in the notations from Fact 3.2 above one has:  $\dim(k) = \delta$  iff the quasi prime  $(d-1)$ -divisors  $\mathfrak{v}$  of  $K|k$  satisfy:*
  - i) *There exist  $\mathfrak{v}$  such that  $I_{\mathfrak{v}}^1$  is non-empty.*
  - ii) *If  $I_{\mathfrak{v}}^1$  is non-empty, then  $\Pi_{K\mathfrak{v}}^1$  endowed with  $(T_{\mathfrak{v}_i}^1)_{\mathfrak{v}_i}$  is curve like.*
- (\*) *If the above conditions are satisfied, a quasi prime divisor  $\mathfrak{v}$  of  $K|k$  is a prime divisor of  $K|k$  if and only if  $\Pi_{K\mathfrak{v}}^1$  endowed with  $(T_{\mathfrak{v}_i}^1)_{\mathfrak{v}_i}$  does not satisfy  $\mathfrak{dim}(\delta')$  for any  $\delta' < \delta$ .*

*Proof.* To 1): First let  $\dim(k) < \delta$ . Then for all quasi prime divisors  $\mathfrak{v}$  of  $K|k$  we have  $\dim(k) \geq \dim(k\mathfrak{v})$ , hence  $\text{td}(K|k) > \delta > \dim(k) \geq \dim(k\mathfrak{v})$ . Thus taking into account that  $\text{td}(K\mathfrak{v}|k\mathfrak{v}) = \text{td}(K|k) - 1 \geq \delta$ , we get:  $\text{td}(K\mathfrak{v}|k\mathfrak{v}) > \dim(k\mathfrak{v})$ , and  $\Pi_{K\mathfrak{v}}^1$  endowed with  $(T_{\mathfrak{v}_i}^1)_{\mathfrak{v}_i}$  satisfies the recipe  $\mathfrak{dim}(\delta')$  with  $\delta' = \dim(k\mathfrak{v}) < \delta$ . Thus  $\mathfrak{v} \notin \mathcal{D}_{\delta}(K)$ , etc. Second, suppose that  $\mathcal{D}_{\delta}(K)$  is empty. Equivalently,  $\Pi_{K\mathfrak{v}}^1$  endowed with  $(T_{\mathfrak{v}_i}^1)_{\mathfrak{v}_i}$  satisfies  $\mathfrak{dim}(\delta')$  for some  $\delta' < \delta$  for each quasi prime divisor  $\mathfrak{v}$ . Hence choosing  $\mathfrak{v}$  to be a prime divisor of  $K|k$  we get:  $k\mathfrak{v} = k$ , and  $K\mathfrak{v}|k$  satisfies  $\mathfrak{dim}(\delta')$  for some  $\delta' < \delta$ . Thus  $\dim(k) = \delta' < \delta$ . The description of the divisors  $\mathfrak{v}$  of  $K|k$  is clear, because  $\mathfrak{v}$  is a prime divisor of  $K|k$  iff  $\mathfrak{v}$  is trivial on  $k$  iff  $\dim(k\mathfrak{v}) = \dim(k)$ .

To 2): First suppose that  $\dim(k) = \delta$ . Recall that a quasi prime divisor  $\mathfrak{v}$  is a prime divisor iff  $\mathfrak{v}$  is trivial on  $k$  iff  $\dim(k) = \dim(k\mathfrak{v})$ . Therefore, if  $\mathfrak{v}$  is a prime divisor, then  $\mathfrak{dim}(\delta')$  is not satisfied by  $\Pi_{K\mathfrak{v}}^1$  endowed with  $(T_{\mathfrak{v}_i}^1)_{\mathfrak{v}_i}$  for any  $\delta' < \delta$ . Second, if  $\mathfrak{v}$  is not a prime divisor, then  $\mathfrak{v}$  is not trivial on  $k$ , and reasoning as above, one gets that  $\mathfrak{dim}(\delta')$  holds for  $\Pi_{K\mathfrak{v}}^1$  endowed with  $(T_{\mathfrak{v}_i}^1)_{\mathfrak{v}_i}$  for  $\delta' = \dim(k\mathfrak{v})$ . Thus finally it follows that  $\mathcal{D}_{\delta}(K)$  consists of exactly all the prime divisors of  $K|k$ . Hence by POP [P2], Introduction, Theorem B, it follows that  $\overline{\mathfrak{In}_{\delta}}(K)$  consists of all the tame inertia at all the  $k$ -valuations of  $K|k$ . Thus reasoning as in the proof of Proposition 3.5 of POP [P4], but using all the divisorial subgroups instead of the quasi divisorial subgroup, and  $\overline{\mathfrak{In}_{\delta}}(K)$  instead of  $\mathfrak{In.tn}(K)$  as in loc.cit., it follows that

the *flags of divisorial subgroups* –as introduced in Definition 3.4 of loc.cit.– can be recovered by a group theoretical recipe from  $\Pi_K$  endowed with  $\overline{\mathfrak{In}}_\delta(K)$ . Hence in the notations from Fact 3.2 above, one has the following: Let  $\mathfrak{v}$  be a quasi  $(d-1)$ -prime divisor of  $K|k$ .

- a) Suppose that  $\mathfrak{v}$  is trivial on  $k$ . Then by the discussion above,  $(T_{v_i})_{v_i}$  is exactly the set of all the divisorial inertia subgroups in  $\Pi_{K\mathfrak{v}}^1$ . By applying Fact 3.1 above, one concludes that  $\Pi_{K\mathfrak{v}}$  endowed with  $(T_{v_i})_{v_i}$  is curve like, etc.
- b) Suppose that  $\mathfrak{v}$  is non-trivial on  $k$ . We claim that the set  $I_{\mathfrak{v}}$  defined at Fact 3.2 above is empty. Indeed, since each non-trivial  $\sigma \in \overline{\mathfrak{In}}_\delta(K)$  is tame inertia element at some  $k$ -valuations of  $K|k$ , it follows that  $\mathfrak{v}_\sigma$  is a  $k$ -valuation. Thus one cannot have  $\mathfrak{v}_\sigma > \mathfrak{v}$ , i.e.,  $\mathfrak{v}_\sigma \notin I_{\mathfrak{v}}$ .

Now suppose that  $\dim(k) > \delta$ . We show that there exist prime  $(d-1)$ -divisors  $\mathfrak{v}$  such that  $I_{\mathfrak{v}}$  is not empty, but  $\Pi_{K\mathfrak{v}}^1$  endowed with  $(T_{v_i})_{v_i}$  is not curve like. Indeed, let  $l \subset k$  be an algebraically closed subfield with  $\text{td}(k|l) = 1$ . Then  $\dim(l) \geq \delta$ , and therefore, if  $\mathfrak{v}$  is a quasi divisor of  $K|k$  which is trivial on  $l$ , we have  $\dim(k\mathfrak{v}) \geq \dim(l) \geq \delta$ . Hence  $\mathfrak{dim}(\delta')$  does not hold for  $K\mathfrak{v}|k\mathfrak{v}$  for all  $\delta' < \delta$ . Therefore,  $\mathfrak{v} \in \mathcal{D}_\delta$ . We then can proceed as in the proof of Proposition 4.4 of POP [P4], but using all the quasi prime divisors of  $K|k$  which are trivial on  $l$  instead of using all the (maximal) quasi prime divisors of  $K|k$ . Namely let  $\mathfrak{In.tm}_l(K)$  be the set of the inertia elements at valuations which are trivial on  $l$ . Then by POP [P2], Introduction, Theorem A, the set  $\mathfrak{In.tm}_l(K)$  is closed in  $\Pi_K$ , and by loc.cit. Theorem B, the set  $\mathfrak{In.tm.q.div}_l(K)$  of tame inertia at the quasi divisors  $\mathfrak{v}$  which are trivial on  $l$  is dense  $\mathfrak{In.tm}_l(K)$ . On the other hand, by the discussion above, every quasi prime divisor  $\mathfrak{v}$  which is trivial on  $l$  lies in  $\mathcal{D}_\delta(K)$ . Hence we conclude that  $\mathfrak{In.tm.q.div}_l(K) \subseteq \overline{\mathfrak{In}}_\delta(K)$ , and so,  $\mathfrak{In.tm}_l(K)$  is contained in the closure  $\overline{\mathfrak{In}}_\delta(K)$ . In other words, if  $\mathfrak{v}$  is any prime  $(d-1)$ -divisor of  $K|k$ , and  $\pi_{\mathfrak{v}} : Z_{\mathfrak{v}} \rightarrow \Pi_{K\mathfrak{v}}$  is the canonical projection, it follows that  $\pi_{\mathfrak{v}}(\overline{\mathfrak{In}}_\delta(K))$  equals all the tame inertia at valuations which are trivial on  $l$ . Therefore, in the notations from Fact 3.2 above, the set  $I_{\mathfrak{v}}$  is non-empty, and  $(T_{v_i})_{v_i}$  consists of the inertia groups of all the quasi prime divisors of  $K\mathfrak{v}|k$  which are trivial on  $l$ . Since  $l \subset k$  strictly, by Fact 3.1 it follows that  $\Pi_{K\mathfrak{v}}$  endowed with all  $(T_{v_i})_{v_i}$  is not curve like.  $\square$

### B) Recovering the generalized prime divisors

In the context of Theorem 1.1 from Introduction, let  $K|k$  and  $L|l$  be function fields with  $k$  and  $l$  algebraically closed of characteristic  $\neq \ell$ . Then using Proposition 3.3, the fact that  $\dim(k) = \delta < \text{td}(K|k)$  is encoded in the total quasi decomposition graph  $\mathcal{G}_{\mathcal{Q}_K^{\text{tot}}}$ , and the recipe to do so is invariant under isomorphisms of total quasi decomposition graphs  $\Phi : \mathcal{G}_{\mathcal{Q}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{Q}_L^{\text{tot}}}$ . i.e., if such an isomorphism  $\Phi$  exists, then one has  $\text{td}(K|k) = \text{td}(L|l)$  and  $\dim(k) = \dim(l)$ . Further, this enables us to single out the divisorial subgroups  $T_v \subset Z_v$  among all the quasi divisorial subgroups  $T_{\mathfrak{v}} \subset Z_{\mathfrak{v}}$  of  $\Pi_K$ , as indicated in Proposition 3.3. Thus we get the following:

**Fact 3.4.** Let  $K|k$  be a function field with  $\text{td}(K|k) > \dim(k)$  with  $k$  algebraically closed of characteristic  $\neq \ell$ , and  $L|l$  a further function field with  $l$  algebraically closed. Let  $\mathfrak{In.tm.div}(K) \subset \Pi_K$  be the set of the divisorial inertia elements in  $\Pi_K$ . Then one has:

- 1) By POP [P2], Introduction, Theorem B, it follows that set of the tame inertia elements  $\mathfrak{In.tm}_k(K) \subset \Pi_K$  at all the valuations  $v$  which are trivial on  $k$  can be recovered from  $\Pi_K^c$  as being the topological closure of  $\mathfrak{In.tm.div}(K)$  in  $\Pi_K$ .

- 2) Let  $\Phi : \mathcal{G}_{\mathcal{Q}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{Q}_L^{\text{tot}}}$  be an isomorphism of total quasi decomposition graphs. Then  $\Phi$  maps the divisorial subgroups  $T_v \subseteq Z_v$  of  $\Pi_K$  bijectively onto the divisorial subgroups  $T_w \subseteq Z_w$  of  $\Pi_L$  of  $L|l$ , respectively  $\mathfrak{In.tm}_k(K) \subset \mathfrak{In.tm.Div}(K)$  homeomorphically onto  $\mathfrak{In.tm}_l(L) \subset \mathfrak{In.tm.Div}(L)$ .

**Definition 3.5.**

- 1) A flag of generalized prime divisors for  $K|k$  is any sequence  $\mathbf{v}_1 \leq \dots \leq \mathbf{v}_r$  such that each  $\mathbf{v}_m$  is a prime  $m$ -divisor of  $K|k$  for  $1 \leq m \leq r$ .
- 2) A flag of generalized divisorial subgroups of  $\Pi_K$ , is the sequence  $Z_{\mathbf{v}_1} \geq \dots \geq Z_{\mathbf{v}_r}$  of the decomposition groups of a flag of generalized prime divisors  $\mathbf{v}_1 \leq \dots \leq \mathbf{v}_r$  endowed with the corresponding sequence of inertia groups  $T_{\mathbf{v}_1} \leq \dots \leq T_{\mathbf{v}_r}$ .

We will next show that using Fact 3.4 above one can recover the flags of generalized divisorial subgroups in  $\Pi_K$  from  $\Pi_K^c$ . In particular, one can recover  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ .

**Proposition 3.6.** *Let  $K|k$  be a function field with  $\text{td}(K|k) > \dim(k)$  and  $k$  algebraically closed of characteristic  $\neq \ell$ , and  $\mathcal{G}_{\mathcal{Q}_K^{\text{tot}}}$  the total quasi decomposition group of  $K|k$ . Then in the usual notations, the following hold:*

- 1) A flag  $Z_1 \geq \dots \geq Z_r, T_1 \leq \dots \leq T_r$ , of generalized quasi divisorial subgroups is a flag of generalized divisorial subgroups if and only if  $T_r \subset \mathfrak{In.tm}_k(K)$ .
- 2) Let  $\mathbf{v}$  be a generalized prime divisor of  $K|k$ , and  $1 \rightarrow T_{\mathbf{v}} \rightarrow Z_{\mathbf{v}} \xrightarrow{\pi} \Pi_{K_{\mathbf{v}}} \rightarrow 1$  be its canonical exact sequence. Then the generalized divisorial subgroups of  $\Pi_{K_{\mathbf{v}}}$  are precisely the images  $\pi(T), \pi(Z)$  of the generalized divisorial subgroups  $T, Z$  in  $\Pi_K$  which satisfy  $Z \subseteq Z_{\mathbf{v}}$  and  $T \supseteq T_{\mathbf{v}}$ .
  - Hence  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  can be recovered from  $\mathcal{G}_{\mathcal{Q}_K^{\text{tot}}}$  as indicated above.
- 3) Let  $L|l$  be a further function field with  $l$  algebraically closed, and  $\Phi : \mathcal{G}_{\mathcal{Q}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{Q}_L^{\text{tot}}}$  be an isomorphism of totally quasi decomposition graphs. Then  $\Phi$  defines an isomorphism of total decomposition graphs  $\Phi : \mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{D}_L^{\text{tot}}}$ .

*Proof.* The proof is identical with the one of POP [P4], Proposition 3.5, but using  $\mathfrak{In.tm}_k(K)$  as given by Fact 3.4 above instead of the total tame inertia set  $\mathfrak{In.tm}(K)$ , as in loc.cit. Thus we will not repeat the argument here and omit the proof.  $\square$

C) *Concluding the proof of Theorem 1.2*

By Proposition 3.6 one can recover  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  from  $\Pi_K^c$ , and the recipe to do so is invariant under isomorphisms  $\Pi_K^c \rightarrow \Pi_L^c$ , i.e., under the hypotheses of Theorem 1.2, every isomorphism  $\Pi_K^c \rightarrow \Pi_L^c$  gives rise to an isomorphism  $\Phi : \mathcal{G}_{\mathcal{Q}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{Q}_L^{\text{tot}}}$  which maps  $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$  isomorphically onto  $\mathcal{G}_{\mathcal{D}_L^{\text{tot}}}$ . Conclude by applying Theorem 2.1.

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# Appendix:

## ON THE ORDER OF THE REDUCTION OF POINTS ON ABELIAN SCHEMES

PETER JOSSEN

We fix an integral scheme  $S$  of finite type over  $\mathrm{Spec}(\mathbb{Z})$ , and a prime number  $\ell$  different from the characteristic of the function field of  $S$ . Our goal is to establish the following

**Theorem 1.** *Let  $A$  be an abelian  $S$ -scheme, and  $P \in A(S)$  be a point of infinite order. Then the set of closed points  $s : \mathrm{Spec}(\kappa) \rightarrow S$  such that  $\ell$  divides the order of the image  $P_s \in A(\kappa)$  of  $P$  has positive Dirichlet density in  $S$ , thus it is Zariski dense as well.*

In the special case where  $S$  is an open subscheme of the ring of integers of a number field, this theorem was proven by PINK [Pk]. The proof rests on two classical theorems: These are the general version of the Mordell–Weil theorem, stating that  $A(S)$  is a finitely generated commutative group, and a generalization of Chebotarev’s density theorem.

**Remarks/Basic Facts.** Since  $\ell$  is different from the characteristic,  $S$  has an open dense subsets on which  $\ell$  is invertible. Thus replacing  $S$  by such an open dense subset, without loss of generality, we can suppose that  $\ell$  is invertible on  $S$ . We fix a geometric point  $\bar{\eta} : \mathrm{Spec}(\bar{k}) \rightarrow S$ , where  $\bar{k}$  is some algebraically closed field, and consider the étale fundamental group of  $S$  based at  $\bar{\eta}$

$$\pi_1 := \pi_1^{\acute{e}t}(S, \bar{\eta}).$$

- a) Given an abelian scheme  $A$  over  $S$  and any point  $P \in A(S)$ , we consider the complex of group schemes

$$M = [u : \mathbb{Z} \rightarrow A], \quad u(1) = P$$

over  $S$ , where  $\mathbb{Z}$  has to be viewed as a constant group scheme. The complex  $M$  is a 1-motive in the sense of Deligne.

- b) With  $M$  as above, one associates in a functorial way a finitely generated free  $\mathbb{Z}_\ell$ -module  $\mathbb{T}_\ell M$  with continuous  $\pi_1$ -action as follows: For  $i \geq 0$  consider the finite  $\mathbb{Z}/\ell^i\mathbb{Z}$ -module

$$M_{\ell^i}(\bar{k}) := \frac{\{(Q, n) \in A(\bar{k}) \times \mathbb{Z} \mid \ell^i Q = nP\}}{\{(nP, \ell^i n) \in A(\bar{k}) \times \mathbb{Z} \mid n \in \mathbb{Z}\}}$$

and put  $\mathbb{T}_\ell M = \varinjlim_i M_{\ell^i}(\bar{k})$ . Notice that because  $\ell$  is invertible on  $S$ , the group  $M_{\ell^i}(\bar{k})$  is actually the group of  $\bar{k}$ -points of a finite étale group scheme  $M_{\ell^i}$  of exponent  $\ell^i$  on  $S$ , as notation suggests.

- c) Hence we have a canonical continuous action of  $\pi_1$  on  $M_{\ell^i}(\bar{k})$  and on  $\mathbb{T}_\ell M$ , and notice that  $M_{\ell^i}(\bar{k})$  fits in a short exact sequence

$$0 \longrightarrow A(\bar{k})[\ell^i] \longrightarrow M_{\ell^i}(\bar{k}) \longrightarrow \mathbb{Z}/\ell^i\mathbb{Z} \longrightarrow 0$$

of  $\pi_1$ -modules. Passing to limits, we find a short exact sequence of  $\mathbb{Z}_\ell$ -modules with continuous  $\pi_1$ -action

$$0 \longrightarrow \mathbb{T}_\ell A \longrightarrow \mathbb{T}_\ell M \longrightarrow \mathbb{Z}_\ell \longrightarrow 0$$

where  $\mathbb{T}_\ell A$  is the usual  $\ell$ -adic Tate module of  $A$ , and where  $\pi_1$  acts trivially on  $\mathbb{Z}_\ell$ . In particular, it follows that  $\mathbb{T}_\ell M$  is free of rank  $2g + 1$  as a  $\mathbb{Z}_\ell$ -module, where  $g$  is the relative dimension of  $A$  over  $S$ .

- d) The operation of  $\pi_1$  on  $\mathbb{T}_\ell M$  is given by a continuous group homomorphism  $\pi_1 \rightarrow \mathrm{GL}(\mathbb{T}_\ell M)$ , and we can consider the  $\ell$ -adic Lie groups

$$L^M := \mathrm{im}(\pi_1 \rightarrow \mathrm{GL}(\mathbb{T}_\ell M)) \quad \text{and} \quad L^A := \mathrm{im}(\pi_1 \rightarrow \mathrm{GL}(\mathbb{T}_\ell A)).$$

By restriction we get a surjective homomorphism from  $L^M$  to  $L^A$ , whose kernel we denote by  $L_A^M$ , that is to say  $L_A^M = \{\sigma \in L^M \mid \sigma|_{\mathbb{T}_\ell A} = \mathrm{id}\}$ .

- e) Finally, we notice that if  $P \in A(S)$  is a torsion point, then the sequence of  $\pi_1$ -modules  $0 \rightarrow \mathbb{T}_\ell A \rightarrow \mathbb{T}_\ell M \rightarrow \mathbb{Z}_\ell \rightarrow 0$  splits after passing to  $\mathbb{Q}_\ell$ -coefficients, and it follows that the group  $L_A^M$  is trivial in that case. The point is that the converse holds as well:

**Proposition 2.** *If  $P \in A(S)$  has infinite order, then the group  $L_A^M$  is not trivial.*

*Proof.* For the proof of this proposition we need two lemmas below.

**Lemma 3.** *If  $P \in A(S)$  has infinite order, then the module of fixed points  $(\mathbb{T}_\ell M)^{\pi_1}$  is trivial.*

*Proof.* Elements of  $\mathbb{T}_\ell M$  can by construction be represented by sequences  $(P_i, n_i)_{i=0}^\infty$ , with  $P_i \in A(\bar{k})$  and  $n_i \in \mathbb{Z}$ , satisfying the relations

$$\ell^i P_i = n_i P \quad \text{and} \quad \ell P_i - P_{i-1} = m_i P \quad \text{and} \quad n_i - n_{i-1} = \ell^{i-1} m_i$$

for suitable  $m_i \in \mathbb{Z}$ . Two such sequences  $(P_i, n_i)_{i=0}^\infty$  and  $(P'_i, n'_i)_{i=0}^\infty$  represent the same element of  $\mathbb{T}_\ell M$  if there exists a sequence  $(m_i)_{i=0}^\infty$  in  $\mathbb{Z}$  such that

$$\ell^i m_i = n_i - n'_i \quad \text{and} \quad m_i P = P_i - P'_i$$

Let now  $x \in \mathbb{T}_\ell M$  be an element which is invariant under the action of  $\pi_1$ , and let  $(P_i, n_i)_{i=0}^\infty$  be a sequence representing  $x$ . Then,  $\sigma x$  is represented by the sequence  $(\sigma P_i, n_i)_{i=0}^\infty$  for all  $\sigma \in \pi_1$ . It follows that  $\sigma P_i = P_i$  for all  $i$  and all  $\sigma$ , hence that  $P_i \in A(S)$  for all  $i$ . By the theorem of Mordell–Weil, see e.g., LANG [La], the commutative group  $A(S)$  is finitely generated. Hence, assuming that  $P$  is not a torsion point, the relation  $\ell^i P_i = n_i P$  implies that the sequence  $(n_i)_{i=0}^\infty$  converges to zero in  $\mathbb{Z}_\ell$ . Replacing  $(P_i, n_i)_{i=0}^\infty$  by an equivalent sequence, we may thus assume that  $n_i = 0$  for all  $i \geq 0$ . Now we find  $P_0 = 0$  and  $\ell P_i = P_{i-1}$  for all  $i \geq 0$ . Again since  $A(S)$  is finitely generated, this can only happen if all the points  $P_i$  are torsion of order prime to  $\ell$ , and it follows that the sequence  $(P_i, 0)_{i=0}^\infty$  is equivalent to the zero sequence.  $\square$

**Lemma 4** (Serre). *The  $\ell$ -adic cohomology group  $H_{\mathrm{cont}}^1(L^A, \mathbb{T}_\ell A)$  is finite.*

*Proof.* Write  $V_\ell A := \mathbb{T}_\ell A \otimes \mathbb{Q}_\ell$ , and  $\mathfrak{t}^A \subseteq \mathrm{End}(V_\ell A)$  for the Lie algebra of  $L^A$ . The  $\mathbb{Z}_\ell$ -module  $H^1(L^A, \mathbb{T}_\ell A)$  is finitely generated by [Se2] Proposition 9, and the canonical map

$$H^1(L^A, \mathbb{T}_\ell A) \otimes \mathbb{Q}_\ell \longrightarrow H^1(L^A, V_\ell A)$$

is an isomorphism. It suffices thus to show that  $H^1(L^A, V_\ell A)$  is trivial. By Proposition 12 in *loc. cit.* it suffices also to show that the Lie cohomology  $H^1(\mathfrak{t}^A, V_\ell A)$  vanishes. Indeed, even  $H^i(\mathfrak{t}^A, V_\ell A)$  vanishes for all  $i \geq 0$ . For abelian varieties over number fields this was shown by SERRE in [Se3]. Serre’s proof works verbatim for abelian varieties over fields which are finitely generated over their prime field, provided  $\ell$  is prime to the characteristic.  $\square$

Coming back to the proof of Proposition 2, we proceed as follows: The short exact sequence  $0 \rightarrow \mathbb{T}_\ell A \rightarrow \mathbb{T}_\ell M \rightarrow \mathbb{Z}_\ell \rightarrow 0$  of continuous  $L^M$ -modules induces a long exact sequence of  $\mathbb{Z}_\ell$ -modules

$$0 \longrightarrow (\mathbb{T}_\ell A)^{L^M} \longrightarrow (\mathbb{T}_\ell M)^{L^M} \longrightarrow \mathbb{Z}_\ell \xrightarrow{\partial} H^1(L^M, \mathbb{T}_\ell A) \longrightarrow H^1(L^M, \mathbb{T}_\ell M) \longrightarrow \dots$$

The  $\mathbb{Z}_\ell$ -module  $(\mathbb{T}_\ell M)^{L^M}$  is trivial by Lemma 3, hence  $\partial$  is injective. In particular  $H^1(L^M, \mathbb{T}_\ell A)$  has rank  $\geq 1$  as a  $\mathbb{Z}_\ell$ -module. The kernel of the restriction map

$$H^1(L^M, \mathbb{T}_\ell A) \longrightarrow H^1(L_A^M, \mathbb{T}_\ell A)$$

is  $H^1(L^A, \mathbb{T}_\ell A)$ , hence finite by Lemma 4. Thus, also the  $\mathbb{Z}_\ell$ -module  $H^1(L_A^M, \mathbb{T}_\ell A)$  has rank  $\geq 1$ , and so  $L_A^M$  can not be trivial.  $\square$

Let  $s = \text{Spec } \kappa \rightarrow S$  be a closed point and let  $\bar{\kappa}$  be an algebraic closure of  $\kappa$ . The resulting geometric point  $\bar{s} : \text{Spec } \bar{\kappa} \rightarrow S$  induces an injection of  $\text{Gal}(\bar{\kappa}|\kappa)$  into  $\pi_1^{\text{ét}}(S, \bar{s})$ . The groups  $\pi_1 = \pi_1^{\text{ét}}(S, \bar{\eta})$  and  $\pi_1^{\text{ét}}(S, \bar{s})$  are isomorphic, by an isomorphism which is canonical up to inner automorphisms. We call *Frobenius element over  $S$*  every element  $F_s \in \pi_1$  which is the image under some composition

$$\text{Gal}(\bar{\kappa}|\kappa) \longrightarrow \pi_1^{\text{ét}}(S, \bar{s}) \xrightarrow{\cong} \pi_1$$

of the Frobenius element  $x \mapsto x^{\#\kappa}$  of  $\text{Gal}(\bar{\kappa}|\kappa)$ . So every element conjugated to a Frobenius element over  $s$  is again a Frobenius element over  $s$ . We shall make use of the following version of Chebotarev's Density Theorem:

**Theorem 5** (Artin–Chebotarev). *The set of all Frobenius elements is dense in  $\pi_1$ .*

This follows in principle from Theorem 7 in [Se1], but see rather HOLSCHBACH [Ho] for a complete proof (of this and other related assertions). The deduction of our Theorem 5 from SERRE's Theorem 7 in [Se1] goes as follows: Let  $\pi_1 \rightarrow G$  be a finite quotient of  $\pi_1$ , corresponding via the defining property of the fundamental group to a finite étale Galois cover  $X$  of  $S$ . Let  $R \subseteq G$  be any subset stable under conjugation. Then the Artin–Chebotarev Density Theorem for the scheme  $S$  states that the set of closed points in  $S$  whose Frobenius conjugacy class in  $G$  is contained in  $R$  has Dirichlet density  $|R|/|G|$ , and in particular it is Zariski dense in  $S$ . Moreover, every element of  $G$  lies in a Frobenius conjugacy class. This being true for all finite quotients of  $\pi_1$ , the statement of Theorem 5 follows.

**Lemma 6.** *Let  $s = \text{Spec } \kappa \rightarrow S$  be a closed point and let  $F_s \in \pi_1$  be a Frobenius element over  $s$ . The order of the image of  $P$  in  $A(\kappa)$  is prime to  $\ell$  if and only if the homomorphism  $(\mathbb{T}_\ell M)^{\langle F_s \rangle} \rightarrow \mathbb{Z}_\ell$  is surjective.*

*Proof.* The order of  $P$  in the finite group  $A(\kappa)$  is prime to  $\ell$  if and only if  $P$  is  $\ell$ -divisible in  $A(\kappa)$ . From the description of elements of  $\mathbb{T}_\ell M$  by sequences as in the proof of lemma 3 it follows that this is the case if and only if  $\mathbb{T}_\ell M$  contains an element fixed by  $F_s$  which is mapped to  $1 \in \mathbb{Z}_\ell$  by the canonical projection  $\mathbb{T}_\ell M \rightarrow \mathbb{Z}_\ell$ .  $\square$

We are done once we have shown that the set  $\Sigma_P := \{\sigma \in \pi_1 \mid (\mathbb{T}_\ell M)^{\langle \sigma \rangle} \rightarrow \mathbb{Z}_\ell \text{ is not surjective}\}$  contains a nonempty open subset of  $\pi_1$ , provided  $P \in A(S)$  has infinite order. We can also work with the image of  $\pi_1$  in  $\text{GL}(\mathbb{T}_\ell M)$  in place of  $\pi_1$ , which we denoted by  $L^M$ .

**Proposition 7.** *If  $P \in A(S)$  has infinite order,  $\Sigma_P$  contains a nonempty open subset of  $L^M$ .*

*Proof.* Let  $G$  be the subgroup of  $\text{GL}(\mathbb{T}_\ell M)$  consisting of those elements which leave invariant the subspace  $\mathbb{T}_\ell A$  of  $\mathbb{T}_\ell M$  and act trivially on the quotient  $(\mathbb{T}_\ell M/\mathbb{T}_\ell A) = \mathbb{Z}_\ell$ . Relative to an appropriate  $\mathbb{Z}_\ell$ -basis of  $\mathbb{T}_\ell M$ , the group  $G$  consists of matrices of the form

$$\left( \begin{array}{c|c} U & v \\ \hline 0 & 1 \end{array} \right) \quad U \in \text{GL}(2g, \mathbb{Z}_\ell), \quad v \in \mathbb{Z}_\ell^{2g}$$

where  $g$  is the relative dimension of  $A$  over  $S$ . The group  $L^M$  is a closed subgroup of  $G$ . By Proposition 2 there exists an element  $\sigma \in L_A^M$  which is not the identity. As a matrix, this element

$\sigma$  is of the form

$$\sigma = \left( \frac{\text{id}_{2g} \mid w}{0 \mid 1} \right) \quad w \in \mathbb{Z}^{2g}, w \neq 0$$

Let  $N \geq 0$  be an integer such that  $\ell^{-N} = |\ell^N|_\ell < \max_i |w_i|_\ell$ , where  $w_i$  are the coefficients of  $w$ . We consider the subset  $X$  of  $G$  consisting of the matrices of the form

$$\left( \frac{\text{id}_{2g} + \ell^N M \mid w + \ell^N v}{0 \mid 1} \right) \quad M \in \text{M}(2g \times 2g, \mathbb{Z}_\ell), \quad v \in \mathbb{Z}^{2g}$$

This set is open and closed in  $G$ , hence  $X \cap L^M$  is open and closed in  $L^M$ . Moreover, the intersection  $X \cap L^M$  is not empty because it contains  $\sigma$ . We are done if we show that  $X \cap L^M$  is contained in  $\Sigma$ , so let us show that for all  $x \in X$ , the map  $(\mathbb{T}_\ell M)^{\langle x \rangle} \rightarrow \mathbb{Z}_\ell$  is not surjective. Let  $t \in (\mathbb{T}_\ell M)^{\langle x \rangle}$ . We claim that the image of  $t$  in  $\mathbb{Z}_\ell$  lies in  $\ell \mathbb{Z}_\ell$ . Indeed, set

$$x = \left( \frac{\text{id}_{2g} + \ell^N M \mid w + \ell^N v}{0 \mid 1} \right) \quad t = \begin{pmatrix} t' \\ \lambda \end{pmatrix} \quad t' \in \mathbb{Z}_\ell^{2g}, \quad \lambda \in \mathbb{Z}_\ell$$

We assume that  $xt = x$ , hence

$$0 = (x - \text{id}_{2g+1})t = \begin{pmatrix} \ell^N(Mt' + \lambda v) + \lambda w \\ 0 \end{pmatrix}$$

However, the equality  $\ell^N(Mt' + \lambda v) + \lambda w = 0$  can only hold if we have  $|\lambda|_\ell < 1$ , because of our choice of  $N$ . The image of  $t$  in  $\mathbb{Z}_\ell$  is  $\lambda$ , and the last inequality shows  $\lambda \in \ell \mathbb{Z}_\ell$ .  $\square$

*Proof of Theorem 1.* We just have to put the pieces together: If  $P \in A(S)$  has infinite order, there exists by Proposition 7 and Theorem 5 a closed point  $s : \text{Spec}(\kappa) \rightarrow S$  and a Frobenius element  $F_s \in \pi_1$  over  $s$ , such that  $(\mathbb{T}_\ell M)^{\langle F_s \rangle} \rightarrow \mathbb{Z}_\ell$  is not surjective. By Lemma 6 that means that the order of the image of  $P$  in  $A(\kappa)$  is divisible by  $\ell$ , so we are done. In fact, the above arguments show that the set  $\Sigma_P$  is precisely the set of all the  $s \in S$  such that the specialization  $P_s$  of  $P$  at  $s$  has order divisible by  $\ell$ . On the other hand, with the notion of Dirichlet density (of closed points) as introduced in SERRE [Se1], it follows that  $\Sigma_P$  has positive Dirichlet density (which in principle can be explicitly given). This concludes the proof of Theorem 1.  $\square$

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