

RECONSTRUCTION OF FUNCTION FIELDS FROM THEIR PRO- ℓ ABELIAN DIVISORIAL INERTIA

— WITH APPLICATIONS TO BOGOMOLOV’S PROGRAM AND I/OM —

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ABSTRACT. Let $\Pi_K^c \rightarrow \Pi_K$ be the maximal pro- ℓ abelian-by-central, respectively abelian, Galois groups of a function field $K|k$ with k algebraically closed and $\text{char} \neq \ell$. We show that $K|k$ can be functorially reconstructed by group theoretical recipes from Π_K^c endowed with the set of divisorial inertia $\mathfrak{In}.\text{div}(K) \subset \Pi_K$. As applications, one gets: First, a group theoretical recipe to reconstruct $K|k$ from Π_K^c , provided either $\text{td}(K|k) > \dim(k) + 1$ or $\text{td}(K|k) > \dim(k) > 1$, where $\dim(k)$ is the Kronecker dimension; thus extending the only known result $\dim(k) = 0$, i.e., k is an algebraic closure of a finite field, see [B-T], [P4]. Second, an application to the *pro- ℓ abelian-by-central* I/OM (Ihara’s question / Oda-Matsumoto conjecture), which in the cases considered here turns out to imply the classical I/OM.

1. INTRODUCTION

We introduce notations which will be used throughout the manuscript as follows: For a fixed prime number ℓ , and function fields $K|k$ over algebraically closed base fields k , we denote by

$$\Pi_K^c := \text{Gal}(K^c|K) \rightarrow \text{Gal}(K'|K) =: \Pi_K$$

the Galois groups of the maximal **pro- ℓ abelian-by-central**, respectively **pro- ℓ abelian** extensions $K^c|K \leftarrow K'|K$ of K (in some fixed algebraic closure, which is not necessary to be explicitly specified). Notice that $\Pi_K^c \rightarrow \Pi_K$ viewed as a morphism of abstract profinite groups can be recovered by a *group theoretical recipe* from Π_K^c , because $\ker(\Pi_K^c \rightarrow \Pi_K) = [\Pi_K^c, \Pi_K^c]$.

Hypothesis. If not otherwise specified, we will also suppose that $\text{char}(k) \neq \ell$.

The present work concerns a program initiated by BOGOMOLOV [Bo] at the beginning of the 1990’s, which aims at giving group theoretical recipes to reconstruct $K|k$ from Π_K^c in a functorial way, provided $\text{td}(K|k) > 1$. Notice that in contrast to Grothendieck’s (birational) anabelian philosophy, where a rich arithmetical Galois action is part of the picture, in Bogomolov’s program there is no arithmetical action (because the base field k is algebraically closed). Note also that the hypothesis $\text{td}(K|k) > 1$ is necessary, because $\text{td}(K|k) = 1$ implies that the absolute Galois group G_K is profinite free on $|k|$ generators (as shown by

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DOUADY, HARBATER, POP), and therefore, Π_K^c encodes only the cardinality $|k|$ of the base field $|k|$, hence that of K as well.

A strategy to tackle Bogomolov's program was proposed by the author in 1999, and along the lines of that strategy, the Bogomolov program was completed for k an algebraic closure of a finite field by BOGOMOLOV–TSCHINKEL [B–T] and POP [P4].

In order to explain the results of this note, we recall a few basic facts as follows.

For a further function field $L|l$ with l algebraically closed, let $\text{Isom}^F(L, K)$ be the set of the isomorphisms of the pure inseparable closures $L^i \rightarrow K^i$ up to Frobenius twists. Further, let $\text{Isom}^c(\Pi_K, \Pi_L)$ be the set of the abelianizations $\Phi : \Pi_K \rightarrow \Pi_L$ of the isomorphisms $\Pi_K^c \rightarrow \Pi_L^c$ modulo multiplication by ℓ -adic units. Notice that given $\phi \in \text{Isom}^F(L, K)$, and any prolongation $\phi' : L' \rightarrow K'$ of ϕ to L' , one gets $\Phi_\phi \in \text{Isom}^c(\Pi_K, \Pi_L)$ defined by $\Phi_\phi(\sigma) := \phi'^{-1} \circ \sigma \circ \phi'$ for $\sigma \in \Pi_K$. Noticing that Φ_ϕ depends on ϕ only, and not on the specific prolongation ϕ' , one finally gets a canonical embedding

$$\text{Isom}^F(L, K) \rightarrow \text{Isom}^c(\Pi_K, \Pi_L), \quad \phi \mapsto \Phi_\phi.$$

Next recall that given a function field $K|k$ as above, a **prime divisor** of $K|k$ is any valuation v of $K|k$ which is trivial on k and its residue field Kv satisfies $\text{td}(K|k) = \text{td}(Kv|k) + 1$. Equivalently, a valuation v of K is a prime divisor of $K|k$ if and only if there exists a normal model X of $K|k$ such that v is a Weil prime divisor of X . For any valuation v of K we denote by $T_v \subseteq Z_v \subset \Pi_K$ the inertia/decomposition groups of some prolongation v to K' (and notice that $T_v \subset Z_v$ depend on v only, because Π_K is abelian). We denote by $\mathcal{D}_{K|k}$ the set of all the prime divisors of $K|k$, and consider the set of **divisorial inertia** in Π_K

$$\mathfrak{In}.div(K) := \cup_{v \in \mathcal{D}_{K|k}} T_v \subset \Pi_K.$$

Finally recall that a natural generalization of prime divisors are the **quasi prime divisors** of $K|k$. These are the valuations \mathfrak{v} of $K|k$ (not necessarily trivial on k), minimal among the valuations of K satisfying: i) $\text{td}(K|k) = \text{td}(K\mathfrak{v}|k\mathfrak{v}) + 1$; ii) $\mathfrak{v}K/\mathfrak{v}k \cong \mathbb{Z}$. [Here, by mere definition, the minimality of \mathfrak{v} means that if w is a valuation of K satisfying i), ii) and the valuation rings satisfy $\mathcal{O}_w \supseteq \mathcal{O}_{\mathfrak{v}}$, then $w = \mathfrak{v}$.] In particular, the prime divisors of $K|k$ are precisely the quasi prime divisors of $K|k$ that are trivial on k . For quasi prime divisors \mathfrak{v} , let $T_{\mathfrak{v}}^1 \subset Z_{\mathfrak{v}}^1 \subset \Pi_K$ be their **minimized inertia/decomposition groups**, see Section 2, A), and/or POP [P5] and TOPAZ [To2], for definitions. We denote by $\mathcal{Q}_{K|k}$ the set of quasi prime divisors of $K|k$, and recall that $T_{\mathfrak{v}}^1 = T_{\mathfrak{v}}$, $Z_{\mathfrak{v}}^1 = Z_{\mathfrak{v}}$ are the usual inertia/decomposition groups if $\text{char}(K\mathfrak{v}) \neq \ell$. By POP [P1] and TOPAZ [T1], there are (*concrete uniform*) *group theoretical recipes* such that given Π_K^c , one can recover the (minimized) quasi divisorial groups $T_{\mathfrak{v}}^1 \subset Z_{\mathfrak{v}}^1 \subset \Pi_K$, thus the set of (**minimized**) **quasi-divisorial inertia**

$$\mathfrak{In}.q.div^1(K) := \cup_{\mathfrak{v} \in \mathcal{Q}_{K|k}} T_{\mathfrak{v}}^1 \subset \Pi_K.$$

Moreover, the recipes to do so are invariant under isomorphisms, i.e., in the above notations, every $\Phi \in \text{Isom}^c(\Pi_K, \Pi_L)$ has the property that $\Phi(\mathfrak{In}.q.div^1(K)) = \mathfrak{In}.q.div^1(L)$.

Unfortunately, for the time being, in the case k is an arbitrary algebraically closed base field, one does not know group theoretical recipes neither to distinguish the divisorial subgroups among the quasi divisorial ones, nor to describe/recover $\mathfrak{In}.div(K) = \cup_v T_v$ inside $\mathfrak{In}.q.div^1(K) = \cup_{\mathfrak{v}} T_{\mathfrak{v}}^1$, using solely the group theoretical information encoded in Π_K^c . (Actually, the latter apparently weaker question turns out to be equivalent to the former one.)

The main result of this note reduces the Bogomolov program about reconstructing function fields $K|k$ with $\text{td}(K|k) > 2$ from the group theoretical information encoded in Π_K^c to giving group theoretical recipes such that given Π_K^c one can answer the following:

Recover $\mathfrak{In}.div(K) = \cup_v T_v$ inside $\mathfrak{In}.q.div^1(K) = \cup_v T_v^1$ using the pro- ℓ group Π_K^c .

The precise result reads as follows:

Theorem 1.1. *In the above notations, there exists a group theoretical recipe about pro- ℓ abelian-by-central groups such that the following hold:*

- 1) *The recipe reconstructs $K|k$ from Π_K^c endowed with $\mathfrak{In}.div(K) \subset \Pi_K$ in a functorial way, provided $\text{td}(K|k) > 2$.*
- 2) *The recipe is invariant under isomorphisms $\Phi \in \text{Isom}^c(\Pi_K, \Pi_L)$ in the sense that if $\Phi(\mathfrak{In}.div(K)) = \mathfrak{In}.div(L)$, then $\Phi = \Phi_\phi$ for a (unique) $\phi \in \text{Isom}^F(L, K)$.*

The proof of the above Theorem 1.1 is quite involved and relies heavily on “specialization” techniques, among other things, Appendix, Theorem 1 by JOSSEN (generalizing PINK [Pk], Theorem 2.8), and previous work by the author POP [P1], [P2], [P3], [P4]. Following referee’s advice, I added at the beginning of the next section, precisely in section 2), A) explanations about the strategy the logical structure of quite complex and involved proof.

Concerning applications of Theorem 1.1 above, the point is that under supplementary conditions on the base field k , one can recover $\mathfrak{In}.div(K)$ inside $\mathfrak{In}.q.div^1(K)$, thus Theorem 1.1 above is applicable. Namely, recall that the Kronecker dimension $\dim(k)$ of k is defined as follows: First, we set $\dim(\mathbb{F}_p) = 0$ and $\dim(\mathbb{Q}) = 1$, and second, if $k_0 \subset k$ is the prime field of k , we define $\dim(k) := \text{td}(k|k_0) + \dim(k_0)$. Hence $\dim(k) = 0$ if and only if k is an algebraic closure of a finite field, and $\dim(k) = 1$ if and only if k is an algebraic closure of a global field, etc. Therefore, $\text{td}(K|k) > 1$ is equivalent to $\text{td}(K|k) > \dim(k) + 1$ in the case k is an algebraic closure of a finite field, whereas $\text{td}(K|k) > \dim(k) + 1$ is equivalent to $\text{td}(K|k) > 2$ if k is an algebraic closure of a global field, etc.

In light of the above discussion and notations, one has the following generalization of the main results of BOGOMOLOV–TSCHINKEL [B–T], POP [P4]:

Theorem 1.2. *Let $K|k$ be an function field with $\text{td}(K|k) > 1$ and k algebraically closed of Kronecker dimension $\dim(k)$. The following hold:*

- 1) *For every nonnegative integer δ , there exists a group theoretical recipe $\mathfrak{dim}(\delta)$ depending on δ , which holds for Π_K^c if and only if $\dim(k) = \delta$ and $\text{td}(K|k) > \dim(k)$.*
- 2) *There is a group theoretical recipe which recovers $\mathfrak{In}.div(K) \subset \Pi_K$ from Π_K^c , provided $\text{td}(K|k) > \dim(k)$. Thus by **Theorem 1.1, 1)**, one can reconstruct $K|k$ functorially from Π_K^c , provided either $\text{td}(K|k) > \dim(k) + 1$, or $\text{td}(K|k) > \dim(k) > 1$.*
- 3) *Both recipes above are invariant under isomorphisms of profinite groups as follows: Suppose that Π_K^c and Π_L^c are isomorphic. Then $\text{td}(K|k) = \text{td}(L|l)$, and one has: $\dim(k) = \delta$ and $\text{td}(K|k) > \dim(k)$ if and only if $\dim(l) = \delta$ and $\text{td}(L|l) > \dim(l)$.*
- 4) *In particular, if either $\text{td}(K|k) > \dim(k) + 1$, or $\text{td}(K|k) > \dim(k) > 1$ holds, then by **Theorem 1.1, 2)**, one concludes that the canonical map below is a bijection:*

$$\text{Isom}^F(L, K) \longrightarrow \text{Isom}^c(\Pi_K, \Pi_L), \quad \phi \mapsto \Phi_\phi.$$

If $\dim(k) = 1$, i.e., k is an algebraic closure of a global field, our methods developed here work as well for the function fields $K = k(X)$ of projective smooth surfaces X with finite (étale) fundamental group, thus for function fields of “generic” surfaces.

An immediate consequence of Theorem 1.2 is a positive answer to a question by Ihara from the 1980’s, which in the 1990’s became a conjecture by Oda–Matsumoto, for short I/OM. The I/OM is about giving a topological/combinatorial description of the absolute Galois group of the rational numbers. See POP [P6], Introduction, for explanations concerning the I/OM. The situation we consider here is as follows: Let k_0 be an arbitrary perfect field, and $k := \bar{k}_0$ an algebraic closure. Let X be a geometrically integral k_0 -variety, $\mathcal{U}_X := \{U_i\}_i$ be a basis of open neighborhoods of the generic point η_X , and $\mathcal{U}_{\bar{X}} = \{\bar{U}_i\}_i$ its base change to k . Set $\Pi_{U_i}^c := \pi_1^c(\bar{U}_i)$ and $\Pi_{U_i} := \pi_1^{\ell, \text{ab}}(\bar{U}_i)$. Then letting $K := k(\bar{X})$ be the function field of \bar{X} , it follows that $\Pi_K^c \rightarrow \Pi_K$ is the projective limit of the system $\Pi_{U_i}^c \rightarrow \Pi_{U_i}$, and there exists a canonical embedding $\text{Aut}^c(\Pi_{\mathcal{U}_X}) \hookrightarrow \text{Aut}^c(\Pi_K)$. Finally let $\text{Aut}_k^F(K) \hookrightarrow \text{Aut}^F(K)$ be the group of all the k -automorphisms of K^i , respectively all the field automorphisms of K^i , modulo Frobenius twists. Note that since $k \subset K^i$ is the unique maximal algebraically closed subfield in K^i , every $\phi \in \text{Aut}^F(K)$ maps k isomorphically onto itself, hence $\text{Aut}^F(K)$ acts on k . Let $k_K \subseteq k_0$ be the corresponding fixed field up to Frobenius twists.

Theorem 1.3. *In the above notations, suppose that $\dim(X) > \dim(k_0) + 1$. Then one has a canonical exact sequence of the form: $1 \rightarrow \text{Aut}_k^F(K) \rightarrow \text{Aut}^c(\Pi_K) \rightarrow \text{Aut}_{k_K}^F(k) \rightarrow 1$.*

Thus if $\text{Aut}_k^F(K) = 1$ and $k_K = k_0$, then $v_K^c : \text{Gal}_{k_0} \rightarrow \text{Aut}^c(\Pi_K)$ is an isomorphism, hence the pro- ℓ abelian-by-central I/OM holds for \mathcal{U}_X , and so does the classical I/OM.

We note that Theorem 1.3 is an immediate consequence of Theorem 1.2: Setting namely $L|l = K|k$, Theorem 1.2 implies that the canonical map $\text{Aut}^F(K) \rightarrow \text{Aut}^c(\Pi_K)$ is an isomorphism of groups. Further, since $k \subset K^i$ is the unique maximal algebraically closed subfield of K^i , every $\phi \in \text{Aut}^F(K)$ maps k isomorphically onto itself. Conclude by noticing that one has an obvious exact sequence of groups

$$1 \rightarrow \text{Aut}_k^F(K) \rightarrow \text{Aut}^F(K) \rightarrow \text{Aut}_{k_K}^F(k) \rightarrow 1.$$

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2. PROOF OF THEOREM 1.1

A) *On the strategy of the proof*

First recall the strategy to tackle Bogomolov’s program as explained in [P3]. Namely, after choosing a throughout fixed identification $\iota : \mathbb{T}_{\mathbb{G}_m, k} = \mathbb{Z}_\ell(1) \rightarrow \mathbb{Z}_\ell$, one gets via Kummer theory an isomorphism, which is canonical up to the choice of the identification ι , as follows:

$$\widehat{K} := \varprojlim_e K^\times / \ell^e \rightarrow \varprojlim_e \text{Hom}(\Pi_K, \mathbb{Z} / \ell^e) = \text{Hom}(\Pi_K, \mathbb{Z}_\ell)$$

Let $j_K : K^\times \rightarrow \widehat{K}$ be the completion functor. Since k^\times is divisible, and K^\times/k^\times is a free abelian group, it follows that $\ker(j_K) = k^\times$, and $j_K(K^\times) = K^\times/k^\times$ is ℓ -adically dense in \widehat{K} . We identify $j_K(K) \subset \widehat{K}$ with $\mathcal{P}(K) := K^\times/k^\times$ and interpret it as the projectivization of the (infinite dimensional) k -vector space $(K, +)$. The 1-dimensional projective subspaces of $\mathcal{P}(K)$ are called **collineations** in $\mathcal{P}(K)$, and notice that the collineations in $\mathcal{P}(K)$ are of the form $\mathfrak{L}_{x,y} := (kx + ky)^\times/k^\times$, where $x, y \in K^\times$ are linearly independent over k .

Using the *Fundamental Theorem of Projective Geometries* FTPG, see e.g. ARTIN [Ar], it follows that one can recover $(K, +)$ from $\mathcal{P}(K)$ endowed with all the collineations $\mathfrak{L}_{x,y}$, $x, y \in K$. Moreover, the atomorphisms of $\mathcal{P}(K)$ which respect all the collineations are semi-linear. Using this fact one shows that the multiplication on $\mathcal{P}(K)$ induced by the group structure on $\widehat{K} = \text{Hom}(\Pi_K, \mathbb{Z})$ is distributive w.r.t. the addition of the group $(K, +)$ recovered via the FTPG. Thus knowing $\mathcal{P}(K) \subset \widehat{K}$ as a subgroup together with all the collineations in $\mathcal{P}(K)$ allows one to finally to recover the function field $K|k$. Furthermore, for an automorphism $\hat{\phi} : \widehat{K} \rightarrow \widehat{K}$ the following are equivalent:

- i) $\hat{\phi}$ is the ℓ -adic completion of an automorphism $\phi \in \text{Aut}^F(K)$.
- ii) $\hat{\phi}(\mathcal{P}(K)) = \mathcal{P}(K)$ and $\hat{\phi}$ maps the set of collineations onto itself.

This leads to the following strategy to tackle Bogomolov's program, see [P3], Introduction, for more details and how ideas evolved in this context.

Give group theoretical recipes which are invariant under isomorphisms of profinite groups and recover/reconstruct from Π_K^c , viewed as abstract profinite group, the following:

- The subset $\mathcal{P}(K) = j_K(K^\times) \subset \widehat{K} = \text{Hom}(\Pi_K, \mathbb{Z}_\ell)$.
- The collineations $\mathfrak{L}_{x,y} \subset \mathcal{P}(K)$ for all k -linearly independent $x, y \in K$.

We will give such group theoretical recipes which work under the hypothesis of Theorem 1.1, that is, recover $\mathcal{P}(K)$ together with the collineations $\mathfrak{L}_{x,y}$ from Π_K^c endowed with $\mathfrak{In.Div}(K)$.

- Step 1. Given Π_K^c , thus $\Pi_K^c \rightarrow \Pi_K$, endowed with $\mathfrak{In.Div}(K) \subset \Pi_K$, recover the **total decomposition graph** $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ for $K|k$.
 - This will be accomplished in Subsection B), see Proposition 2.2.
- Step 2. Given the total decomposition graph $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ for $K|k$, reconstruct a *first approximation* $\mathcal{L}_K \subset \widehat{K}$ of $\mathcal{P}(K)$ inside \widehat{K} , called the (canonical) **divisorial \widehat{U}_K -lattice** of $K|k$.
 - This will be accomplished in Subsection C), see Proposition 2.6.
- Step 3. Given Π_K^c and $\mathcal{L}_K \subset \widehat{K}$, reconstruct a *better approximation* $\mathcal{L}_K^0 \subset \mathcal{L}_K$ of $\mathcal{P}(K)$ inside \mathcal{L}_K , called the **quasi arithmetical \widehat{U}_K -lattice** in \widehat{K} , satisfying the following: $\mathcal{P}(K) \subseteq \mathcal{L}_K^0$ and $\mathcal{L}_K^0/(\widehat{U}_K \cdot \mathcal{P}(K))$ is torsion.
 - This will be accomplished in Subsection D), see Proposition 2.22.
- Step 4. Given $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ endowed with \mathcal{L}_K^0 , for all $x \in K$ with $k(x) \subset K$ relatively algebraically closed, recover the (rational) **projection** $pr_{k(x)} : \Pi_K \rightarrow \Pi_{k(x)}$.
 - This will be accomplished in Subsection E), see Proposition 2.25.

Finally, given $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ endowed with \mathcal{L}_K and all the rational projections $\Pi_K \rightarrow \Pi_{k(x)}$ as above, conclude by applying the Main Theorem from POP [P3], Introduction.

B) *Generalities about decomposition graphs*

Let k be an algebraically closed field with $\text{char}(k) \neq \ell$, and $K|k$ be a function field with $d := \text{td}(K|k) > 1$. We begin by recalling briefly basics about the (quasi) prime divisors of the function field $K|k$, see POP [P1], Section 3, for more details.

A flag of generalized prime divisors of $K|k$ is a chain of k -valuations $\tilde{v}_1 \leq \dots \leq \tilde{v}_r$ of K such that \tilde{v}_1 is a prime divisor of $K|k$ and inductively, $\tilde{v}_{i+1}/\tilde{v}_i$ is a prime divisor of the function field $K\tilde{v}_i|k$. In particular, $r \leq \text{td}(K|k)$, and we also say that \tilde{v}_r is a prime r -divisor of $K|k$. By abuse of language, we will say that the trivial valuation is the prime 0-divisor of $K|k$. A flag of generalized quasi prime divisors $\tilde{\mathfrak{v}}_1 \leq \dots \leq \tilde{\mathfrak{v}}_r$ of $K|k$ is defined in a similar way, but replacing prime by quasi prime. In particular, $\tilde{\mathfrak{v}}_r$ will also be called a quasi prime r -divisor, or a generalized quasi prime divisors of $K|k$ if r is irrelevant for the context. Note that the prime r -divisors of $K|k$ are precisely the quasi prime r -divisors of $K|k$ which are trivial on k .

The total prime divisor graph $\mathcal{D}_K^{\text{tot}}$ of $K|k$ is the half-oriented graph defined as follows:

- a) Vertices: The vertices of $\mathcal{D}_K^{\text{tot}}$ are the residue fields $K\tilde{v}$ of all the generalized prime divisors \tilde{v} of $K|k$ viewed as distinct function fields.
- b) Edges: If $\tilde{v} = \tilde{w}$, the trivial valuation $\tilde{v}/\tilde{w} = \tilde{w}/\tilde{v}$ is the only edge from $K\tilde{v} = K\tilde{w}$ to itself, and it by definition a non-oriented edge. If $\tilde{v} \neq \tilde{w}$, and \tilde{w}/\tilde{v} is a prime divisor of $K\tilde{v}$, then $w := \tilde{w}/\tilde{v}$ is the only edge from $K\tilde{v}$ to $K\tilde{w}$, and by definition, oriented. Otherwise there are no edges from $K\tilde{v}$ to $K\tilde{w}$.

The total quasi prime divisor graph $\mathcal{Q}_K^{\text{tot}}$ of $K|k$ is defined in a totally parallel way, but considering as vertices all the generalized quasi prime divisors.

Notice that $\mathcal{D}_K^{\text{tot}} \subset \mathcal{Q}_K^{\text{tot}}$ is a full subgraph, and that the following functoriality holds:

- 1) *Embeddings.* Let $L|l \hookrightarrow K|k$ be an embedding of function fields which maps l isomorphically onto k . Then the canonical restriction map of valuations $\text{Val}_K \rightarrow \text{Val}_L$, $v \mapsto v|_L$, gives rise to a surjective morphism of the total (quasi) prime divisor graphs $\varphi_i : \mathcal{D}_K^{\text{tot}} \rightarrow \mathcal{D}_L^{\text{tot}}$ and $\varphi_i : \mathcal{Q}_K^{\text{tot}} \rightarrow \mathcal{Q}_L^{\text{tot}}$.
- 2) *Restrictions.* Given a generalized prime divisor \tilde{v} of $K|k$, let $\mathcal{D}_{\tilde{v}}^{\text{tot}}$ be the set of all the generalized prime divisors \tilde{w} of $K|k$ with $\tilde{v} \leq \tilde{w}$. Then the map $\mathcal{D}_{\tilde{v}}^{\text{tot}} \rightarrow \text{Val}_{K\tilde{v}}$, $\tilde{w} \mapsto \tilde{w}/\tilde{v}$, is an isomorphism of $\mathcal{D}_{\tilde{v}}^{\text{tot}}$ onto the total prime divisor graph of $K\tilde{v}|k$. Similarly, the corresponding assertion for generalized quasi prime divisors holds as well.

• **Decomposition graphs** [See POP [P3], Section 3, for more details.]

Let $K|k$ be as above. For every valuation v of K , let $1 + \mathfrak{m}_v =: U_v^1 \subset U_v := \mathcal{O}_v^\times$ be the principal v -units, respectively the v -units in K^\times . By POP [P1] and TOPAZ [To1], it follows that the decomposition field K_v^Z of v is contained in $K^{Z^1} := K[\ell^\infty \sqrt{U_v^1}]$, and the inertia field K_v^T of v is contained in $K^{T^1} := K[\ell^\infty \sqrt{U_v}]$. We denote $T_v^1 := \text{Gal}(K'|K^{T^1}) \subseteq T_v$ and $Z_v^1 := \text{Gal}(K'|K^{Z^1}) \subseteq Z_v$ and call $T^1 \subseteq Z^1$ the minimized inertia/decomposition groups of v . Recalling that $Kv^\times = U_v/U_v^1$, by Kummer theory one gets that

$$\Pi_{Kv}^1 := Z_v^1/T^1 = \text{Hom}^{\text{cont}}(U_v/U_v^1, \mathbb{Z}_\ell(1)) = \text{Hom}^{\text{cont}}(Kv^\times, \mathbb{Z}_\ell(1)).$$

By abuse of language, we say that Π_{Kv}^1 is the minimized residue Galois group at v . We notice that $K^Z = K^{Z^1}$, $K^{T^1} = K^T$, $\Pi_{Kv}^1 = \Pi_{Kv}$, provided $\text{char}(Kv) \neq \ell$. On the other hand, if

$\text{char}(Kv) = \ell$, then one must have $\text{char}(k) = 0$, and in this case $T_v^1 \subseteq Z_v^1 \subseteq T_v$, hence the residue field of K^{Z^1} contains $(Kv)'$, thus $\Pi_{Kv}^1 \subseteq T_v/T_v^1$ has trivial image in $\Pi_{Kv} = Z_v/T_v$.

Anyway, for generalized prime divisors \tilde{v} of K , one has: $T_{\tilde{v}} = T_{\tilde{v}}^1$, $Z_{\tilde{v}} = Z_{\tilde{v}}^1$, $\Pi_{K\tilde{v}}^1 = \Pi_{K\tilde{v}}$.

Recall that a generalized quasi prime divisors $\tilde{\mathfrak{v}}$ is a quasi prime r -divisor iff $T_{\tilde{\mathfrak{v}}}^1 \cong \mathbb{Z}_\ell^r$. Moreover, any of the equalities $K_{\tilde{\mathfrak{v}}}^{Z^1} = K_{\tilde{\mathfrak{v}}}^Z$, $K_{\tilde{\mathfrak{v}}}^{T^1} = K_{\tilde{\mathfrak{v}}}^T$, $\Pi_{K\tilde{\mathfrak{v}}}^1 = \Pi_{K\tilde{\mathfrak{v}}}$, is equivalent to $\text{char}(K\tilde{\mathfrak{v}}) \neq \ell$. Further, for generalized quasi prime divisors $\tilde{\mathfrak{v}}_1$ and $\tilde{\mathfrak{v}}_2$ one has: $Z_{\tilde{\mathfrak{v}}_1}^1 \cap Z_{\tilde{\mathfrak{v}}_2}^1 \neq 1$ iff $T_{\tilde{\mathfrak{v}}_1}^1 \cap T_{\tilde{\mathfrak{v}}_2}^1 \neq 1$, and if $T_{\tilde{\mathfrak{v}}_1}^1 \cap T_{\tilde{\mathfrak{v}}_2}^1 \neq 1$, there exists a unique generalized quasi prime divisor $\tilde{\mathfrak{v}}$ of $K|k$ with $T_{\tilde{\mathfrak{v}}}^1 = T_{\tilde{\mathfrak{v}}_1}^1 \cap T_{\tilde{\mathfrak{v}}_2}^1$. And $\tilde{\mathfrak{v}}$ is also the unique generalized quasi prime divisor of $K|k$ maximal with the property $Z_{\tilde{\mathfrak{v}}_1}^1, Z_{\tilde{\mathfrak{v}}_2}^1 \subseteq Z_{\tilde{\mathfrak{v}}}^1$. Finally, $\tilde{\mathfrak{v}}$ is trivial on k , provided $\min(\tilde{\mathfrak{v}}_1, \tilde{\mathfrak{v}}_2)$ is so.

In particular, for generalized (quasi) prime divisors $\tilde{\mathfrak{v}}$ and $\tilde{\mathfrak{w}}$ of $K|k$ one has: $\tilde{\mathfrak{v}} = \tilde{\mathfrak{w}}$ iff $T_{\tilde{\mathfrak{v}}}^1 = T_{\tilde{\mathfrak{w}}}^1$ iff $Z_{\tilde{\mathfrak{v}}}^1 = Z_{\tilde{\mathfrak{w}}}^1$. Further, $\tilde{\mathfrak{v}} < \tilde{\mathfrak{w}}$ iff $T_{\tilde{\mathfrak{v}}}^1 \subset T_{\tilde{\mathfrak{w}}}^1$ strictly iff $Z_{\tilde{\mathfrak{v}}}^1 \supset Z_{\tilde{\mathfrak{w}}}^1$ strictly. And if $\tilde{\mathfrak{v}} < \tilde{\mathfrak{w}}$ is are a (quasi) prime r -divisor, respectively a (quasi) prime s -divisor, then $T_{\tilde{\mathfrak{w}}}^1/T_{\tilde{\mathfrak{v}}}^1 \cong \mathbb{Z}_\ell^{s-r}$.

We conclude that the partial ordering on the set of all the generalized (quasi) prime divisors $\tilde{\mathfrak{v}}$ of $K|k$ is encoded in the set of their minimized inertia/decomposition groups $T_{\tilde{\mathfrak{v}}}^1 \subseteq Z_{\tilde{\mathfrak{v}}}^1$. In particular, the existence of the trivial, respectively nontrivial, edge from $K\tilde{\mathfrak{v}}$ to $K\tilde{\mathfrak{w}}$ in $\mathcal{Q}_K^{\text{tot}}$ (and/or $\mathcal{D}_K^{\text{tot}}$) is equivalent to $T_{\tilde{\mathfrak{v}}}^1 = T_{\tilde{\mathfrak{w}}}^1$, respectively to $T_{\tilde{\mathfrak{v}}}^1 \subset T_{\tilde{\mathfrak{w}}}^1$ and $T_{\tilde{\mathfrak{w}}}^1/T_{\tilde{\mathfrak{v}}}^1 \cong \mathbb{Z}_\ell$.

Via the Galois correspondence and the functorial properties of the Hilbert decomposition theory for valuations, we attach to the total prime divisor graph $\mathcal{D}_K^{\text{tot}}$ of $K|k$ a graph $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ whose vertices and edges are in bijection with those of $\mathcal{D}_K^{\text{tot}}$, as follows:

- a) The vertices of $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ are the pro- ℓ groups $\Pi_{K\tilde{v}}$, viewed as distinct pro- ℓ groups, where \tilde{v} are all the generaliazed prime divisors of $K|k$.
- b) If an edge from $K\tilde{v}$ to $K\tilde{w}$ exists, the corresponding edge from $\Pi_{K\tilde{v}}$ to $\Pi_{K\tilde{w}}$ is endowed with the pair of groups $T_{\tilde{w}/\tilde{v}} = T_{\tilde{w}}/T_{\tilde{v}} \subseteq Z_{\tilde{w}}/T_{\tilde{v}} = Z_{\tilde{w}/\tilde{v}}$, viewed as subgroups of the residue Galois group $\Pi_{K\tilde{v}} = Z_{\tilde{w}}/T_{\tilde{v}}$, and notice that in this case $\Pi_{K\tilde{w}} = Z_{\tilde{w}/\tilde{v}}/T_{\tilde{w}/\tilde{v}}$.

The graph $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ will be called the **total decomposition graph** of $K|k$, or of Π_K .

In a similar way, we attach to $\mathcal{Q}_K^{\text{tot}}$ the **total quasi decomposition graph** of $K|k$, which we denote $\mathcal{G}_{\mathcal{Q}_K^{\text{tot}}}$, but using the minimized inertia/decomposition/residue groups $T_{\tilde{\mathfrak{v}}}^1$, $Z_{\tilde{\mathfrak{v}}}^1$, $\Pi_{K\tilde{\mathfrak{v}}}^1$ instead of the inertia/decomposition/residue Galois groups (which are the same for generalized prime divisors, because $\text{char}(k) \neq \ell$). Clearly, $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ is a full subgraph of $\mathcal{G}_{\mathcal{Q}_K^{\text{tot}}}$.

The functorial properties of the total graphs of (quasi) prime divisors translate in the following functorial properties of the total (quasi) decomposition graphs:

- 1) *Embeddings*. Let $\iota : L|l \hookrightarrow K|k$ be an embedding of function fields which maps l isomorphically onto k . Then the canonical projection homomorphism $\Phi_\iota : \Pi_K \rightarrow \Pi_L$ is an open homomorphism, and moreover, for every generalized (quasi) prime divisor \mathfrak{v} of $K|k$ and its restriction \mathfrak{v}_L to L one has: $\Phi_\iota(Z_{\mathfrak{v}}^1) \subseteq Z_{\mathfrak{v}_L}^1$ is an open subgroup, and $\Phi_\iota(T_{\mathfrak{v}}^1) \subseteq T_{\mathfrak{v}_L}^1$ satisfies: $\Phi_\iota(T_{\mathfrak{v}}^1) = 1$ iff \mathfrak{v}_L has divisible value group, e.g., \mathfrak{v}_L is the trivial valuation. Therefore, Φ_ι gives rise to **morphisms** of total (quasi) decomposition graphs

$$\Phi_\iota : \mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{D}_L^{\text{tot}}}, \quad \Phi_\iota : \mathcal{G}_{\mathcal{Q}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{Q}_L^{\text{tot}}}.$$

- 2) *Restrictions*. Given a generalized (quasi) prime divisor \mathfrak{v} of $K|k$, let $pr_{\mathfrak{v}} : Z_{\mathfrak{v}}^1 \rightarrow \Pi_{K\mathfrak{v}}^1$ be the canonical projection. Then for every $\mathfrak{w} \geq \mathfrak{v}$ we have: $T_{\mathfrak{w}}^1 \subseteq Z_{\mathfrak{w}}^1$ are mapped onto

$T_{\mathfrak{w}/\mathfrak{v}}^1 := T_{\mathfrak{w}}^1/T_{\mathfrak{v}}^1 \subseteq Z_{\mathfrak{w}}^1/T_{\mathfrak{v}}^1 =: Z_{\mathfrak{w}/\mathfrak{v}}^1$. Therefore, the total (quasi) decomposition graph for $K\mathfrak{v}|k\mathfrak{v}$ can be recovered from the one for $K|k$ in a canonical way via $pr_{\mathfrak{v}} : Z_{\mathfrak{v}}^1 \rightarrow \Pi_{K\mathfrak{v}}^1$.

Remark 2.1. By the discussion above, *reconstructing the total decomposition graph $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ is equivalent to describing the set of all the generalized divisorial groups $T_{\mathfrak{v}} \subset Z_{\mathfrak{v}}$ in Π_K .*

Proposition 2.2. *Let $\mathfrak{In.tm}_k(K) \subset \Pi_K$ be the topological closure of $\mathfrak{In.div}(K)$ in Π_K . Further, for closed subgroups $\Delta \subset \Pi_K$, let $\Delta'' \subset \Pi_K^c$ be their presages under the canonical projection $\Pi_K^c \rightarrow \Pi_K$. Then $\text{td}(K|k)$ is the maximal integer d such that there exist subgroups $\Delta \cong \mathbb{Z}_{\ell}^d$ of Π_K with $\Delta'' \subset \Pi_K^c$ abelian. Further, the following hold:*

- 1) *For sequences of subgroups $Z_1 \supset \cdots \supset Z_r$, $T_1 \subset \cdots \subset T_r$ with $T_m \subseteq Z_m$, $m = 1, \dots, r$, the following are equivalent:*
 - i) *There exist prime m -divisors \tilde{v}_m such that $T_m = T_{\tilde{v}_m} \subset Z_{\tilde{v}_m} = Z_m$ for $m = 1, \dots, r$.*
 - ii) *$Z_1 \supset \cdots \supset Z_r$, $T_1 \subset \cdots \subset T_r$ are maximal w.r.t. inclusion satisfying:*
 - Z_r *contains subgroups $\Delta \cong \mathbb{Z}_{\ell}^{\text{td}(K|k)}$ with $\Delta'' \subset \Pi_K^c$ abelian.*
 - $T_r \subset \mathfrak{In.tm}_k(K)$, $T_m \cong \mathbb{Z}_{\ell}^m$ and $T_m'' \subset Z_m''$ *is the center of Z_m'' for $m = 1, \dots, r$.*
- 2) *In particular, the total decomposition graph $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ for $K|k$ can be reconstructed from $\Pi_K^c \rightarrow \Pi_K$ endowed with divisorial inertia $\mathfrak{In.div}(K) \subset \Pi_K$.*
- 3) *Moreover, the group theoretical recipe to recover $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ is invariant under isomorphisms as follows: Let $\Phi \in \text{Isom}(\Pi_K, \Pi_L)$ satisfy $\Phi(\mathfrak{In.div}(K)) = \mathfrak{In.div}(L)$. Then Φ maps the generalized divisorial groups $T_{\mathfrak{v}} \subset Z_{\mathfrak{v}}$ of Π_K onto the generalized divisorial groups $T_{\tilde{v}} \subset Z_{\tilde{v}}$ of Π_L , thus defines an isomorphism $\Phi : \mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{D}_L^{\text{tot}}}$.*

Proof. The proof of the assertions 1), 2) follows instantly from the characterization of the generalized divisorial groups $T_{\mathfrak{v}} \subset Z_{\mathfrak{v}}$ given at ii). Therefore, it is sufficient to prove the equivalence of assertions i), ii). First, the implication i) \Rightarrow ii) holds even for a more general class of valuations, see [P1], Proposition 4.2. The proof of the converse implication ii) \Rightarrow i), is identical with the proof of the corresponding assertion from [P4], Proposition 3.5, the only (formal) change necessary being to replace $\mathfrak{In.tm}(K)$ from [P4] by $\mathfrak{In.tm}_k(K)$ in the present situation. \square

C) Fundamental groups and divisorial lattices

We begin by recalling here a few basic facts from POP [P3] and proving a little bit more precise/stronger results about the (pro- ℓ abelian) fundamental group of quasi projective normal k -varieties, see the discussion from [P3], Appendix, section 7.3 for some of the details. These stronger and more precise results will be needed later on, e.g., in Subsection D) when discussing specializations techniques. The facts were not known to the author and those whom he asked at the time [P3] was written.

- *On the fundamental group of sets of prime divisors*

Let D be a set of prime divisors of $K|k$. We denote by $T_D \subseteq \Pi_K$ the closed subgroup generated by all the T_v , $v \in D$, and say that $\Pi_{1,D} := \Pi_K/T_D$ is the **fundamental group** of the set D . In the case D equals the set of all the prime divisors $D = D_{K|k}$ of $K|k$, we say that $\Pi_{1,D_{K|k}} =: \Pi_{1,K}$ is the (birational) **fundamental group** for $K|k$. Recall that a set D of prime divisors of $K|k$ is called **geometric**, if there exists a normal model $X \rightarrow k$ of

$K|k$ such that $D = D_X$ is the set of Weil prime divisors of X . (If so, there are always quasi-projective normal models X with $D = D_X$.) In particular, if X is a normal model of $K|k$ and $\Pi_1(X)$ denotes the maximal pro- ℓ abelian fundamental group of X , one has canonical surjective projections $\Pi_{1,D_X} \rightarrow \Pi_1(X)$ and $\Pi_{1,D_X} \rightarrow \Pi_{1,K}$. Moreover, if $U \subset X$ is an open *smooth* k -subvariety, then $\Pi_{1,D_U} \rightarrow \Pi_1(U)$ is an isomorphism (by the purity of the branch locus). In particular, $\Pi_{1,D_U} = \Pi_1(U)$ is a finite \mathbb{Z}_ℓ -module, and since one has the canonical surjective homomorphisms $\Pi_{1,D_U} \rightarrow \Pi_{1,D_X} \rightarrow \Pi_{1,K}$, it follows that $\Pi_{1,D_X} = \Pi_{1,D}$ and $\Pi_{1,K}$ are finite \mathbb{Z}_ℓ -modules. Further, it was shown in [P3], Appendix, 7.3, see especially Fact 57, that there exist (quasi projective) normal models X such that $\Pi_{1,D_X} \rightarrow \Pi_{1,K}$ is an isomorphism. Nevertheless, it is not clear that for every geometric set D there exists quasi projective normal models X such that $\Pi_{1,D_X} \rightarrow \Pi_1(X)$ is an isomorphism.

We begin by recalling two fundamental facts concerning *alterations* as introduced by DE JONG and developed by GABBER, TEMKIN, and many others, see e.g. [ILO], Expose X.

Let D be a fixed geometric set of prime divisors for $K|k$, X_0 be any projective normal model of $K|k$ with $D \subseteq D_{X_0}$, and $S_0 \subset X_0$ a fixed closed proper subset. Usually S_0 will be chosen to define D in the sense that $D = D_{X_0 \setminus S_0}$. By [ILO], Expose X, Theorem 2.1, there exist **prime to ℓ alterations above S_0** , i.e., projective generically finite separable morphisms $Y \rightarrow X_0$ satisfying the following:

- Y is a projective smooth k -variety.
- $T := f_0^{-1}(S_0)$ is a NCD (normal crossings divisor) in Y .
- $[k(Y) : K]$ is prime to ℓ .

We denote by D_0 the restriction of D_Y to K , and notice that $D_{X_0} \subseteq D_0$, because $Y \rightarrow X_0$ is surjective. Hence finally $D \subseteq D_{X_0} \subseteq D_0$.

Second, let $X_1 \rightarrow X_0$ be a dominant morphism with X_1 a projective normal model of $K|k$ such that $D_0 \subseteq D_{X_1}$, thus we have $D \subseteq D_{X_0} \subseteq D_0 \subseteq D_{X_1}$. Then by DE JONG's theory of alterations, see e.g., [ILO], Expose X, Lemma 2.2, there exists a **generically normal finite alteration** of X_1 , i.e., a projective dominant k -morphism $Z \rightarrow X_1$ satisfying the following:

- Z is a projective smooth k -variety.
- The field extension $K = k(X_1) \hookrightarrow k(Z) =: M$ is a finite and normal.
- $\text{Aut}(M|K)$ acts on Z and $Z \rightarrow X_1$ is $\text{Aut}(M|K)$ -invariant.

By a standard scheme theoretical construction (recalled below), there exists a projective normal model X for $K|k$ and a dominant k -morphism $X \rightarrow X_1$ such that $Z \rightarrow X_1$ factors through $X \rightarrow X_1$, and the resulting k -morphism $Z \rightarrow X$ is finite. In particular, since Z is smooth, thus normal, $Z \rightarrow X$ is the normalization of X in the field extension $K \hookrightarrow M$.

We briefly recall the standard scheme theoretical construction, which is as follows: Let $Z \rightarrow \text{Aut}(M|K) \backslash Z =: Z_i$ be the quotient of Z by $\text{Aut}(M|K)$. Then $Z \rightarrow Z_i$ is a finite generically Galois morphism, and its function field $M_i := k(Z_i)$ satisfies: $M|M_i$ is Galois, and $M_i|K$ purely inseparable. Hence there exists $e > 0$ such that $M_i^{(e)} := \text{Frob}^e(M_i) \subset K$, and $M_i^{(e)}$ is the function field of the e^{th} Frobenius twist $Z_i^{(e)}$ of Z_i . And notice that $Z_i^{(e)}$ is a projective normal model for the function field $M_i^{(e)}|k$. Further, the normalization of $Z_i^{(e)}$ in the finite field extension $M_i^{(e)} \hookrightarrow M_i$ is nothing but Z_i . Finally, let X be the normalization of $Z_i^{(e)}$ in the function field extension $M_i^{(e)} \hookrightarrow K$. Then X is a projective normal k -variety because $Z_i^{(e)}$ was so, and $k(X) = K$, thus X is a projective normal model of $K|k$. Further,

by the transitivity of normalization, it follows that the normalization of X in $K \hookrightarrow M_i$ equals the normalization of $Z_i^{(e)}$ in the field extension $M_i^{(e)} \hookrightarrow M_i$, and that normalization is Z_i . Finally, using the transitivity of normalization again, it follows that the normalization of X in $K \hookrightarrow M_1$ is Z itself. Finally, to prove that $Z \rightarrow X_1$ factors through $Z \rightarrow X$, we proceed as follows: First, since X_1 and X are both projective normal models of $K|k$, there is a canonical rational map $X \dashrightarrow X_1$. We claim that $X \rightarrow X_1$ is actually a morphism. Indeed, let $x \in X$ be a fixed point, and $Z_x \subset Z$ be the preimage of x under $Z \rightarrow X$. Then $\text{Aut}(M|K)$ acts transitively on Z_x , and since $Z \rightarrow X_1$ is $\text{Aut}(M|K)$ -invariant, it follows that the image of Z_x under $Z \rightarrow X_1$ consists of a single point, say $x_1 \in X_1$. Now let $V_1 := \text{Spec } R_1 \subset X_1$ be an affine open subset containing x_1 , and $W := \text{Spec } S \subset Z$ be an $\text{Aut}(M|K)$ invariant open subset of Z containing Z_x , and $V := \text{Spec } R \subset X$ be the image of $W = \text{Spec } S$ under $Z \rightarrow X$. Then identifying R_1, R and S_1 with the corresponding k -subalgebras of finite type of K , respectively of $M = k(Z)$, it follows that $W \rightarrow V_1$ and $W \rightarrow V$ are defined by the k -embeddings $R_1 \hookrightarrow S$, respectively $R \hookrightarrow S$, defined via the inclusion $K \hookrightarrow M$. Now since S is the normalization of R , it follows that R is mapped isomorphically onto $K \cap S$. Thus since $K \hookrightarrow M$ maps R_1 into $K \cap S$, we conclude that $R_1 \subset R$. That in turns shows that $V \dashrightarrow V_1$ is defined by the k -inclusion $R_1 \hookrightarrow R$, thus it is a morphism, and therefore, defined at $x \in V$. Conclude that $X \dashrightarrow X_1$ is actually a k -morphism.

Preparation/Notations 2.3. Summarizing the discussion above, for a geometric set D of prime divisors for $K|k$, and X_0 a projective normal model of $K|k$ with $D \subseteq D_{X_0}$, we let $S_0 \subset X_0$ be a closed subset with $D = D_{X_0 \setminus S_0}$, and can/will consider the following:

- a) A prime to ℓ alteration $Y \rightarrow X_0$ above S_0 . We denote by $T \subset Y$ the preimage of $S_0 \subset X_0$ under $Y \rightarrow X_0$, and by D_0 the restriction of D_Y to K . Hence $D \subseteq D_{X_0} \subseteq D_0$.
- b) A morphism of projective normal models $X \rightarrow X_0$ with $D_0 \subseteq D_X$. For closed subsets $S'_0 \subset X_0$, let $T' \subset Y$ and $S' \subset X$ be the corresponding preimages of $S'_0 \subset X_0$.
- c) A smooth projective k -variety Z together with a dominant finite k -morphism $Z \rightarrow X$ such that $K = k(X) \hookrightarrow k(Z) =: M$ is normal and $\text{Aut}(M|K)$ acts on Z .

Proposition 2.4. *In the above notations, the following hold:*

- 1) *The canonical projection $\Pi_{1, D_{X \setminus S'}} \rightarrow \Pi_1(X \setminus S')$ is an isomorphism. Hence one has canonical surjective projections $\Pi_{1, D_{X_0 \setminus S'_0}} \rightarrow \Pi_{1, D_{X \setminus S'}} \rightarrow \Pi_1(X \setminus S') \rightarrow \Pi_1(X_0 \setminus S'_0)$.*
- 2) *Suppose that $D_0 \subseteq D_{X \setminus S'}$. Then $\Pi_{1, D_{X \setminus S'}} \rightarrow \Pi_1(X \setminus S') \rightarrow \Pi_{1, K}$ are isomorphisms.*

Proof. To 1): The existence and the surjectivity of the projections is clear. Thus it is left to show that $\Pi_{1, D_{X \setminus S'}} \rightarrow \Pi_1(X \setminus S')$ is injective. Equivalently, one has to prove the following: Let $\tilde{K}|K$ be an abelian ℓ -power degree extension, and $\tilde{X} \rightarrow X$ be the normalization of X in $K \hookrightarrow \tilde{K}$. Then $\tilde{X} \rightarrow X$ is etale above S' if and only if none of the prime divisors $v \in D_{X \setminus S'}$ has ramification in $\tilde{K}|K$. Clearly, this is equivalent to the corresponding assertion for all cyclic sub extensions of $\tilde{K}|K$, thus without loss of generality, we can suppose that $\tilde{K}|K$ is cyclic, thus $\text{Gal}(\tilde{K}|K)$ is cyclic. For points $\tilde{x} \mapsto x$ under $\tilde{X} \rightarrow X$ and valuations $\tilde{v}|v$ of $\tilde{K}|K$, we denote by $T_{\tilde{x}}$, respectively $T_{\tilde{v}}$ the corresponding inertia groups.

Claim 1. *Suppose that $\tilde{G} := \text{Gal}(\tilde{K}|K) = \langle \tilde{g} \rangle$ is cyclic. Then for every $\tilde{x} \in \tilde{X}$ there exists a prime divisor \tilde{w} of $\tilde{K}|k$ with $T_{\tilde{x}} \subseteq T_{\tilde{w}}$ and \tilde{x} in the closure of the center $x_{\tilde{w}}$ of \tilde{w} on \tilde{X} .*

Proof of Claim 1. Compare with POP [P2], Proof of Theorem B. Recalling the cover $Z \rightarrow X$ with $M := k(Z)$, let $\tilde{K}' := M\tilde{K}$ be the compositum of \tilde{K} and $M = k(Z)$, and $\tilde{X}' \rightarrow X$ be the normalization of X in the function field extension $K \hookrightarrow \tilde{K}'$ and the resulting canonical factorizations $\tilde{X}' \rightarrow \tilde{X} \rightarrow X$ and $\tilde{X}' \rightarrow Z \rightarrow X$. Further, choosing a preimage \tilde{x}' of \tilde{x} under $\tilde{X}' \rightarrow \tilde{X}$, consider $\tilde{x}' \mapsto \tilde{x} \mapsto x$ and $\tilde{x}' \mapsto z \mapsto x$ under the above factorizations of $\tilde{X}' \rightarrow X$. Then by the functoriality of the Hilbert decomposition/ramification theory, there are surjective canonical projections $T_{\tilde{x}'} \rightarrow T_{\tilde{x}}$ and $T_{\tilde{x}'} \rightarrow T_z$. Thus given a generator \tilde{g} of $T_{\tilde{x}}$, there exists $\tilde{g}' \in T_{\tilde{x}'}$ which maps to \tilde{g} under $T_{\tilde{x}'} \rightarrow T_{\tilde{x}}$. And if \tilde{w}' is a prime divisor of $\tilde{K}'|k$ such that $\tilde{g}' \in T_{\tilde{w}'}$, then setting $\tilde{w} := \tilde{w}'|_{\tilde{x}}$, it follows that $\tilde{g} \in T_{\tilde{w}}$. Hence letting $K' := M^{\tilde{g}}$ be the fixed field of \tilde{g} in M , and replacing $\tilde{K}|K$ by $\tilde{K}'|K'$, we can suppose that from the beginning we have $M \subset \tilde{K}$, and $\tilde{K}|K$ is cyclic with Galois group $\tilde{G} = \langle \tilde{g} \rangle$, and $M|K$ is a cyclic subextension, say with Galois group $G = \langle g \rangle$, where $g = \tilde{g}|_M$. And notice that $\tilde{G} = T_{\tilde{x}}$ and $G = T_z$. Thus G acts on the local ring \mathcal{O}_z of z , and \tilde{G} acts on the local ring $\mathcal{O}_{\tilde{x}}$ of \tilde{x} .

Case 1. $K = M$. Then $X = Z$ is a projective smooth model for $K|k$, thus $\mathcal{O}_x = \mathcal{O}_z$ is a regular ring. Let \mathcal{O}_x^n be the normalization of \mathcal{O}_x in $K \hookrightarrow \tilde{K}$, and $\text{Spec } \mathcal{O}_x^n \rightarrow \text{Spec } \mathcal{O}_x$ the restriction of $\tilde{X} \rightarrow X$ to $\text{Spec } \mathcal{O}_x^n$. Since $\text{Spec } \mathcal{O}_x$ is regular, by the purity of the branch locus it follows that $T_{\tilde{x}}$ is generated by inertia groups of the form $T_{\tilde{w}}$, with \tilde{w} prime divisors of $\tilde{K}|k$ having the center on $\text{Spec } \mathcal{O}_x^n \subset \tilde{X}$. Since $T_{\tilde{x}}$ is cyclic, it follows that there exists \tilde{w} with $T_{\tilde{x}} \subseteq T_{\tilde{w}}$, etc.

Case 2. $K \subset M$ strictly. Then letting $(\mathcal{O}, \mathfrak{m})$ be the local ring $(\mathcal{O}_z, \mathfrak{m}_z)$ at z , one has $T_z = G$. Then proceeding as in the in the proof of Theorem B, explanations after Fact 2.2, (the proof of) Lemma 2.4 of loc.cit. is applicable, and by Step 3 of that proof, it follows by Lemma 2.6 of loc.cit. that there exists a local ring $(\mathcal{O}', \mathfrak{m}')$ which has the properties:

- G acts on $(\mathcal{O}', \mathfrak{m}')$, and $(\mathcal{O}', \mathfrak{m}')$ dominates $(\mathcal{O}_z, \mathfrak{m}_z)$.
- There exist local parameters (t'_1, \dots, t'_d) of \mathcal{O}' , and a primitive character χ of G such that $\sigma(t'_i) = t'_i$ for $1 \leq i < d$ and $\sigma(t'_d) = \chi(\sigma)t'_d$.

Since \mathcal{O}' dominates \mathcal{O}_z , it follows that $\mathcal{O}^T := \mathcal{O}'^G$ dominates $\mathcal{O}_z^G = \mathcal{O}_x$, and \mathcal{O}^T is a regular ring by Step 4 of loc.cit. Further $G = T_z$ is a quotient of the ramification group $T_{\mathcal{O}^T}$ of \mathcal{O}^T in $K \hookrightarrow \tilde{K}$, and since $K \hookrightarrow \tilde{K}$ is cyclic of ℓ -power order, it follows that $T_{\mathcal{O}^T} = \text{Gal}(\tilde{K}|K) = T_{\tilde{x}}$. Thus \mathcal{O}^T has a unique prolongation $\tilde{\mathcal{O}}^T$ to \tilde{K} . And since \mathcal{O}^T dominates \mathcal{O}_x , it follows that $\tilde{\mathcal{O}}^T$ dominates $\mathcal{O}_{\tilde{x}}$ (which is the unique prolongation of \mathcal{O}_x to \tilde{K}). One concludes in the same way as at Case 1 above.

Claim 2. *Let $\tilde{K}|K$ be an abelian ℓ -power Galois extension as above. Then a prime divisor w of $K|k$ ramifies in $\tilde{K}|K$ iff w has a prolongation w_L to L which ramifies in $\tilde{L}|L$.*

Proof of Claim 2. Let $\hat{K}|K$ be a (minimal) finite normal extension with $L, \tilde{K} \subseteq \hat{K}$, and $\text{Gal}(\hat{K}|K) =: \hat{G} \rightarrow \tilde{G} := \text{Gal}(\tilde{K}|K)$, $\hat{g} \mapsto g$, be the corresponding projection of Galois groups. Setting $\tilde{L} = L\tilde{K}$ inside \hat{K} , it follows that $\text{Gal}(\tilde{L}|L) =: H \rightarrow \tilde{G}$ is an isomorphism, and $\hat{H} := \text{Gal}(\hat{K}|L) \rightarrow H$ is surjective, because \tilde{K} and L are linearly disjoint over K . Thus there exists a Sylow ℓ -group \hat{H}_ℓ of \hat{H} with $\hat{H}_\ell \rightarrow H$ surjective. Since $(\hat{G} : \hat{H}) = [L : K]$ is prime to ℓ , it follows that \hat{H}_ℓ is a Sylow ℓ -group of \hat{G} as well. Let $\hat{w}|\tilde{w}|w$ denote the prolongations of w to \hat{K} , respectively \tilde{K} , and $T_{\hat{w}} \rightarrow T_{\tilde{w}}$ be the corresponding surjective projections of the inertia groups. For every $\sigma \in T_{\tilde{w}}$, let $\hat{\sigma} \in T_{\hat{w}}$ be a preimage of order a

power of ℓ . Then $\hat{\sigma}$ is contained in a Sylow ℓ -group of \hat{G} , hence there exists a conjugate $\hat{\sigma}^{\hat{\tau}}$ which lies in \hat{H}_{ℓ} . Thus replacing $\hat{w}|\hat{w}|w$ by $\hat{w}^{\hat{\tau}}|\hat{w}^{\hat{\tau}}|w$, and $\sigma, \hat{\sigma}$ by $\sigma^{\hat{\tau}}, \hat{\sigma}^{\hat{\tau}}$, respectively $\hat{\sigma}^{\hat{\tau}}$, without loss of generality, we can suppose that $\hat{\sigma} \in T_{\hat{w}} \subseteq \hat{H}_{\ell}$ is a preimage of $\sigma \in T_{\tilde{w}}$. Thus setting $\tilde{w}_L := \hat{w}|_{\tilde{L}}$ and $w_L := \hat{w}|_L$, it follows that $\tilde{w}_L|w_L$ are prolongations of $\tilde{w}|w$ to \tilde{L} , respectively L . And taking into account that $\hat{H} \rightarrow \tilde{G}$ factors as $\hat{H} \rightarrow H \rightarrow \tilde{G}$, it follows that the image $\sigma_L \in H$ of $\hat{\sigma} \in \hat{H}$ under $\hat{H} \rightarrow H$ lies in $T_{\tilde{w}_L} \subset H$. This concludes the proof of Claim 2.

Coming back to assertion 1) of Proposition 2.4, we prove the following more precise result:

Lemma 2.5. *Let $\tilde{K}|K$ be a finite abelian pro- ℓ field extension, and set $\tilde{L} := L\tilde{K}$. Then for the corresponding normalization $\tilde{X} \rightarrow X$ and $\tilde{Y} \rightarrow Y$, the following assertions are equivalent:*

- i) $\tilde{X} \rightarrow X$ is étale above S' .
- ii) All $v \in D_{X \setminus S'} \cap D_0$ are unramified in $\tilde{K}|K$.
- iii) All $w \in D_{Y \setminus T'}$ are unramified in $\tilde{L}|L$.
- iv) $\tilde{Y} \rightarrow Y$ is étale above T' .

Proof of Lemma 2.5. First, i) implies ii) in an obvious way. For the implication ii) \Rightarrow iii), let $v_L \in D_{Y \setminus T'}$ have restriction $v := (v_L)|_K$ to K . Then $v \in D_0 \cap D_{X \setminus S'}$, as observed above. On the other hand, by Claim 2) above one has that v does not ramify in $\tilde{K}|K$ iff all its prolongations to L do not ramify in $\tilde{L}|L$. Hence v_L does not ramify in $\tilde{L}|L$. Next, since Y is smooth, thus regular, assertions iii), iv) are equivalent by the purity of the branch locus. Thus it is left to prove that iv) implies i). By contradiction, suppose that $\tilde{X} \rightarrow X$ is branched at some point $x \in X \setminus S'$. Then by the discussion before Claim 1 combined with Claim 1, it follows that there exists a prime divisor w of $K|k$ which is ramified in $\tilde{K}|K$ and x lies in the closure of the center x_w of w . Thus since $x \in X \setminus S'$ and $S' \subset X$ is closed, it follows that $x_w \in X \setminus S'$. By Claim 2 there exists a prolongation w_L of w to L which is ramified in $\tilde{L}|L$, and let y_w be the center of w_L on Y . Then by the compatibility of center of valuations with (separated) morphism, it follows that w_L and its restriction w to K have the same center x_0 on X_0 , and that center satisfies $y_w \mapsto x_0, x_w \mapsto x_0$. Now let $S'_0 \subset X_0$ be the closed subset such that $S' \subset X$ is the preimage of S'_0 , and let $T' \subset Y$ be the preimage of S'_0 . Then $x_0 \in X_0 \setminus S'_0$ (because $x \in X \setminus S'$), and $y_w \in Y \setminus T'$. Since w_L is branched in $\tilde{Y} \rightarrow Y$ and has center $y_L \in Y \setminus T'$, it follows that $\tilde{Y} \rightarrow Y$ is not étale above T' , contradiction! This concludes the proof of Lemma 2.5, thus of assertion 1) of Proposition 2.4.

To 2): Let $S'_0 \subset X_0$ be such that $S' \subset X$ is the preimage of S'_0 , and $T' \subset Y$ be the preimage of S'_0 in Y . In the notations from the proof of assertion 2) above, suppose that all $v \in D_{X \setminus S'}$ are unramified in $\tilde{K}|K$. We claim that all prime divisors w of $K|k$ are unramified in $\tilde{K}|K$. Indeed, by Claim 2 above and the discussion there after, it follows that a prime divisor w of $K|k$ is ramified in $\tilde{K}|K$ iff w has a prolongation w_L to L which is ramified in $\tilde{L}|L$ iff there exists $v_L \in D_Y$ which is ramified in $\tilde{L}|L$ iff the restriction $v := (v_L)|_K$ is ramified in $\tilde{K}|K$. In other words, $\Pi_{1, D_Y} = \Pi_{1, L}$ if and only if $\Pi_{1, D_{X \setminus S'}} = \Pi_{1, K}$, etc. \square

- *Reviewing divisorial lattices*

Let $D = D_X$ be a geometric set for $K|k$, where X is a quasi projective normal model of $K|k$. Then $\Gamma(X, \mathcal{O}_X)^{\times} =: \mathbb{G}_m(D)$, the group of principal divisors $\mathcal{H}_K(X) := K^{\times}/\mathbb{G}_m(D)$,

and $\text{Div}(X)$ depend on D only, and not on the specific X with $D_X = D$. Hence the exact sequence $0 \rightarrow \mathcal{H}_K(X) \xrightarrow{\text{div}} \text{Div}(X) \xrightarrow{\text{pr}} \mathfrak{C}\mathfrak{I}(X) \rightarrow 0$ depends only on D . To stress that, write:

$$0 \rightarrow \mathcal{H}_K(D) \xrightarrow{\text{div}} \text{Div}(D) \xrightarrow{\text{pr}} \mathfrak{C}\mathfrak{I}(D) \rightarrow 0.$$

Tensoring the above exact sequence with \mathbb{Z}/n for all $n = \ell^e$, $e > 0$, one gets exact sequence of \mathbb{Z}/n -modules $0 \rightarrow {}_n\mathfrak{C}\mathfrak{I}(D) \rightarrow \mathcal{H}_K(D)/n \rightarrow \text{Div}(D)/n \rightarrow \mathfrak{C}\mathfrak{I}(D)/n \rightarrow 0$, where ${}_n\mathfrak{C}\mathfrak{I}(D)$ is the n -torsion of $\mathfrak{C}\mathfrak{I}(D)$. Since $1 \rightarrow \mathbb{G}_m(D)/k^\times \rightarrow K^\times/k^\times \rightarrow \mathcal{H}_K(D) \rightarrow 1$ is an exact sequence of free abelian groups, $1 \rightarrow \mathbb{G}_m(D)/n \rightarrow K^\times/n \rightarrow \mathcal{H}_K(D)/n \rightarrow 1$ is exact as well. Hence if $U(D)/n \subset K^\times/n$ is the preimage of ${}_n\mathfrak{C}\mathfrak{I}(D)$ under $K^\times/n \rightarrow \mathcal{H}_K(D)/n$, it follows that the resulting sequence $1 \rightarrow \mathbb{G}_m(D)/n \rightarrow U(D)/n \rightarrow {}_n\mathfrak{C}\mathfrak{I}(D) \rightarrow 0$ is exact, thus the above long exact sequence above give rise canonically to a long exact sequence:

$$1 \rightarrow U(D)/n \rightarrow K^\times/n \rightarrow \text{Div}(D)/n \rightarrow \mathfrak{C}\mathfrak{I}(D)/n \rightarrow 0.$$

Taking projective limits over $n = \ell^e \rightarrow \infty$, we get the long exact sequence of ℓ -adic modules:

$$(\dagger) \quad 1 \rightarrow \widehat{U}(D) \hookrightarrow \widehat{K} \xrightarrow{\text{div}} \widehat{\text{Div}}(D) \xrightarrow{\text{pr}} \widehat{\mathfrak{C}\mathfrak{I}}(D) \rightarrow 0.$$

A geometric set D is called **complete regular like**, if for every geometric set $\tilde{D} \supseteq D$ one has that $\widehat{U}(D) = \widehat{U}(\tilde{D})$ and $\widehat{\mathfrak{C}\mathfrak{I}}(\tilde{D}) \cong \widehat{\mathfrak{C}\mathfrak{I}}(D) \oplus \mathbb{Z}_\ell^r$, where $r := |\tilde{D} \setminus D|$. The following hold:

- The complete regular like sets of prime divisors are quite abundant, namely: First, every geometric set is contained in a complete regular like set D . Second, if D complete regular like, then every geometric set containing D is complete regular like as well.
- For a complete regular like geometric set D , let $\mathfrak{C}\mathfrak{I}'^0(D) \subseteq \mathfrak{C}\mathfrak{I}'(D)$ be the maximal divisible, respectively ℓ -divisible subgroups of $\mathfrak{C}\mathfrak{I}(D)$. Then by structure of $\mathfrak{C}\mathfrak{I}(D)$, see e.g. [P3], Appendix, 7.3, it follow that $\mathfrak{C}\mathfrak{I}'(D)/\mathfrak{C}\mathfrak{I}'^0(D)$ is a (finite) torsion group of order prime to ℓ . Thus if $\text{Div}^0(D) \subseteq \text{Div}'(D)$ are the preimages of $\mathfrak{C}\mathfrak{I}'^0(D) \subseteq \mathfrak{C}\mathfrak{I}'(D)$ in $\text{Div}(D)$, it follows that $\text{Div}^0(D)_{(\ell)} = \text{Div}'(D)_{(\ell)}$ inside $\text{Div}(X)_{(\ell)}$.¹
- Both $\widehat{U}_K := \widehat{U}(D)$ and $\text{Div}'(D)_{(\ell)}$ are birational invariants of $K|k$.

We let $\mathcal{L}_K \subset \widehat{K}$ be the preimage of $\text{Div}'(D)_{(\ell)} \subset \widehat{\text{Div}}(D)$ under $\text{div} : \widehat{K} \rightarrow \widehat{\text{Div}}(D)$, and we call \mathcal{L}_K the **canonical divisorial** \widehat{U}_K -lattice for $K|k$. Notice that $\text{div}(\mathcal{L}_K) = \text{Div}'(D)_{(\ell)}$.

Next we recall the Galois theoretical counterpart of the above exact sequence (\dagger) .

Let D be a geometric (complete regular like) set of prime divisors of $K|k$. Recall the closed subgroup $T_D \subseteq \Pi_K$ generated by T_v , $v \in D$, and the resulting $1 \rightarrow T_D \rightarrow \Pi_K \rightarrow \Pi_{1,D} \rightarrow 1$. Interpreting the elements of $\mathcal{P}(K) = K^\times/k^\times \subset \widehat{K} = \text{Hom}(\Pi_K, \mathbb{Z}_\ell)$ as function on Π_K , one gets: Each T_v has a *unique* generator $\tau_v \in T_v$, called **canonical**, such that for uniformizing parameters $t_v \in \mathcal{O}_v$ one has: $t_v(\tau_v) = 1$. Then $j^v : \widehat{K} = \text{Hom}(\Pi_K, \mathbb{Z}_\ell) \rightarrow \text{Hom}(T_v, \mathbb{Z}_\ell) = \mathbb{Z}_\ell$, $\varphi \mapsto \varphi|_{T_v}$, is identified with the ℓ -adic completion of the valuation $v : K^\times \rightarrow \mathbb{Z}$.

Finally, the system of the maps $j^v : \widehat{K} \rightarrow \mathbb{Z}_\ell$, $v \in D$, gives rise by restriction canonically to

$$j^D = \bigoplus_v j^v : \mathcal{L}_K \rightarrow \text{Div}_D := \bigoplus_v \mathbb{Z}_\ell.$$

Let \mathfrak{F}_D be the free abelian pro- ℓ group on $(\tau_v)_{v \in D}$. The canonical projection $\mathfrak{F}_D \rightarrow T_D \subseteq \Pi_K$ gives rise to a long exact sequence $1 \rightarrow \mathfrak{R}_D \rightarrow \mathfrak{F}_D \rightarrow \Pi_K \rightarrow \Pi_{1,K} \rightarrow 1$ of pro- ℓ abelian groups, where $\mathfrak{R}_D \subset \mathfrak{F}_D$ is the relation module for the system of inertia generators $(\tau_v)_v$,

¹Recall that for an abelian group A , we denote $A_{(\ell)} := A \otimes \mathbb{Z}_{(\ell)}$.

and the first three terms of the exact sequence have no torsion. Since the ℓ -adic dual of \mathfrak{F}_D is canonically isomorphic to $\widehat{\text{Div}}_D$, by taking ℓ -adic duals, one gets an exact sequence:

$$(\dagger)_{\text{Gal}} \quad 1 \rightarrow \widehat{U}_D \hookrightarrow \widehat{K} \xrightarrow{\widehat{j}^D} \widehat{\text{Div}}_D \xrightarrow{\text{can}} \widehat{\mathfrak{C}}\mathfrak{I}_D \rightarrow 0,$$

in which each of the terms is the ℓ -adic dual of the corresponding pro- ℓ group, \widehat{j}^D is the ℓ -adic completion of the map j^D given above, and the other morphisms are canonical.

Now recalling that $D = D_X$ for some quasi projective normal model of $K|k$, it is obvious that the above exact sequences (\dagger) and $(\dagger)_{\text{Gal}}$ are canonically isomorphic, and in particular one has $\widehat{U}_D = \widehat{U}(D)$ inside \widehat{K} , and $\widehat{\mathfrak{C}}\mathfrak{I}_D \cong \widehat{\mathfrak{C}}\mathfrak{I}(D)$ canonically. And notice that the exact sequence $(\dagger)_{\text{Gal}}$ was constructed from Π_K endowed with the canonical system $(\tau_v)_{v \in D}$ of the inertia groups $(T_v)_{v \in D}$ only. *Unfortunately*, we do not have a group theoretical recipe (yet?) to identify the canonical system of inertia generators $(\tau_v)_v$ —say, up to simultaneous raising to some power $\epsilon \in \mathbb{Z}_\ell^\times$, from Π_K^c endowed with $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$. Nevertheless, using the theory of geometric decomposition graphs as developed in [P3], one proves the following:

Proposition 2.6. *In the above notations, the following hold:*

- 1) *There exist group theoretical recipes to recover/reconstruct from $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ the following:*
 - a) *The geometric sets of prime divisors D of $K|k$. Moreover, the group theoretical recipes single out the complete regular like sets of prime divisors.*
 - b) *Given a complete regular like set of prime divisors D , the recipes reconstruct the divisorial \widehat{U}_K -lattices of the form $\epsilon \cdot \mathcal{L}_K \subset \widehat{K}$, or equivalently, the subgroups of the form $\epsilon \cdot \text{Div}'(D)_{(\ell)} \subset \widehat{\text{Div}}(D)$, and systems of inertia generators $(\tau_v^\epsilon)_{v \in D}$ for all $\epsilon \in \mathbb{Z}_\ell^\times / \mathbb{Z}_{(\ell)}^\times$. Thus finally the recipes reconstruct the exact sequences of the form*

$$1 \rightarrow \widehat{U}_K \rightarrow \epsilon \cdot \mathcal{L}_K \xrightarrow{\widehat{j}^D} \epsilon \cdot \text{Div}(D)_{(\ell)} \xrightarrow{\text{can}} \widehat{\mathfrak{C}}\mathfrak{I}(D), \quad \epsilon \in \mathbb{Z}_\ell^\times.$$

- c) *Replacing any complete regular like set D by any geometric subset $D' \subset D$, one gets by restriction the corresponding $1 \rightarrow \widehat{U}_{D'} \rightarrow \epsilon \cdot \mathcal{L}_{D'} \xrightarrow{\widehat{j}^{D'}} \epsilon \cdot \text{Div}(D')_{(\ell)} \xrightarrow{\text{can}_{D'}} \widehat{\mathfrak{C}}\mathfrak{I}(D')$.*
- 2) *The recipes to recover the above information from $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ are invariant under isomorphisms of total decomposition graphs $\Phi : \mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{D}_L^{\text{tot}}}$ as follows: Φ defines bijections $D_K \rightarrow D_L$, say $v \mapsto w$, between the (complete regular like) geometric sets D_K for $K|k$, and D_L for $L|l$. Further, there exist $\epsilon \in \mathbb{Z}_\ell^\times$ and $\epsilon_v \in \epsilon \cdot \mathbb{Z}_{(\ell)}^\times$, $v \in D_K$, satisfying:*
 - a) $\Phi(\tau_v)_{v \in D_K} = (\tau_w^{\epsilon_v})_{w \in D_L}$, hence Φ gives rise to an isomorphism of $\mathbb{Z}_{(\ell)}$ -modules $\text{Div}(D_L)_{(\ell)} \rightarrow \epsilon \cdot \text{Div}(D_K)_{(\ell)}$ defined by via $j^w \mapsto \epsilon_v \cdot j^v$ for $v \mapsto w$.
 - b) *The Kummer isomorphism $\hat{\phi} : \widehat{L} \rightarrow \widehat{K}$ of Φ , i.e., the ℓ -adic dual of $\Phi : \Pi_K \rightarrow \Pi_L$, satisfies: $\hat{\phi}(\widehat{U}_L) = \hat{\phi}(\widehat{U}_K)$, $\hat{\phi}(\mathcal{L}_L) = \epsilon \cdot \mathcal{L}_K$.*

Proof. First, the assertion 1) follows from [P3], Propositions 22 and 23 (where the above facts were formulated in terms of geometric decomposition groups, rather than geometric sets of prime divisors). Second, for assertion 2), see [P3], Proposition 30, etc. \square

D) *Specialization techniques*

We begin by recalling briefly the specialization/reduction results concerning projective integral/normal/smooth k -varieties and (finite) k -morphisms between such varieties, see MUMFORD [Mu], II, §§ 7–8, ROQUETTE [Ro], and GROTHENDIECK-MURRE [G–M]. We first recall the

general facts and then come back with specifics in our situation. We consider the following context: k is an algebraically closed field, and Val_k is the space of all the valuations v of k . For $v \in \text{Val}_k$ we denote by $\mathcal{O}_v \subset k$ its valuation ring, and let $\mathcal{O}_v \rightarrow kv$, $a \mapsto \bar{a}$, be its residue field. One has the **reduction map** $\mathcal{O}_v[T_1, \dots, T_N] \rightarrow kv[T_1, \dots, T_N] =: \mathcal{R}_v$, $f \mapsto \bar{f}$, defined by mapping each coefficient of f to its residue in kv . In particular, the reduction map above gives rise to a reduction of the ideals $\mathfrak{a} \subset k[T_1, \dots, T_N]$ to ideals $\mathfrak{a}_v \subset \mathcal{R}_v$ defined by $\mathfrak{a} \mapsto \mathfrak{A} := \mathfrak{a} \cap \mathcal{O}_v[T_1, \dots, T_N]$ followed by $\mathfrak{A} \rightarrow \mathfrak{a}_v$, $f \mapsto \bar{f}$. Finally, we recall that Val_k carries the Zariski topology τ^{Zar} which has as a basis the subsets of the form

$$\mathcal{U}_A = \{v \in \text{Val}_k \mid v(a) = 0 \ \forall a \in A\}, \quad A \subset k^\times \text{ finite subsets.}$$

We notice the trivial valuation v_0 on k belongs to every Zariski open non-empty set, thus τ^{Zar} is a prefilter on Val_k . We will denote by \mathfrak{D} ultrafilters on Val_k with $\tau^{\text{Zar}} \subset \mathfrak{D}$.

Remarks 2.7. In the above notations, we consider/remark the following:

- 1) Let $*k := \prod_v kv / \mathfrak{D}$ be the **ultraproduct** of $(kv)_v$, and consider the **canonical embedding** of k into $*k$, defined by $a \mapsto (a_v) / \mathfrak{D}$, where $a_v = \bar{a}$ if a is a v -unit, and $a_v = 0$ else.
- 2) Let $\mathfrak{a} := (f_1, \dots, f_r) \subset R := k[T_1, \dots, T_N]$ be an ideal. The fact that \mathfrak{a} is a prime ideal, respectively that $V(\mathfrak{a}) = \text{Spec } R/\mathfrak{a} \subseteq \mathbb{A}_k^N$ is normal/smooth is an open condition involving the coefficients if f_1, \dots, f_r as parameters. Therefore, there exists a Zariski open subset $\mathcal{U}_\mathfrak{a} \subset \text{Val}_k$ such that for all $v \in \mathcal{U}_\mathfrak{a}$ the following hold:
 - a) $f_1, \dots, f_r \in \mathfrak{A}$, and $\mathfrak{a}_v = (\bar{f}_1, \dots, \bar{f}_r)$.
 - b) If \mathfrak{a} is prime, then so is $\mathfrak{a}_v = (\bar{f}_1, \dots, \bar{f}_r)$.
 - c) If $V(\mathfrak{a}) \hookrightarrow \mathbb{A}_k^N$ is a normal/smooth k -subvariety, then so is $V(\mathfrak{a}_v) \hookrightarrow \mathbb{A}_{kv}^N$.

Definition/Remark 2.8. In the context of Remark 2.7, we introduce notations as follows:

- 1) $\mathfrak{a}_* := \prod_v \mathfrak{a}_v / \mathfrak{D} \subset \prod_v \mathcal{R}_v =: \mathcal{R}_*$ is the **\mathfrak{D} -ultraproduct** of the ideals $(\mathfrak{a}_v)_v$. One has a **canonical embedding** $\mathfrak{a} \hookrightarrow \mathfrak{a}_*$ defined by $f \mapsto f_*$, where f_* is the image of f under $R \hookrightarrow \mathcal{R}_*$. In particular, $\mathfrak{a} = \mathfrak{a}_* \cap R$ inside \mathcal{R}_* .
- 2) $\mathfrak{a}k_* = (f_{1*}, \dots, f_{r*}) \subset Rk_* = k_*[T_1, \dots, T_N]$ is called the **finite part** of \mathfrak{a}_* . Notice that $V(\mathfrak{a}k_*) \hookrightarrow \mathbb{A}_{k_*}^N$ is nothing but the base change of $V(\mathfrak{a}) \hookrightarrow \mathbb{A}_k^N$ under $k \hookrightarrow k_*$.
- 3) Finally, if \mathfrak{a} is a prime ideal, then so \mathfrak{a}_* , and the canonical inclusion of k -algebras $R/\mathfrak{a} \hookrightarrow \mathcal{R}_*/\mathfrak{a}_*$ gives rise to inclusions of their quotient fields $\kappa(\mathfrak{a}) \hookrightarrow \kappa(\mathfrak{a}_*)$, and one has: The algebraic closure of $\overline{\kappa(\mathfrak{a})}$ of $\kappa(\mathfrak{a})$ and $\kappa(\mathfrak{a}_*)$ are linearly disjoint over $\kappa(\mathfrak{a})$.

Let V be a projective reduced k -scheme, and $V = \text{Proj } k[T_0, \dots, T_N] / \mathfrak{p} \hookrightarrow \mathbb{P}_k^N$ be a fixed projective embedding, where $\mathfrak{p} = (f_1, \dots, f_r) \subset k[T_0, \dots, T_N]$ is a homogeneous prime ideal. Then $\mathfrak{P} := \mathfrak{p} \cap \mathcal{O}_v[T_0, \dots, T_N]$ is a homogeneous prime ideal of $\mathcal{O}_v[T_0, \dots, T_N]$ and therefore, $\mathcal{V} := \text{Proj } \mathcal{O}_v[T_0, \dots, T_N] / \mathfrak{P} \hookrightarrow \mathbb{P}_{\mathcal{O}_v}^N$ is a projective \mathcal{O}_v -scheme, etc.

Fact 2.9. In the above notations, the following hold:

- 1) $\mathcal{V} \hookrightarrow \mathbb{P}_{\mathcal{O}_v}^N$ is the scheme theoretical closure of V under $V \hookrightarrow \mathbb{P}_k^N \hookrightarrow \mathbb{P}_{\mathcal{O}_v}^N$, and $V \hookrightarrow \mathbb{P}_k^N$ is the generic fiber of $\mathcal{V} \hookrightarrow \mathbb{P}_{\mathcal{O}_v}^N$. Further, the special fiber $\mathcal{V}_v \hookrightarrow \mathbb{P}_{kv}^N$ is reduced.
- 2) If $V = \cup_i V_i$ with V_i closed subsets, then $\mathcal{V} = \cup_i \mathcal{V}_i$, and $\mathcal{V}_v = \cup_i \mathcal{V}_{i,v}$. Further, if V is connected, so is \mathcal{V}_v , and if V is integral, then \mathcal{V}_v is of pure dimension equal to $\dim(V)$.
- 3) If V is integral and normal, then the local ring at any generic point η_i of \mathcal{V}_v is a valuation ring \mathcal{O}_{v,η_i} of $k(V)$ dominating \mathcal{O}_v .

Definition 2.10.

- 1) The valuation v_{η_i} with valuation ring $\mathcal{O}_{v_{\eta_i}}$ introduced at Fact 2.9, c), above is called a **Deuring constant reduction** of $k(V)$ at v .
- 2) The v -**reduction/specialization map** for closed subsets of V is defined by

$$\mathrm{sp}_v : \{S \subseteq V \mid S \text{ closed}\} \rightarrow \{\mathcal{S}'_v \subseteq \mathcal{V}_v \mid \mathcal{S}'_v \text{ closed}\}, \quad S \mapsto \mathcal{S}_v.$$

- 3) In particular, if $P \subset V$ is a prime Weil divisor, then $\mathcal{P} \subset \mathcal{V}$ is a relative Weil divisor of \mathcal{V} . Finally, sp_v gives rise to a **Weil divisor reduction/specialization homomorphism**

$$\mathrm{sp}_v : \mathrm{Div}(V) \rightarrow \mathrm{Div}(\mathcal{V}_v), \quad P \mapsto \sum_i \mathcal{P}_{v,i}$$

where $\mathcal{P}_{v,i}$ are the irreducible components of the special fiber \mathcal{P}_v of \mathcal{P} .

Fact 2.11. There exists a Zariski open nonempty set $\mathcal{U} \subset \mathrm{Val}_k$ such that each $v \in \mathcal{U}$ satisfies:

- a) If V is integral/normal/smooth, then so are \mathcal{V} and \mathcal{V}_v . In particular, if V is integral and normal, then \mathcal{V}_v is integral and normal, and v_{η_v} is called the **canonical (Deuring) constant reduction** of $k(V)$ at v , and we denote it by $v_{k(V)}$.
- b) If $S_1, \dots, S_r \subseteq V$ are distinct closed subsets of V , then $\mathrm{sp}_v(S_1), \dots, \mathrm{sp}_v(S_r)$ are distinct closed subsets of \mathcal{V}_v . And if $\mathcal{P} = \cup_i P_i$ is a NCD (normal crossings divisor) in V , then so is $\mathcal{P}_v = \cup_i \mathrm{sp}_v(P_i)$ in \mathcal{V}_v .
- c) $\mathrm{sp}_v : \mathrm{Div}(V) \rightarrow \mathrm{Div}(\mathcal{V}_v)$ is compatible with principal divisors, and therefore gives rise to a **reduction/specialization homomorphism** $\mathrm{sp}_v : \mathfrak{Cl}(V) \rightarrow \mathfrak{Cl}(\mathcal{V}_v)$.
- d) If $H \subset \mathbb{P}_k^N$ is a general hyperplane, then $\mathcal{H} \subset \mathbb{P}_{\mathcal{O}_v}^N$ is a general relative hyperplane and $\mathcal{H}_v \subset \mathbb{P}_{kv}^N$ is a general hyperplane, and $(\mathcal{H} \cap \mathcal{V})_v = \mathcal{H}_v \cap \mathcal{V}_v$ inside \mathbb{P}_{kv}^N .
- e) Let $\mathbb{G}_m(V) \subset k(V)$ be the group of invertible global section on V , and $\mathbb{G}_m(\mathcal{V}_v)$ be correspondingly defined. Then $\mathbb{G}_m(V)/k^\times \rightarrow \mathbb{G}_m(\mathcal{V}_v)/kv^\times$ is an isomorphism.

In the above context and notations, let $W \rightarrow V$ be a dominant morphism of projective integral normal k -varieties, which is generically finite, and $K := k(V) \hookrightarrow k(W) =: L$ be the corresponding finite field extension. Let $\tilde{L}|L$ be a finite Galois extension with $[\tilde{L} : L]$ relatively prime to $[L : K]$, and $\tilde{K}|K$ be the maximal Galois subextension of $K \hookrightarrow \tilde{L}$ having degree $[\tilde{K} : K]$ relatively prime to $[L : K]$. Then the canonical projection of Galois groups $H := \mathrm{Gal}(\tilde{L}|L) \rightarrow \mathrm{gal}(\tilde{K}|K) =: G$ is surjective. Further, let $\tilde{V} \rightarrow V$ and $\tilde{W} \rightarrow W$ be the normalizations of V in $K \hookrightarrow \tilde{K}$, respectively of W in $L \hookrightarrow \tilde{L}$. Then by the universal property of normalization, $\tilde{W} \rightarrow W \rightarrow V$ factors as $\tilde{W} \rightarrow \tilde{V} \rightarrow V$.

Finally, let $V \hookrightarrow \mathbb{P}_k^N$, etc., be projective k -embeddings, and consider the corresponding projective \mathcal{O}_v -schemes $V \hookrightarrow \mathcal{V}$, etc., as defined above. Then by general scheme theoretical non-sense, $W \rightarrow V$, etc., give rise to dominant rational \mathcal{O}_v -maps $\mathcal{W} \dashrightarrow \mathcal{V}$, etc.

Fact 2.12. There exists a Zariski open nonempty set $\mathcal{U} \subset \mathrm{Val}_k$ such that each $v \in \mathcal{U}$ satisfies:

- a) The rational maps \mathcal{O}_v -maps $\mathcal{W} \dashrightarrow \mathcal{V}$ above are actually morphisms of \mathcal{O}_v -schemes, and one has commutative diagrams of morphisms of projective \mathcal{O}_v -schemes

$$\begin{array}{ccc} \tilde{W} & \rightarrow & W \\ \downarrow & & \downarrow \\ \tilde{V} & \rightarrow & V \end{array} \quad \begin{array}{ccc} \tilde{\mathcal{W}} & \rightarrow & \mathcal{W} \\ \downarrow & & \downarrow \\ \tilde{\mathcal{V}} & \rightarrow & \mathcal{V} \end{array} \quad \begin{array}{ccc} \tilde{\mathcal{W}}_v & \rightarrow & \mathcal{W}_v \\ \downarrow & & \downarrow \\ \tilde{\mathcal{V}}_v & \rightarrow & \mathcal{V}_v \end{array}$$

where the LHS diagrams is the generic fiber of the middle one, respectively RHS diagrams is the special fiber of the middle one.

- b) Moreover, the degrees of the morphism which correspond to each other in the above diagrams are equal. Thus the canonical constant reductions $v_L, v_{\tilde{K}}, v_{\tilde{L}}$ are the unique prolongations of v_K to $L, \tilde{K},$ and \tilde{L} , respectively.
- c) The residue field extension $\tilde{K}v_{\tilde{K}}|Kv_K$ is the maximal Galois subextension of $\tilde{L}v_{\tilde{L}}|Lv_L$ which has degree relatively prime to $[L : K] = [Lv_L|Kv_K]$.

Fact 2.13. Let W be a projective smooth k -variety, and $T' \subset W$ be either empty, or the support of a normal crossings divisor in W , and set $\mathcal{W}' := W \setminus T'$. For $v \in \text{Val}_k$ define correspondingly $\mathcal{T}' \hookrightarrow \mathcal{W}, \mathcal{T}'_v \hookrightarrow \mathcal{W}_v$, thus $\mathcal{W}' = \mathcal{W} \setminus \mathcal{T}', \mathcal{W}'_v = \mathcal{W}_v \setminus \mathcal{T}'_v$, and let $D_{\mathcal{W}'}$ and $D_{\mathcal{W}'_v}$ be the corresponding geometric sets of Weil prime divisors. Then there exists a Zariski open nonempty set $\mathcal{U} \subset \text{Val}_k$ such that for $v \in \mathcal{U}$ the special fiber \mathcal{W}_v is smooth, \mathcal{T}'_v either empty or a NCD, and the following hold: First, by the purity of the branch locus, and second, by GROTHENDICK–MURRE [G–M] applied to \mathcal{W}'_v , the canonical maps of pro- ℓ abelian fundamental groups below are isomorphisms:

$$\Pi_{1,D_{\mathcal{W}'}} \rightarrow \Pi_1(W') \rightarrow \Pi_1(\mathcal{W}') \leftarrow \Pi_1(\mathcal{W}'_v) \leftarrow \Pi_{1,D_{\mathcal{W}'_v}}.$$

We now come back to the context of Theorem 1.1. Recall the notations and the context from Preparation/Notations 2.3, and the k -morphisms $X \rightarrow X_0 \leftarrow Y, Z \rightarrow X$, further the closed subset $S_0 \subset X_0$ and the algebraic set of prime divisors $D = D_{X_0 \setminus S_0}$, and finally, the preimages $S \rightarrow S_0 \leftarrow T$ of S_0 under $X \rightarrow X_0 \leftarrow Y$. Finally, by Proposition 2.4, one has canonical identifications, respectively a surjective canonical projection as below:

$$\Pi_{1,D_{X \setminus S}} = \Pi_1(X \setminus S) \rightarrow \Pi_1(X) = \Pi_{1,D_X} = \Pi_{1,K}.$$

After choosing projective embeddings for each of the k -varieties X_0, X, Y, Z , we get for every $v \in \text{Val}_k$ the corresponding \mathcal{O}_v -schemes $X_0 \hookrightarrow \mathcal{X}_0 \leftarrow \mathcal{X}_{0v}$, etc., and for every of the k -morphisms above, we get corresponding dominant rational \mathcal{O}_v -maps, etc. Then applying Facts 2.9–2.13, one has the following:

Fact/Notations 2.14. There exists a Zariski open subset $\mathcal{U} \subset \text{Val}_k$ of valuations v satisfying the following: The special fiber of each of the above k -varieties is irreducible and normal/smooth if the generic fiber was so. Further, all the dominant rational \mathcal{O}_v -maps under discussion above are actually \mathcal{O}_v -morphisms such that the degrees of the generic fiber k -morphisms and the corresponding special fiber kv -morphisms are equal. In particular, the canonical constant reductions v_L and v_M are the unique prolongations of v_K to L , respectively M . Finally, let $S'_0 \subset X_0$ be either empty or equal to S_0 , hence its preimages $S' \subset X$ and $T' \subset Y$ are either empty or equal to S , respectively T . We set $X'_0 := X_0 \setminus S'_0, X' := X \setminus S', Y' := Y \setminus T'$, and for $v \in \mathcal{U}$ consider the corresponding $X_0 \rightarrow \mathcal{X}_0 \leftarrow \mathcal{X}_{0v}, S'_0 \rightarrow \mathcal{S}'_0 \leftarrow \mathcal{S}'_{0v}$ and the resulting $X'_0 \rightarrow \mathcal{X}'_0 \leftarrow \mathcal{X}'_{0v}$, etc. Then the resulting kv -morphisms satisfy the following:

- a) $\mathcal{S}_v \subset \mathcal{X}_v, \mathcal{T}_v \subset \mathcal{Y}_v$ are the preimages of \mathcal{S}_{0v} under $\mathcal{X}_v \rightarrow \mathcal{X}_{0v} \leftarrow \mathcal{Y}_v$, and the restriction \mathcal{D}_0 of $D_{\mathcal{Y}_v}$ to Kv_K satisfies $D_{\mathcal{X}_{0v}} \subseteq \mathcal{D}_0 \subseteq D_{\mathcal{X}_v}$.
- b) Further, setting $\mathcal{D} := D_{\mathcal{X}_{0v} \setminus \mathcal{S}_{0v}}$, one has that $\mathcal{D} \subseteq \text{sp}_v(D)$, and further:
 - $\mathcal{Y}_v \rightarrow \mathcal{X}_{0v}$ is a prime to ℓ alteration above \mathcal{S}_{0v} , hence \mathcal{T}_v is NCD in \mathcal{Y}_v .
 - $\mathcal{X}_v \rightarrow \mathcal{X}_{0v}$ are projective normal models of $Kv_K|kv$, and $\mathcal{D} \subseteq \mathcal{D}_0 \subseteq D_{\mathcal{X}_v}$.
 - $\mathcal{Z}_v \rightarrow \mathcal{X}_v$ is a finite generically normal alteration with $\text{Aut}(Mv_M|Kv_K) = \text{Aut}(M|K)$.

- c) Applying Proposition 2.4 to the fibers, one has:
- The canonical maps $\Pi_{1,D_{X'}} \rightarrow \Pi_1(X')$ and $\Pi_{1,D_{\mathcal{X}'_v}} \rightarrow \Pi_1(\mathcal{X}'_v)$ are isomorphisms.
 - $\Pi_{1,D_X} \rightarrow \Pi_1(X) \rightarrow \Pi_{1,K}$ and $\Pi_{1,D_{\mathcal{X}_v}} \rightarrow \Pi_1(\mathcal{X}_v) \rightarrow \Pi_{1,Kv_K}$ are isomorphisms.
- d) By the functoriality of fundamental group and Fact 2.13 one has:
- $X' \hookrightarrow \mathcal{X}' \hookleftarrow \mathcal{X}'_v$ give rise to surjective projections $\Pi_1(X') \rightarrow \Pi_1(\mathcal{X}') \leftarrow \Pi_1(\mathcal{X}'_v)$.
 - $Y' \hookrightarrow \mathcal{Y}' \hookleftarrow \mathcal{Y}'_v$ give rise to canonical isomorphisms $\Pi_1(Y') \rightarrow \Pi_1(\mathcal{Y}') \leftarrow \Pi_1(\mathcal{Y}'_v)$.

Proposition 2.15. *The canonical maps $\Pi_1(X') \rightarrow \Pi_1(\mathcal{X}') \leftarrow \Pi_1(\mathcal{X}'_v)$ are isomorphisms.*

Proof. We consider the canonical commutative diagram of \mathcal{O}_v -morphism, and the corresponding maps of fundamental groups:

$$\begin{array}{ccccc}
 Y' & \hookrightarrow & \mathcal{Y}' & \hookleftarrow & \mathcal{Y}'_v \\
 (*) \downarrow & & \downarrow & & \downarrow \\
 X' & \hookrightarrow & \mathcal{X}' & \hookleftarrow & \mathcal{X}'_v
 \end{array}
 \qquad
 \begin{array}{ccccc}
 \Pi_1(Y') & \rightarrow & \Pi_1(\mathcal{Y}') & \leftarrow & \Pi_1(\mathcal{Y}'_v) \\
 (\dagger) \downarrow & & \downarrow & & \downarrow \\
 \Pi_1(X') & \rightarrow & \Pi_1(\mathcal{X}') & \leftarrow & \Pi_1(\mathcal{X}'_v)
 \end{array}$$

The homomorphisms in these diagrams satisfy: First, since the vertical morphism in the diagram (*) are finite, having degree equal to $[L : K]$, the vertical homomorphisms in the diagram (\dagger) have open images of index dividing $[L : K]$. Thus since $[L : K]$ is prime to ℓ , it follows that the vertical maps in the diagram (\dagger) are actually surjective. Further, the morphisms in the first row of the diagram (\dagger) are isomorphisms, and those of the second row are surjective by Fact/Notations 2.14, d). We have to prove that the homomorphisms in the second row of the diagram (\dagger) are isomorphisms too.

Claim 1. $\Pi_1(X') \rightarrow \Pi_1(\mathcal{X}')$ is injective.

Indeed, let $\tilde{X} \rightarrow X'$ any finite abelian ℓ -power degree etale cover of X' , $\tilde{K} = k(\tilde{X})$ its function field, and $\tilde{\mathcal{X}} \rightarrow \mathcal{X}'$ the normalization of \mathcal{X}' in $K \hookrightarrow \tilde{K}$. We claim that $\tilde{\mathcal{X}} \rightarrow \mathcal{X}'$ is etale. Since being etale is an open condition, it is sufficient to prove that the cover of the special fiber $\tilde{\mathcal{X}}_v \rightarrow \mathcal{X}'_v$ is etale. (Notice that we do not know yet whether $\tilde{\mathcal{X}}_v$ is reduced.) Let $\tilde{L} := L\tilde{K}$, and $\tilde{Y} \rightarrow Y'$ be the normalization of Y' in $L \hookrightarrow \tilde{L}$. Since $\Pi_1(Y') \rightarrow \Pi_1(X')$ is surjective, and $\tilde{X} \rightarrow X'$ is etale, it follows that $\text{Gal}(\tilde{L}|L)$ is a quotient of $\Pi_1(Y')$. Thus the normalization $\tilde{\mathcal{Y}} \rightarrow \mathcal{Y}'$ of \mathcal{Y}' in $L \hookrightarrow \tilde{L}$ is etale. Further, since the morphisms in the first row of the diagram (\dagger) are isomorphisms, it follows that the corresponding $\tilde{\mathcal{Y}} \rightarrow \mathcal{Y}'$, $\tilde{\mathcal{Y}}_v \rightarrow \mathcal{Y}'_v$ are etale covers of integral \mathcal{O}_v -schemes satisfying $\deg(\tilde{Y} \rightarrow Y) = \deg(\tilde{\mathcal{Y}}_v \rightarrow \mathcal{Y}_v)$. In particular, the canonical constant reduction v_L has a unique prolongation to $v_{\tilde{L}}$ to \tilde{L} , and $v_{\tilde{L}}|v_L$ is totally inert. Since $v_L|v_K$ is totally inert by the fact that $v \in \mathcal{U}$, it follows that $v_{\tilde{L}}|v_K$ is totally inert as well. Thus v_K has a unique prolongation $v_{\tilde{K}}$ to \tilde{K} as well, and $v_{\tilde{K}}|v_K$ is totally inert, i.e., $\text{Gal}(\tilde{K}|K)$ equals the decomposition group above v_K , and the inertia group above v_K is trivial. Equivalently, $\tilde{\mathcal{X}}_v$ is integral, and $Kv_K = \kappa(\mathcal{X}'_v) \hookrightarrow \kappa(\tilde{\mathcal{X}}_v) = \tilde{K}v_{\tilde{K}}$ is Galois extension with $\text{Gal}(\tilde{K}v_{\tilde{K}}|Kv_K) = \text{Gal}(\tilde{K}|K)$. Hence we have the following: $\tilde{K}v_{\tilde{K}}|Kv_K$ is an abelian ℓ -power degree extension such that $\tilde{L}v_{\tilde{L}}$ is the compositum $\tilde{L}v_{\tilde{L}} = Lv_L\tilde{K}v_{\tilde{K}}$, and $\tilde{\mathcal{Y}}_v \rightarrow \mathcal{Y}'_v$ is etale. By Proposition 2.4, it follows that that $\tilde{\mathcal{X}}_v \rightarrow \mathcal{X}'_v$ is etale as well. Thus $\tilde{\mathcal{X}} \rightarrow \mathcal{X}'$ is etale, and Claim 1 is proved. Conclude that $\Pi_1(X') \rightarrow \Pi_1(\mathcal{X}')$ is an isomorphism.

Claim 2. $\Pi_1(\mathcal{X}'_v) \rightarrow \Pi_1(\mathcal{X}')$ is injective.

In order to prove the claim, via the canonical isomorphisms $\Pi_1(Y') \rightarrow \Pi_1(\mathcal{Y}') \leftarrow \Pi_1(\mathcal{Y}'_v)$, we can identify these groups with a finite \mathbb{Z}_ℓ -module Π , thus the canonical surjective projections

$\Pi \rightarrow \Pi_1(\mathcal{X}'_v) \rightarrow \Pi_1(\mathcal{X}')$ are defined as quotients of Π by $\Delta := \ker(\Pi \rightarrow \Pi_1(\mathcal{X}'))$ and $\Delta_v := \ker(\Pi \rightarrow \Pi_1(\mathcal{X}'_v))$, which are finite \mathbb{Z}_ℓ -submodules of Π . The following conditions are obviously equivalent:

- i) $\Pi_1(\mathcal{X}'_v) \rightarrow \Pi_1(\mathcal{X}')$ is an isomorphism
- ii) $\Delta_v = \Delta$
- iii) $\ell\Delta + \Delta_v = \Delta$

where iii) \Rightarrow ii) follows by Nakayama's Lemma, because $\Delta \subseteq \Pi_1(Y)$ is a finite \mathbb{Z}_ℓ -module. On the other hand, since Δ is a finite \mathbb{Z}_ℓ -module, there exist only finitely many subgroups Σ such that $\ell\Delta \subseteq \Sigma \subseteq \Delta$. Thus for all sufficiently large ℓ^e [precisely, for $\Pi \rightarrow \bar{\Pi} := \Pi/\ell^e$, one must have that $\Delta/\ell \rightarrow \bar{\Delta}/\ell$ is injective], the above conditions are equivalent as well to:

- iv) $\bar{\Delta}_v = \bar{\Delta}$.
- v) $\Pi_1(\mathcal{X}'_v)/\ell^e \rightarrow \Pi_1(\mathcal{X}')/\ell^e$ is an isomorphism.

Thus we conclude that it is sufficient to show that given $e > 0$, there exists a Zariski open non-empty subset $\mathcal{U}_e \subset \mathcal{U}$ of valuations v such that condition v) above is satisfied. By contradiction, suppose that this is not the case, hence for every Zariski open subset $\mathcal{U}' \subset \mathcal{U}$, there exists some $v \in \mathcal{U}'$ such that condition v) does not hold at v . We notice that Π/ℓ^e is finite, thus it has only finitely many quotients. Therefore, there exist a quotient $\Pi/\ell^e \rightarrow G$ such that the set $\Sigma_G := \{v \in \mathcal{U} \mid G = \Pi_1(\mathcal{X}'_v)/\ell^e \rightarrow \Pi_1(\mathcal{X}')/\ell^e \text{ is not an isomorphism}\}$ is in \mathfrak{D} . For $v \in \Sigma$, and the etale cover $\tilde{\mathcal{X}}_v \rightarrow \mathcal{X}'_v$, it follows that the base change $\tilde{\mathcal{X}}_v \times_{\mathcal{X}'_v} \mathcal{Y}'_v \rightarrow \mathcal{Y}'_v$ of $\tilde{\mathcal{X}}_v \rightarrow \mathcal{X}'_v$ to \mathcal{Y}'_v is etale and has Galois group G . Since the morphisms in the first row of the diagram (†) are isomorphisms, there exists a unique etale cover $\tilde{\mathcal{Y}} \rightarrow \mathcal{Y}'$ with $\text{Gal}(\tilde{\mathcal{Y}}|\mathcal{Y}') = G$ and special fiber $\tilde{\mathcal{Y}}_v = \tilde{\mathcal{X}}_v \times_{\mathcal{X}'_v} \mathcal{Y}'_v$. Further, if $\tilde{Y} \rightarrow Y$ is the generic fiber of $\tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$, recalling the discussion/notations from Definition/Remark 2.8, one has the following: Let $\kappa(\mathcal{X}'_*) := \prod_v \kappa(\mathcal{X}'_v)/\mathfrak{D}$, and $\kappa(\tilde{\mathcal{X}}_*)$, $\kappa(\mathcal{Y}'_*)$ and $\kappa(\tilde{\mathcal{Y}}_*)$ be correspondingly defined. Then by general principles of ultraproducts of fields one has that $\kappa(\tilde{\mathcal{Y}}'_*) = \kappa(\tilde{\mathcal{X}}'_*)L$, $\kappa(\tilde{\mathcal{Y}}_*) = \kappa(\tilde{\mathcal{X}}'_*)\tilde{L}$, and $\kappa(\mathcal{X}'_*) \rightarrow \kappa(\tilde{\mathcal{X}}_*)$, $\kappa(\mathcal{Y}'_*) \rightarrow \kappa(\tilde{\mathcal{Y}}_*)$ are Galois field extensions with Galois group G . On the other hand, by Definition/Remark 2.8, c), the fields \bar{K} and $\kappa(\mathcal{X}'_*)$ are linearly disjoint over K . Therefore, there exists a unique finite abelian ℓ -power degree extension $\tilde{K}|K$ such that $\kappa(\tilde{\mathcal{X}}_*) = \kappa(\mathcal{X}'_*)\tilde{K}$. But then the linear disjointness of \bar{K} and $\kappa(\mathcal{X}'_*)$ over K implies that $\tilde{L} = L\tilde{K}$. And since $\tilde{Y} \rightarrow Y'$ is etale with Galois group G , by Lemma 2.5, it follows that the normalization $\tilde{X} \rightarrow X'$ of X' in $K \hookrightarrow \tilde{K}$ is etale (with Galois group G). Hence the corresponding $\tilde{\mathcal{X}} \rightarrow \mathcal{X}'$ is an etale cover with Galois group G , etc., contradiction! This concludes the proof of Proposition 2.15. \square

Fact 2.16. We finally notice the following:

- a) For $v \in \mathcal{U}$, let $H \subset \mathbb{P}_{kv}^N$ be a general hyperplane. Then $\mathcal{X}_H := \mathcal{X}_v \cap H$ is a projective normal kv -variety, and a Weil prime divisor of \mathcal{X}_v , whose valuation we denote by v_H . And $\mathcal{S}_H := \mathcal{S}_v \cap H$ is a closed subset of \mathcal{X}_H , and we let $D_{\mathcal{X}_H \setminus \mathcal{S}_H}$ be the corresponding geometric set of Weil prime divisors of the function field $\kappa(\mathcal{X}_H)|kv$ of \mathcal{X}_H .
- b) For a valuation \mathfrak{v} of K , we let $U_{\mathfrak{v}} := \mathcal{O}_{\mathfrak{v}}^\times \subset K^\times$ be the \mathfrak{v} -units, and $\hat{j}_{\mathfrak{v}} : \hat{U}_{\mathfrak{v}} \rightarrow \widehat{K\mathfrak{v}}$ be the ℓ -adic completion of the \mathfrak{v} -reduction homomorphism $j_{\mathfrak{v}} : U_{\mathfrak{v}} \rightarrow K\mathfrak{v}^\times$.

Proposition 2.17. *Let \mathcal{U} be the Zariski open non-empty set from Fact/Notations 2.14. In the above notations, for $v \in \mathcal{U}$ we set $\mathfrak{v} := v_H \circ v_K$, thus $K\mathfrak{v} = (Kv_K)v_H$ is a function field over*

kv , and consider the geometric sets of prime divisors: $D_{X \setminus S}$ of $K|k$, $D_{\mathcal{X}_v \setminus \mathcal{S}_v}$ of the function field $Kv_K|kv$, respectively $D_{\mathcal{X}_H \setminus \mathcal{S}_H}$ of the function field $K\mathfrak{v}|kv$. Then one has:

- 1) $\widehat{U}_K \subseteq \widehat{U}_{D_{X \setminus S}} \subseteq \widehat{U}_{v_K}$, and \hat{j}_{v_K} maps $\widehat{U}_K \subseteq \widehat{U}_{D_{X \setminus S}}$ isomorphically onto $\widehat{U}_{Kv_K} \subseteq \widehat{U}_{D_{\mathcal{X}_v \setminus \mathcal{S}_v}}$.
- 2) $\mathfrak{v} := v_H \circ v_K$ is a quasi prime divisor of $K|k$ satisfying $\mathfrak{v}|_k = v$ and $K\mathfrak{v} = \kappa(\mathcal{X}_v)$.
- 3) $\widehat{U}_K \subseteq \widehat{U}_{D_{X \setminus S}} \subseteq \widehat{U}_{\mathfrak{v}}$, and $\hat{j}_{\mathfrak{v}}$ maps $\widehat{U}_K \subseteq \widehat{U}_{D_{X \setminus S}}$ isomorphically onto $\widehat{U}_{K\mathfrak{v}} \subseteq \widehat{U}_{D_{\mathcal{X}_H \setminus \mathcal{S}_H}}$.

Proof. To 1): The ℓ -adic duals of the isomorphisms $\Pi_1(X) \leftarrow \Pi_1(\mathcal{X}_v)$, $\Pi_1(X \setminus S) \leftarrow \Pi_1(\mathcal{X}_v \setminus \mathcal{S}_v)$ are the isomorphism $\widehat{U}_K \rightarrow \widehat{U}_{Kv_K}$, respectively $\widehat{U}(X \setminus S) \rightarrow \widehat{U}(\mathcal{X}_v \setminus \mathcal{S}_v)$. On the other hand, by Kummer theory it follows that the last two isomorphisms are defined by the ℓ -adic completion of the residue homomorphism $j_{v_K} : U_{v_K} \rightarrow Kv_K^\times$. This completes the proof of assertion 1).

The proof of assertion 2) is clear, because w_H is trivial on kv , thus the restriction of \mathfrak{v} to k equals the one of v_K , which is v by the definition of v_K .

Finally, assertion 3) is proved actually in POP [P3], at the end of the proof of assertion 1) of loc.cit., Proposition 23, and we omit that argument here. \square

We next make a short preparation for the second application of the specialization techniques. Let $W \rightarrow V$ be a finite morphism of projective normal k -varieties with function fields $K := k(V)$ and $M := k(W)$. Consider the inclusion $\mathcal{I} : \text{Div}(V) \rightarrow \text{Div}(W)$ and the norm $\mathcal{N} : \text{Div}(W) \rightarrow \text{Div}(V)$ maps, which map principal divisors to principal divisors and $\mathcal{N} \circ \mathcal{I} = [M : K] \cdot \text{id}_{\text{Div}(V)}$ on $\text{Div}(V)$. Thus one has a commutative diagram of the form:

$$\begin{array}{ccccccc} 1 & \rightarrow & \mathcal{H}(M) & \hookrightarrow & \text{Div}(W) & \longrightarrow & \mathfrak{Cl}(W) \rightarrow 0 \\ & & \uparrow \downarrow & & \uparrow \downarrow & & \uparrow \downarrow \\ 1 & \rightarrow & \mathcal{H}(K) & \hookrightarrow & \text{Div}(V) & \longrightarrow & \mathfrak{Cl}(V) \rightarrow 0 \end{array}$$

where the upper arrows are defined by \mathcal{I} , the down arrows are defined by \mathcal{N} , and the composition $\downarrow \circ \uparrow$ is the multiplication by $[M : K]$. Thus \mathcal{I} and $\mathcal{N} \circ \mathcal{I}$ map elements of infinite order to such, and if $[x] \in \mathfrak{Cl}(X)$ has finite order $o_{[x]}$, the other of $\mathcal{I}([x])$ divides $o_{[x]}$, and the order of $\mathcal{N} \circ \mathcal{I}([x])$ is precisely $o_{[x]}/n$, where $n := \text{g.c.d.}(o_{[x]}, [M : K])$.

Fact 2.18. There exists a Zariski open nonempty set $\mathcal{U} \subset \text{Val}_k$ such that each $v \in \mathcal{U}$ satisfies:

- 1) The inclusion/norm morphisms $\mathcal{I} : \text{Div}(V) \rightarrow \text{Div}(W)$, $\mathcal{N} : \text{Div}(W) \rightarrow \text{Div}(V)$ are compatible with the specialization maps $\text{sp}_{W,v}$ and $\text{sp}_{V,v}$, and with the inclusion/norm morphisms $\mathcal{I}_v : \text{Div}(\mathcal{W}_v) \rightarrow \text{Div}(\mathcal{V}_v)$ and $\mathcal{N}_v : \text{Div}(\mathcal{W}_v) \rightarrow \text{Div}(\mathcal{V}_v)$, i.e., one has:

$$\text{sp}_{W,v} \circ \mathcal{I} = \text{sp}_{V,v} \circ \mathcal{I}_v, \quad \mathcal{N}_v \circ \text{sp}_{W,v} = \mathcal{N} \circ \text{sp}_{V,v}$$

- 2) The specialization maps sp_v are compatible with principal divisors, hence define specialization morphisms $\text{sp}_v : \mathfrak{Cl}(V) \rightarrow \mathfrak{Cl}(\mathcal{V}_v)$, $\text{sp}_v : \mathfrak{Cl}(W) \rightarrow \mathfrak{Cl}(\mathcal{W}_v)$ which are compatible with the inclusion/norm morphisms, thus one has commutative diagrams:

$$\begin{array}{ccccccc} \mathcal{H}(M) & \xrightarrow{\text{sp}_v} & \mathcal{H}(Mv_M) & & \text{Div}(W) & \xrightarrow{\text{sp}_v} & \text{Div}(\mathcal{W}_v) & & \mathfrak{Cl}(W) & \xrightarrow{\text{sp}_v} & \mathfrak{Cl}(\mathcal{W}_v) \\ \uparrow \downarrow & & \uparrow \downarrow \\ \mathcal{H}(K) & \xrightarrow{\text{sp}_v} & \mathcal{H}(Kv_K) & & \text{Div}(V) & \xrightarrow{\text{sp}_v} & \text{Div}(\mathcal{V}_v) & & \mathfrak{Cl}(V) & \xrightarrow{\text{sp}_v} & \mathfrak{Cl}(\mathcal{V}_v) \end{array}$$

in which the vertical maps are the defined by the inclusion/norm homomorphisms, and the horizontal maps are the corresponding specialization homomorphism (as introduced in Fact 2.12).

- 3) Recalling that $\mathfrak{C}\ell^0(\bullet) \subseteq \mathfrak{C}\ell'(\bullet)$ are the maximal divisible, respectively ℓ -divisible, subgroups of $\mathfrak{C}\ell(\bullet)$, it follows that the specialization morphism $\text{sp}_v : \mathfrak{C}\ell(V) \rightarrow \mathfrak{C}\ell(\mathcal{V}_v)$ satisfies: $\text{sp}_v(\mathfrak{C}\ell^0(V)) \subseteq \mathfrak{C}\ell^0(\mathcal{V}_v)$, $\text{sp}_v(\mathfrak{C}\ell'(V)) \subseteq \mathfrak{C}\ell'(\mathcal{V}_v)$, and correspondingly for W .
- 4) Finally, for $[x] \in \mathfrak{C}\ell(V)$ and $[y] := \mathcal{I}([x])$, one has:
 - a) The order of $\text{sp}_v[x]$ is infinite if and only if the order of $\text{sp}_v[y]$ is infinite.
 - b) If $\text{sp}_v[y]$ has finite order, then $\text{sp}_v[x]$ has finite order, and their orders satisfy:

$$o_{[M:K] \cdot \text{sp}_v[x]} \mid o_{\text{sp}_v[y]} \mid o_{\text{sp}_v[x]}.$$

We conclude this preparation by mentioning the following fundamental fact about specialization of points of abelian k -varieties, which follows from Appendix, Theorem 1 by JOSSEN, which generalizes an earlier result by PINK [Pk] to arbitrary finitely generated base fields.

Fact 2.19. Let A be an abelian variety over k , and $x \in A(k)$. Then there exists $\mathcal{U}_A \subset \text{Val}_k$ Zariski open such for $v \in \mathcal{U}_A$, there exists an abelian \mathcal{O}_v -scheme \mathcal{A} with generic fiber A and special fiber \mathcal{A}_v , and the specialization $x_v \in \mathcal{A}_v(kv)$ of $x \in A(k)$ satisfying:

- 1) If x has finite order o_x , then the order o_{x_v} of x_v is finite, and $o_x = o_{x_v}$.
- 2) If o_x is infinite, then for every $\ell^e > 0$ there exists $\Sigma_x \subset \mathcal{U}_A$ Zariski dense satisfying:
 - i) kv is an algebraic closure of a finite field with $\text{char}(kv) \neq \ell$.
 - ii) o_{x_v} is divisible by ℓ^e .

Proof. Let $R \subset k$ be a finitely generated \mathbb{Z} -algebra, say $R = \mathbb{Z}[a_1, a_1^{-1}, \dots, a_n, a_n^{-1}]$, $a_i \in k^\times$, over which A and $x \in A(k)$ are defined, i.e., there exists an R -abelian scheme \mathcal{A}_R and an R -point $x_R \in \mathcal{A}_R(R)$, such that A and $x \in A(k)$ are the base changes of \mathcal{A}_R and $x_R \in \mathcal{A}_R(R)$ under the canonical inclusion $R \hookrightarrow k$. We denote by \mathcal{A}_s and $x_s \in \mathcal{A}_s(\kappa(s))$ the fibers of \mathcal{A}_R , respectively x_R , at $s \in \text{Spec } R$, and notice that if the order o_x of x is finite, then $o_x = o_{x_R}$. Thus the set of all the points $s \in \text{Spec } R$ such that $o_x = o_{x_R} = o_{x_s}$ is a Zariski open subset in $\text{Spec } R$. Thus replacing R by a $R[a^{-1}]$ for a properly chosen $a \in R$, $a \neq 0$, we can suppose without loss of generality that $o_x = o_{x_R} = o_{x_s}$ for all $s \in \text{Spec } R$.

Let \mathcal{U}_A be the set of all the valuations v of k with $R \subset \mathcal{O}_v$. Notice that $v \in \mathcal{U}_A$ if and only if v has a center $s \in \text{Spec } R$ via the inclusion $R \hookrightarrow k$ if and only if a_i is a v -unit for $i = 1, \dots, n$. In particular, the base change $\mathcal{A}_{\mathcal{O}_v} := \mathcal{A}_R \times_R \mathcal{O}_v$ is an abelian \mathcal{O}_v -scheme with generic fiber A , and the base change $x_{\mathcal{O}_v}$ of x_R under $R \hookrightarrow \mathcal{O}_v$ is an \mathcal{O}_v -point of $\mathcal{A}_{\mathcal{O}_v}$ whose generic fiber is x , and whose special fiber $x_v \in \mathcal{A}_v(kv)$ is the base change of x_s under the canonical embedding $\kappa(s) \hookrightarrow kv$. Thus the assertions 1) and 2) of Fact 2.19 follow from the discussion above, whereas the more difficult assertion 3) follows from Appendix, Theorem 1 by JOSSEN applied to the abelian R -scheme \mathcal{A}_R over $\text{Spec } R$ (recalling that R is a \mathbb{Z} -algebra of finite type). Namely, the set Σ_{x_R} of all $s \in \text{Spec } R$ with o_{x_s} divisible by ℓ^e is Zariski dense. Hence the set Σ_x of all the v which have center in Σ_{x_R} is Zariski dense in Val_k , etc. \square

We now define the quasi arithmetical \widehat{U} -lattice $\mathcal{L}_K^0 \subseteq \mathcal{L}_K$ mentioned at Step 3 in subsection A). Recall that a \mathbb{Z}_ℓ -submodule $\Delta \subset \widehat{K}$ is said to have finite co-rank, if there exists some geometric set of prime divisors D such that $\Delta \subseteq \widehat{U}_D$. Since the family of geometric sets for $K|k$ is closed under intersection (and union), it follows that the set of all the finite corank \mathbb{Z}_ℓ -submodules $\Delta \subset \widehat{K}$ is filtered w.r.t. inclusion, and that their union \widehat{K}_{fin} is given by:

$$\widehat{K}_{\text{fin}} = \bigcup_D \widehat{U}_D \subset \widehat{K}, \quad D \text{ geometric.}$$

Clearly, \widehat{K}_{fin} is a birational invariant of $K|k$, and if $\mathcal{L}'_K \subset \widehat{K}$ is any divisorial \widehat{U}_K -lattice, then $\mathcal{L}'_K \subset \widehat{K}_{\text{fin}}$ and $\widehat{K}_{\text{fin}} = \mathcal{L}'_K \otimes \mathbb{Z}_\ell$. Further, if $L|l$ is another function field over an algebraically closed field l , then every embedding $L|l \hookrightarrow K|k$ induces an embedding $\widehat{L}_{\text{fin}} \hookrightarrow \widehat{K}_{\text{fin}}$.

Remarks 2.20. In the context/notations from subsection B) above and Proposition 2.17, let $J_K : K^\times \rightarrow \widehat{K}$, $J_{K\mathfrak{v}} : K\mathfrak{v}^\times \rightarrow \widehat{K\mathfrak{v}}$ be the ℓ -adic completion homomorphisms. Let $\mathcal{L}'_K \subset \widehat{K}$, $\mathcal{L}'_{K\mathfrak{v}} \subset \widehat{K\mathfrak{v}}$ be fixed divisorial lattices, and $\mathcal{L}_K \subset \widehat{K}$, $\mathcal{L}_{K\mathfrak{v}} \subset \widehat{K\mathfrak{v}}$ be the canonical ones. Then by mere definitions, there exist unique $\epsilon, \epsilon_{\mathfrak{v}} \in \mathbb{Z}_\ell^\times / \mathbb{Z}_{(\ell)}^\times$ such that $\mathcal{L}'_K = \epsilon \cdot \mathcal{L}_K$ and $\mathcal{L}'_{K\mathfrak{v}} = \epsilon_{\mathfrak{v}} \cdot \mathcal{L}_{K\mathfrak{v}}$.

1) For every $\eta \in \mathbb{Z}_\ell^\times$ the following conditions are equivalent:²

- i) $\eta \cdot J_K(K^\times)_{(\ell)} \subset \mathcal{L}'_K$.
- i)' $\eta \cdot J_K(K^\times)_{(\ell)} \cap \mathcal{L}'_K$ is non-trivial.
- ii) $\eta \cdot \mathcal{L}_K = \mathcal{L}'_K$
- ii)' $\eta \cdot \mathcal{L}_K \cap \mathcal{L}'_K$ contains \widehat{U}_K strictly.

Actually, $\eta := \epsilon$ is the unique $\eta \in \mathbb{Z}_\ell^\times / \mathbb{Z}_{(\ell)}^\times$ satisfying the above equivalent conditions.

2) Correspondingly, the same is true about $\mathcal{L}'_{K\mathfrak{v}}$, and $\eta_{\mathfrak{v}} := \epsilon_{\mathfrak{v}}$ is the unique $\eta_{\mathfrak{v}} \in \mathbb{Z}_\ell^\times / \mathbb{Z}_{(\ell)}^\times$ such that $\epsilon_{\mathfrak{v}} \cdot J_{K\mathfrak{v}}(K\mathfrak{v}^\times) \subseteq \mathcal{L}'_{K\mathfrak{v}}$, etc.

3) Since $J_{\mathfrak{v}}(U_{\mathfrak{v}} \cap J_K(K^\times)) = J_{K\mathfrak{v}}(K\mathfrak{v}^\times)$, it follows that $\epsilon/\epsilon_{\mathfrak{v}} \in \mathbb{Z}_\ell^\times / \mathbb{Z}_{(\ell)}^\times$ is unique such that $J_{\mathfrak{v}}(\mathcal{L}'_K \cap \widehat{U}_{\mathfrak{v}})$ and $(\epsilon/\epsilon_{\mathfrak{v}}) \cdot \mathcal{L}'_{K\mathfrak{v}}$ are equal modulo $J_{\mathfrak{v}}(\widehat{U}_K) \cdot \widehat{U}_{K\mathfrak{v}} \subset \widehat{K\mathfrak{v}}_{\text{fin}}$.

4) Conclude that for every $\eta_{\mathfrak{v}} \in \mathbb{Z}_\ell^\times / \mathbb{Z}_{(\ell)}^\times$ the following conditions are equivalent:

- j) $\eta_{\mathfrak{v}} \cdot J_{K\mathfrak{v}}(K\mathfrak{v}^\times)_{(\ell)} \subseteq \widehat{J}_{\mathfrak{v}}(\mathcal{L}'_K \cap \widehat{U}_{\mathfrak{v}})$.
- jj) $\eta_{\mathfrak{v}} \cdot J_{K\mathfrak{v}}(K\mathfrak{v}^\times)_{(\ell)} \cap \widehat{J}_{\mathfrak{v}}(\mathcal{L}'_K \cap \widehat{U}_{\mathfrak{v}})$ is non-trivial.

Moreover, $\eta_{\mathfrak{v}} := \epsilon/\epsilon_{\mathfrak{v}}$ is the unique $\eta_{\mathfrak{v}} \in \mathbb{Z}_\ell^\times / \mathbb{Z}_{(\ell)}^\times$ satisfying the conditions j), jj).

5) For every finite corank \mathbb{Z}_ℓ -submodule $\Delta \subset \widehat{K}_{\text{fin}}$ with $\widehat{U}_K \subset \Delta$, one has: There exists a Zariski open non-empty subset $\mathcal{U}_\Delta \subset \text{Val}_k$ such that for every $v \in \mathcal{U}_\Delta$ there exist “many” quasi prime divisors \mathfrak{v} with $\mathfrak{v}|_k = v$ and the following hold:

- a) $\Delta \subset \widehat{U}_{\mathfrak{v}}$, and $J_{\mathfrak{v}}$ is injective on Δ and maps Δ into $\widehat{K\mathfrak{v}}_{\text{fin}}$.
- b) For $\epsilon, \epsilon_{\mathfrak{v}}$ from 1), 2) above one has: $\epsilon \cdot (\Delta \cap \mathcal{L}_K) \subset \mathcal{L}'_K$ iff $\epsilon_{\mathfrak{v}} \cdot J_{\mathfrak{v}}(\Delta \cap \mathcal{L}_K) \subset \mathcal{L}'_{K\mathfrak{v}}$.

Definition 2.21. Let \mathcal{L}_K be the canonical divisorial \widehat{U}_K -lattice for $K|k$. For finite corank \mathbb{Z}_ℓ -modules $\Delta \subset \widehat{K}_{\text{fin}}$ with $\widehat{U}_K \subset \Delta$, let \mathcal{D}_Δ be the quasi prime divisors \mathfrak{v} such that $k\mathfrak{v}$ is algebraic over a finite field, $\text{char}(k\mathfrak{v}) \neq \ell$, and \mathfrak{v} satisfies conditions 5a), 5b) above. We set

$$\Delta(\mathfrak{v}) := \widehat{U}_K \cdot \{x \in \Delta \cap \mathcal{L}_K \mid \widehat{J}_{\mathfrak{v}}(x) \in J_{K\mathfrak{v}}(K\mathfrak{v}^\times)_{(\ell)}\}, \quad \Delta^0 := \bigcap_{\mathfrak{v} \in \mathcal{D}_\Delta} \Delta(\mathfrak{v}),$$

and call $\mathcal{L}_K^0 := \bigcup_{\Delta} \Delta^0$ the quasi arithmetical \widehat{U}_K -lattice of $\mathcal{G}_{\mathcal{D}_K}$.

Proposition 2.22. *In the above notations, suppose that $\text{td}(K|k) > 2$. The following hold:*

- 1) $\widehat{U}_K \cdot J_K(K^\times) \subseteq \mathcal{L}_K^0$ and $\mathcal{L}_K^0 / (\widehat{U}_K \cdot J_K(K^\times))$ is a torsion $\mathbb{Z}_{(\ell)}$ -module.
- 2) Let $\Phi \in \text{Isom}^c(\Pi_K, \Pi_L)$ define an isomorphism $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{D}_L^{\text{tot}}}$, and $\widehat{\phi} : \widehat{L} \rightarrow \widehat{K}$ be its Kummer isomorphism. Then for all $\epsilon \in \mathbb{Z}_\ell^\times$ one has: $\widehat{\phi}(\mathcal{L}_L) = \epsilon \cdot \mathcal{L}_K$ iff $\widehat{\phi}(\mathcal{L}_L^0) = \epsilon \cdot \mathcal{L}_K^0$.

² Recall that for every abelian group A we denote $A_{(\ell)} := A \otimes \mathbb{Z}_{(\ell)}$.

Proof. To 1): First, the inclusion $\widehat{U}_K \cdot J_K(K^\times) \subseteq \mathcal{L}_K^0$ is clear, because $\hat{j}_v(J_K(K^\times)) \subseteq J_{Kv}(Kv^\times)$ for all v , and therefore $\hat{j}_v(J_K(K^\times)_{(\ell)}) \subseteq J_{Kv}(Kv^\times)_{(\ell)}$ as well. The torsion assertion is much more involved. Let D_K be a complete regular like set of prime divisors for $K|k$, and recalling the notations from Proposition 2.6, consider the corresponding canonical exact sequence:

$$1 \rightarrow \widehat{U}_K \rightarrow \mathcal{L}_K \xrightarrow{\hat{j}^{D_K}} \text{Div}(D_K)_{(\ell)} \xrightarrow{\text{can}} \widehat{\mathfrak{C}}\mathfrak{I}(D_K).$$

The fact that $\mathcal{L}_K^0 / (\widehat{U}_K \cdot J_K(K^\times))$ is a torsion $\mathbb{Z}_{(\ell)}$ -module is equivalent to the fact that for every $x \in \mathcal{L}_K^0 \subset \mathcal{L}_K$, there exists some positive integer $n > 0$ such that $\hat{j}^{D_K}(x^n) \in \hat{j}^{D_K}(K^\times)$, thus principal. Let $x \in \mathcal{L}_K^0$ be a fixed element throughout.

- *By contradiction*, suppose that $\hat{j}^{D_K}(x^n)$ is not a principal divisor for any $n > 0$.

Let $D' \subseteq D_K$ be any geometric set of prime divisors for $K|k$ such that $x \in \widehat{U}_{D'}$. Then $x \in \widehat{U}_D$ for every geometric set $D \subseteq D'$. Hence considering a projective normal variety X_0 with $D_K \subset D_{X_0}$, we can choose a closed subset $S_0 \subset X_0$ such that $D := D_{X_0 \setminus S_0} \subseteq D'$, and $X_0 \setminus S_0$ is smooth. Recalling the context from Fact/Notations 2.14, we consider $X \rightarrow X_0$, and the preimage $S \subset X$ of S_0 under $X \rightarrow X_0$, one has: D_X is a complete regular like set of prime divisors for $K|k$ (because it contains D_K which is so). Further, since $X_0 \setminus S_0$ is smooth, one has $\Pi_{1, X_0 \setminus S_0} = \Pi_1(X_0 \setminus S_0)$, and by Proposition 2.4, it follows that all the canonical projections below are isomorphisms:

$$\Pi_{1, D} = \Pi_{1, D_{X_0 \setminus S_0}} \rightarrow \Pi_{1, D_X \setminus S} \rightarrow \Pi_1(X \setminus S) \rightarrow \Pi_1(X_0 \setminus S_0).$$

Therefore, the corresponding ℓ -adic duals are isomorphic as well, hence $\widehat{U}_D = \widehat{U}_{X \setminus S}$ inside \widehat{K} . Since \mathcal{L}_K is a birational invariant of $K|k$, considering the canonical exact sequence

$$1 \rightarrow \widehat{U}_K \rightarrow \mathcal{L}_K \xrightarrow{\hat{j}^{D_X}} \text{Div}(X)_{(\ell)} \xrightarrow{\text{can}} \widehat{\mathfrak{C}}\mathfrak{I}(X),$$

one has that $x \in \mathcal{L}_K^0 \subset \mathcal{L}_K$ satisfies: $x \in \widehat{U}_D = \widehat{U}_{X \setminus S}$, and therefore $x \in \mathcal{L}_K^0 \cap \widehat{U}_{X \setminus S}$ inside \widehat{K} . Further, $\hat{j}^{D_X}(x^n)$ is not a principal divisor for any positive integer $n > 0$.

Next recall that $\mathfrak{C}\mathfrak{I}^0(X) \subset \mathfrak{C}\mathfrak{I}'(X)$ are the maximal divisible, respectively ℓ -divisible, subgroups of $\mathfrak{C}\mathfrak{I}(X)$, and that $\text{Div}^0(X) \subseteq \text{Div}'(X)$ are their preimages in $\text{Div}(X)$, respectively. Since D_X is complete regular like, by mere definitions one has that $\hat{j}^{D_X}(\mathcal{L}_K) = \text{Div}'(X)_{(\ell)}$. Therefore, $\hat{j}^{D_X}(x) \in \text{Div}'(X)_{(\ell)}$, thus its divisor class $[x]$ satisfies $[x] \in \mathfrak{C}\mathfrak{I}'(X)$. And since $\hat{j}^{D_X}(x^n)$ is not principal for any $n > 0$, it follows that $[x] \in \mathfrak{C}\mathfrak{I}'(X)$ has infinite order, thus the same is true for every multiple $m[x]$, $m \neq 0$. On the other hand, by the structure of $\mathfrak{C}\mathfrak{I}(X)$, see e.g. POP [P3], Appendix, it follows that $\mathfrak{C}\mathfrak{I}'(X)/\mathfrak{C}\mathfrak{I}^0(X)$ is a finite prime-to- ℓ torsion group. Thus we conclude that some multiple $m[x]$ lies in $\mathfrak{C}\mathfrak{I}^0(X)$ and has infinite order. *Mutatis mutandis*, we can replace $x \in \mathcal{L}_K^0$ by its power $x^m \in \mathcal{L}_K^0$, thus without loss we can suppose that $[x] \in \mathfrak{C}\mathfrak{I}^0(X)$ has infinite order.

Now let $\mathcal{U} \subset \text{Val}_k$ be the Zariski open subset introduced at Fact/Notations 2.14, and consider $\mathcal{U}^{\max} := \{v \in \mathcal{U} \mid kv = \overline{\mathbb{F}}_q \text{ for some prime } q \neq \ell\}$. Then by general valuation theoretical non-sense, it follows that \mathcal{U}^{\max} is Zariski dense in Val_k .

Step 1. We claim that there exists a Zariski dense subset Σ_x of Val_k with $\Sigma_x \subset \mathcal{U}^{\max}$ such that $\text{sp}_v[x] \in \mathfrak{C}\mathfrak{I}(\mathcal{X}_v)$ has order divisible by ℓ . Indeed, in order to do so, recall the finite generically normal alteration $Z \rightarrow X$ of X from Fact/Notations 2.14. Since Z is a projective smooth k -variety, the connected component Pic_Z^0 of Pic_Z is an abelian k -variety. Further, the image $[y] = \mathcal{I}([x])$ of $[x]$ under $\mathcal{I} : \mathfrak{C}\mathfrak{I}(X) \rightarrow \mathfrak{C}\mathfrak{I}(Z)$ satisfies: First, $[y]$ lies in the divisible

part $\mathfrak{C}\ell^0(Z)$ of $\mathfrak{C}\ell(Z)$, because $[x]$ lies in the divisible part $\mathfrak{C}\ell^0(X)$ of $\mathfrak{C}\ell^0(X)$, and second, $[y]$ has infinite order by Fact 2.18. On the other hand, the divisible part $\mathfrak{C}\ell^0(Z)$ is nothing but the k -rational points of the abelian variety Pic_Z^0 . Thus by Fact 2.19 above, it follows that for every given positive integer $e > 0$, there exists $\Sigma_x \subset \mathcal{U}^{\max}$ which is dense in Val_k such that all $v \in \Sigma_x$ satisfy: The specialization $\text{sp}_v([y]) \in \text{Pic}_{Z_v}^0(kv) = \mathfrak{C}\ell^0(Z_v)$ has order divisible by ℓ^e . On the other hand, by Fact 2.18, 4), one has that the order of $\text{sp}_v([x])$ is divisible by the order of $\text{sp}_v([y])$, thus by ℓ^e . We thus conclude that $o_{\text{sp}_v([x])}$ is divisible by ℓ , as claimed.

Step 2. Recalling that $\widehat{U}_{D_{X \setminus S}} = \widehat{U}_D$ and $x \in \mathcal{L}_K^0 \cap \widehat{U}_D$, set $\Delta := \widehat{U}_{D_{X \setminus S}}$ and keep in mind the definition of \mathcal{D}_Δ . We claim that for every $v \in \mathcal{U}^{\max}$ there exist some $\mathfrak{v} \in \mathcal{D}_\Delta$ such that $v = \mathfrak{v}|_K$ (or in other words, \mathcal{U}^{\max} is contained in the restriction of \mathcal{D}_Δ to K). Indeed, for every $v \in \mathcal{U}^{\max}$, consider the general hyperplane sections $\mathcal{X}_H := H \cap \mathcal{X}_v$ and $\mathcal{S}_H := H \cap \mathcal{S}_v$, thus $\mathcal{S}_H \subset \mathcal{X}_H$ is a closed subset. Then \mathcal{X}_H is a projective normal variety over kv , and $\mathcal{X}_H \subset \mathcal{X}_v$ is a Weil prime divisor, say with valuation v_H . Then setting $D_H := D_{\mathcal{X}_H \setminus \mathcal{S}_H}$, and $\mathfrak{v} := v_H \circ v_K$, by Proposition 2.17, 2), 3), it follows that $\mathfrak{v} \in \mathcal{D}_\Delta$.

Step 3. Consider $\mathfrak{v} = v_H \circ v_K$ with $v \in \mathcal{U}^{\max}$ and v_H as above. Then by Proposition 2.17 it follows that $\widehat{U}_K \subseteq \widehat{U}_{D_{X \setminus S}} \subseteq \widehat{U}_{v_K}$, and \hat{j}_{v_K} maps $\widehat{U}_K \subseteq \widehat{U}_{D_{X \setminus S}}$ bijectively onto $\widehat{U}_{Kv_K} \subseteq \widehat{U}_{D_{\mathcal{X}_v \setminus \mathcal{S}_v}}$, and $\widehat{U}_K \subseteq \widehat{U}_{D_{X \setminus S}} \subseteq \widehat{U}_{\mathfrak{v}}$, and $\hat{j}_{\mathfrak{v}}$ maps $\widehat{U}_K \subseteq \widehat{U}_{D_{X \setminus S}}$ bijectively onto $\widehat{U}_{K\mathfrak{v}} \subseteq \widehat{U}_{D_{\mathcal{X}_H \setminus \mathcal{S}_H}}$. And setting $x_H := \hat{j}_{\mathfrak{v}}(x) = \hat{j}_{v_H}(\hat{j}_{v_K}(x))$, it follows that $x_H \in \widehat{U}_{K\mathfrak{v}} \cdot J_{K\mathfrak{v}}(K\mathfrak{v}^\times)$ by the fact that $x \in \Delta^0 \subseteq \Delta(\mathfrak{v})$. Thus in particular, x_H has a trivial divisor class $[x_H] = 0$.

On the other hand, if $v \in \Sigma_x$, then the divisor class of $[x_v]$ in $\mathfrak{C}\ell(\mathcal{X}_v)$ has finite order $o_{v,x}$ divisible by ℓ . Let $\text{div}_v(x_v) = \sum_i n_i P_i$ with distinct Weil prime divisors P_i of \mathcal{X}_v . Since $\mathcal{X}_H := H \cap \mathcal{X}_v$ is a general hyperplane section, it follows that $Q_i := H \cap P_i$ are distinct prime Weil divisors of \mathcal{X}_H , and the divisor class of $[x_H] := [\sum_i n_i Q_i]$ in $\mathfrak{C}\ell(\mathcal{X}_H)$ has order equal to the order of $[x]$. This contradicts the fact proved above that $[x_H] = 0$.

To 2): Since $\text{td}(K|k) > 2$, it follows that for all quasi prime divisors \mathfrak{v} of $K|k$, one has $\text{td}(K\mathfrak{v}|k\mathfrak{v}) > 1$. Let $\mathcal{V}_{\mathfrak{v}}$ be the generalized quasi prime $(d-1)$ -divisors $\tilde{\mathfrak{v}}$ with $\mathfrak{v} < \tilde{\mathfrak{v}}$. One has:

a) We claim that $k\mathfrak{v}$ is an algebraic closure of a finite field iff $k\tilde{\mathfrak{v}}$ is an algebraic closure of a finite field for all $\tilde{\mathfrak{v}} \in \mathcal{V}_{\mathfrak{v}}$. Indeed, if $k\mathfrak{v}$ is not an algebraic closure of a finite field, then one has: First, $K\mathfrak{v}|k\mathfrak{v}$ is a function field with $\text{td}(K\mathfrak{v}|k\mathfrak{v}) = \text{td}(K|k) - 1$. Second, choosing any prime $(d-2)$ -divisor $\tilde{\mathfrak{w}}$ of the function field $K\mathfrak{v}|k\mathfrak{v}$, it follows that the valuation theoretical composition $\tilde{\mathfrak{v}} := \tilde{\mathfrak{w}} \circ \mathfrak{v}$ is a quasi prime $(d-1)$ -divisor of $K|k$ with $K\tilde{\mathfrak{v}} = K\tilde{\mathfrak{w}}$, thus $k\mathfrak{v} = k\tilde{\mathfrak{v}}$. Hence $k\mathfrak{v}$ is an algebraic closure of a finite field iff $k\tilde{\mathfrak{v}}$ is so, as claimed.

b) Let $\mathcal{T}^1(K) \subset \Pi_K$ be the topological closure of the minimized quasi divisorial inertia $\cup_{\mathfrak{v} \in \mathcal{Q}_{K|k}} T_{\mathfrak{v}}^1 \subset \Pi_K$. For every given $\tilde{\mathfrak{v}} \in \mathcal{V}_{\mathfrak{v}}$, let $\mathcal{T}^1(K\tilde{\mathfrak{v}}) \subset \Pi_{K\tilde{\mathfrak{v}}}$ be the image of $\mathcal{T}^1 \cap Z_{\tilde{\mathfrak{v}}}^1$ under the canonical projection $Z_{\tilde{\mathfrak{v}}}^1 \rightarrow \Pi_{\tilde{\mathfrak{v}}}^1 = Z_{\tilde{\mathfrak{v}}}^1/T_{\tilde{\mathfrak{v}}}^1$. Then [P5], Theorem 4.2, a), gives a group theoretical recipe to decide whether $k\tilde{\mathfrak{v}}$ is an algebraic closure of a finite field. In particular, combining this with the discussion at point a) above, it follows that there are group theoretical recipes to characterize the quasi prime divisors \mathfrak{v} having $k\mathfrak{v}$ an algebraic closure of a finite field in terms of the set $\mathcal{T}^1(K)$. Moreover, if $k\mathfrak{v}$ is an algebraic closure of a finite field, the quasi prime divisors and the prime divisors of $K\mathfrak{v}|k\mathfrak{v}$ coincide (because $k\mathfrak{v}$ being algebraic over a finite field, has non non-trivial valuations). Thus we conclude that the minimized $(r-1)$ -divisorial groups in $\Pi_{K\mathfrak{v}}^1$ are precisely the images $T_{\tilde{\mathfrak{v}}}^1/T_{\mathfrak{v}}^1 \subset Z_{\tilde{\mathfrak{v}}}^1/T_{\mathfrak{v}}^1$ of the minimized quasi r -divisorial groups $T_{\tilde{\mathfrak{v}}}^1 \subset Z_{\tilde{\mathfrak{v}}}^1$ of the quasi prime r -divisors $\tilde{\mathfrak{v}} > \mathfrak{v}$. On the

other hand, by [P4], Propostion 3.5 and TOPAZ, [To1], Theorem 8, there are group theoretical recipes which recover the groups $T_{\mathfrak{v}}^1 \subset Z_{\mathfrak{v}}^1 \subset \Pi_K$ from $\Pi_K^c \rightarrow \Pi_K$, provided $r < d := \text{td}(K|k)$. To recover $T_{\mathfrak{v}}^1 \subset Z_{\mathfrak{v}}^1$ for $\mathfrak{v} > \mathfrak{v}$ for quasi prime d -divisors \mathfrak{v} , one uses [P5], Theorem 1.2, as follows: First, by mere definitions, for any given quasi prime r -divisor $\mathfrak{v} > \mathfrak{v}$, there exists a exists chain of generalized quasi prime divisors $\mathfrak{v} =: \mathfrak{v}_1 < \dots < \mathfrak{v}_r := \mathfrak{v}$. And since $k\mathfrak{v}$ is an algebraic closure of a finite field, all the $\mathfrak{v}_m/\mathfrak{v}$ are trivial on $k\mathfrak{v}$, hence $\mathfrak{v}_m/\mathfrak{v}$ are generalized *prime divisors* of $K\mathfrak{v}|k\mathfrak{v}$. In particular, if $r = d$, after setting $w := \mathfrak{v}_{d-1}$, in the terminology from [P5], Theorem 1.2, one has: The quasi prime d -divisors $\mathfrak{v}_d := \mathfrak{v}$ above are actually c.r. w -divisors of $K|k$. Hence by [P5], Theorem 1.2, one can recover the minimized d -divisorial groups $T_{\mathfrak{v}}^1 \subset Z_{\mathfrak{v}}^1 \subset \Pi_K$ from Π_K endowed with $\mathcal{T}^1(K)$ and $T_{\mathfrak{v}_{d-1}}^1 \subset Z_{\mathfrak{v}_{d-1}}^1$.

We thus finally conclude that for all generalized quasi prime divisors $\mathfrak{v} > \mathfrak{v}$, one can recover the generalized quasi divisor ail groups $T_{\mathfrak{v}}^1 \subset Z_{\mathfrak{v}}^1 \subset \Pi_K$. Hence via the canonical projection $Z_{\mathfrak{v}}^1 \rightarrow \Pi_{K\mathfrak{v}}^1 = Z_{\mathfrak{v}}^1/Z_{\mathfrak{v}}^1$ one can recover all the generalized quasi divisorial groups $T_{\mathfrak{v}}^1/T_{\mathfrak{v}}^1 \subset Z_{\mathfrak{v}}^1/T_{\mathfrak{v}}^1 \subset \Pi_{K\mathfrak{v}}^1$ as well, for all $\mathfrak{v} > \mathfrak{v}$. Thus finally the total decomposition graph $\mathcal{G}_{\mathcal{D}_{K\mathfrak{v}}^{\text{tot}}}$, can be recovered from Π_K^c .

c) Moreover, the group theoretical recipes to, first, check whether $k\mathfrak{v}$ is an algebraic closure of a finite field, and second, if so, reconstruct $\mathcal{G}_{\mathcal{D}_{K\mathfrak{v}}^{\text{tot}}}$, are invariant under all isomorphisms $\Phi \in \text{Isom}^c(\Pi_K, \Pi_L)$. Indeed, every such isomorphism Φ maps the minimized quasi r -divisorial groups $T_{\mathfrak{v}}^1 \subset Z_{\mathfrak{v}}^1$ for $K|k$ isomorphically onto the minimized quasi r -divisorial groups $T_{\mathfrak{w}}^1 \subset Z_{\mathfrak{w}}^1$ for $L|l$, provided $r < \text{td}(K|k)$. Hence Φ maps $\mathcal{T}^1(K)$ homeomorphically onto $\mathcal{T}^1(L)$. Thus if \mathfrak{v} and \mathfrak{w} correspond to each other under Φ , i.e., $\Phi(T_{\mathfrak{v}}^1) = T_{\mathfrak{w}}^1$ and $\Phi(Z_{\mathfrak{v}}^1) = Z_{\mathfrak{w}}^1$, then Φ gives rise to a residual isomorphism

$$\Phi_{\mathfrak{v}} : \Pi_{K\mathfrak{v}}^1 = Z_{\mathfrak{v}}^1/T_{\mathfrak{v}}^1 \rightarrow Z_{\mathfrak{w}}^1/T_{\mathfrak{w}}^1 = \Pi_{L\mathfrak{w}}^1 .$$

Further, $\Phi_{\mathfrak{v}}$ maps $\mathcal{T}^1(K\mathfrak{v})$ homeomorphically onto $\mathcal{T}^1(L\mathfrak{w})$, etc., hence finally gives rise to an isomorphism of total decomposition graphs $\Phi_{\mathfrak{v}} : \mathcal{G}_{\mathcal{D}_{K\mathfrak{v}}^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{D}_{L\mathfrak{w}}^{\text{tot}}}$.

d) Now suppose that $k\mathfrak{v}$ is an algebraic closure of a finite field of characteristic $\neq \ell$. Then by Theorems 2.2 and 2.1 of POP [P4], it follows that there exists $\epsilon_{\mathfrak{v}} \in \mathbb{Z}_{\ell}^{\times}$ such that $\Phi_{\epsilon_{\mathfrak{v}}} := \epsilon_{\mathfrak{v}} \cdot \Phi_{\mathfrak{v}}$ is defined by an isomorphism of the pure inseparable closures $\iota_{\mathfrak{v}} : L\mathfrak{w}^{\ell} \rightarrow K\mathfrak{v}^{\ell}$ of $L\mathfrak{w}$ and $K\mathfrak{v}$ respectively. Moreover, $\epsilon_{\mathfrak{v}}$ is unique up to multiplication by powers $p_{\mathfrak{v}}^m$, $m \in \mathbb{Z}$, of $p_{\mathfrak{v}} := \text{char}(k\mathfrak{v})$. In particular, the Kummer isomorphism $\hat{\phi}_{\epsilon_{\mathfrak{v}}} : \widehat{L\mathfrak{w}} \rightarrow \widehat{K\mathfrak{v}}$ of $\Phi_{\epsilon_{\mathfrak{v}}}$ is nothing but the ℓ -adic completion $\hat{\phi}_{\epsilon_{\mathfrak{v}}} = \hat{\iota}_{\mathfrak{v}}$ of $\iota_{\mathfrak{v}}$.

e) Next we consider the case that $k\mathfrak{v}$ is an algebraic closure of \mathbb{F}_{ℓ} . Then one does not have a result corresponding loc.cit. above. We claim that, nevertheless, the fact that $\text{char}(k\mathfrak{v}) = \ell$ can be distinguished from the case $\text{char}(k\mathfrak{v}) \neq \ell$ as follows: First, applying the reconstruction recipe from loc.cit., one either gets the function field $K\mathfrak{v}|k\mathfrak{v}$ (and experts think that this is the case), or maybe the reconstruction recipe does not work, and the reconstruction result is not a function field. In the former case, one has: Raising to the power ℓ is a field morphism on the reconstructed function field $K\mathfrak{v}|k\mathfrak{v}$ iff $\text{char}(k\mathfrak{v}) = \ell$. In the latter case, we must have $\text{char}(k\mathfrak{v}) = \ell$, because otherwise the reconstruction result is a function field, namely the function field $K\mathfrak{v}|k\mathfrak{v}$. We thus conclude that one can check whether $\text{char}(k\mathfrak{v}) = \ell$.

Note also that the case e) above can occur only if $\text{char}(k) = 0$, because by hypothesis of Theorem 1.1, one has $\text{char}(k) \neq \ell$.

(*) Finally, putting everything together, we conclude the following: Let \mathfrak{v} be a quasi prime divisor of $K|k$ and \mathfrak{w} a quasi prime divisor of $L|l$ which correspond to each other under Φ . The by the discussion at a), b), c), e), above, it follows that $k\mathfrak{v}$ is an algebraic closure of a finite field of characteristic $\neq \ell$ if and only if $l\mathfrak{w}$ is an algebraic closure of a finite field of characteristic $\neq \ell$. If so, then there exists an isomorphism $\phi_{\epsilon_{\mathfrak{v}}} : K\mathfrak{v}^i|k\mathfrak{v} \rightarrow L\mathfrak{w}^i|l\mathfrak{w}$ defining $\Phi_{\epsilon_{\mathfrak{v}}} := \epsilon_{\mathfrak{v}} \cdot \Phi_{\mathfrak{v}}$ for some $\epsilon_{\mathfrak{v}} \in \mathbb{Z}_{\ell}^{\times}$, which is unique up to powers p^m , $m \in \mathbb{Z}$, of $p_{\mathfrak{v}} := \text{char}(k\mathfrak{v})$.

Next recall that since Φ maps $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ isomorphically onto $\mathcal{G}_{\mathcal{D}_L^{\text{tot}}}$, it follows by subsection B), Proposition 2.6, 2), b) that replacing Φ by some multiple $\epsilon^{-1} \cdot \Phi$ with $\epsilon \in \mathbb{Z}_{\ell}^{\times}$ properly chosen, *mutatis mutandis*, we can suppose that the Kummer isomorphism $\hat{\phi} : \hat{L} \rightarrow \hat{K}$ of Φ maps \mathcal{L}_L isomorphically onto \mathcal{L}_K . In particular, since \mathcal{L}_L is the unique divisorial lattice for $L|l$ which intersects $J_L(L^{\times})_{(\ell)}$ non-trivially, it follows that \mathcal{L}_K is the unique divisorial lattice for $K|k$ which intersects $\hat{\phi}(J_L(L^{\times})_{(\ell)})$ non-trivially.

We claim that for \mathfrak{v} and \mathfrak{w} being as at the conclusion (*) above, one actually has $\epsilon_{\mathfrak{v}} \in \mathbb{Z}_{(\ell)}$. Indeed, by Remarks 2.20 above we have: \mathcal{L}_L is the unique divisorial lattice for $L|l$ such that $J_{\mathfrak{w}}(\mathcal{L}_L)$ intersects $J_{L\mathfrak{w}}(L\mathfrak{w}^{\times})$ non-trivially, and \mathcal{L}_K is the unique divisorial lattice for $K|k$ such that $J_{\mathfrak{v}}(\mathcal{L}_K)$ intersects $J_{K\mathfrak{v}}(K\mathfrak{v}^{\times})_{(\ell)}$ non-trivially. On the other hand, since $\hat{\phi}_{\mathfrak{v}} \circ \hat{j}_{\mathfrak{w}} = \hat{j}_{\mathfrak{v}} \circ \hat{\phi}$, and $\hat{\phi}(\mathcal{L}_L) = \mathcal{L}_K$, and $\hat{\phi}_{\mathfrak{v}}(J_{L\mathfrak{w}}(L\mathfrak{w}^{\times})_{(\ell)}) = \epsilon_{\mathfrak{v}} \cdot J_{K\mathfrak{v}}(K\mathfrak{v}^{\times})_{(\ell)}$, one has:

- i) $J_{K\mathfrak{v}}(K\mathfrak{v}^{\times})_{(\ell)} \subseteq \hat{j}_{\mathfrak{v}}(\mathcal{L}_K)$
- ii) $\epsilon_{\mathfrak{v}} \cdot J_{K\mathfrak{v}}(K\mathfrak{v}^{\times})_{(\ell)} = \hat{\phi}_{\mathfrak{v}}(J_{L\mathfrak{w}}(L\mathfrak{w}^{\times})_{(\ell)}) \subseteq \hat{\phi}_{\mathfrak{v}}(\hat{j}_{\mathfrak{w}}(\mathcal{L}_L)) = \hat{j}_{\mathfrak{v}}(\mathcal{L}_K)$

Hence by Remarks 2.20, 4), j), jj), above, it follows that $\epsilon_{\mathfrak{v}} \in \mathbb{Z}_{(\ell)}$, as claimed.

Finally, let $\Xi \subset \hat{L}$ be some \mathbb{Z}_{ℓ} -submodule, and $\Delta = \hat{\phi}(\Xi)$ be its image under $\hat{\phi}$. Since $\hat{\phi}(\hat{U}_L) = \hat{U}_K$ and $\hat{\phi}$ maps the finite corank \mathbb{Z}_{ℓ} -submodules of \hat{L} isomorphically onto the ones of \hat{K} , one has: Ξ has finite corank and $\hat{U}_L \subset \Xi$ iff Δ has finite corank and $\hat{U}_K \subset \Delta$. Now suppose that $\Delta = \hat{\phi}(\Xi)$ satisfy these conditions. Then in the notations from Definition 2.21, it follows that if \mathfrak{v} and \mathfrak{w} are quasi-prime divisors of $K|k$, respectively $L|l$ which correspond to each other under Φ , and satisfy the conclusion (*) above, then $\mathfrak{v} \in \mathcal{D}_{\Delta}$ iff $\mathfrak{w} \in \mathcal{D}_{\Xi}$. Hence $\hat{\phi}$ maps $\Xi(\mathfrak{w})$ isomorphically onto $\Delta(\mathfrak{v})$, thus $\hat{\phi}$ maps $\Xi^0 := \bigcap_{\mathfrak{w} \in \mathcal{D}_{\Xi}} \Xi(\mathfrak{w})$ isomorphically onto $\Delta^0 := \bigcap_{\mathfrak{v} \in \mathcal{D}_{\Delta}} \Delta(\mathfrak{v})$. Thus finally, $\hat{\phi}$ maps $\mathcal{L}_L^0 := \bigcup_{\Xi} \Xi^0$ isomorphically onto $\mathcal{L}_K^0 := \bigcup_{\Delta} \Delta^0$. \square

E) Recovering the rational quotients

We recall briefly the notion of an **abstract quotient** of the total decomposition graph, see POP [P3], Sections 4, 5 for details (where quotients of decomposition graphs were discussed).

First, let \mathcal{G}_{α} be the pair consisting of a pro- ℓ abelian group G_{α} endowed with a system of pro-cyclic subgroups $(T_{v_{\alpha}})_{v_{\alpha}}$ which makes G_{α} **curve like** of genus $g = 0$, i.e., there exists a system of generators $(\tau_{v_{\alpha}})_{\alpha}$ of the $(T_{v_{\alpha}})_{\alpha}$ such that the following are satisfied:

- i) $\prod_{\alpha} \tau_{v_{\alpha}} = 1$, and this is the only pro-relation satisfied by $(\tau_{v_{\alpha}})_{\alpha}$.
- ii) G_{α} is topologically generated by $(\tau_{v_{\alpha}})_{\alpha}$.

Notice that if $\mathfrak{T}' := (\tau'_{v_{\alpha}})_{v_{\alpha}}$ is a further system of generators of $(T_{v_{\alpha}})_{v_{\alpha}}$ satisfying i), ii) above, then there exists a unique ℓ -adic unit $\epsilon \in \mathbb{Z}_{\ell}^{\times}$ such that $\mathfrak{T}' = \mathfrak{T}^{\epsilon}$. Let $\hat{\mathcal{L}}_{\mathcal{G}_{\alpha}} := \text{Hom}_{\text{cont}}(G_{\alpha}, \mathbb{Z}_{\ell})$ the ℓ -adic dual of G_{α} , and $\mathcal{L}_{\mathcal{G}_{\alpha}} \subset \hat{\mathcal{L}}_{\mathcal{G}_{\alpha}}$ be the $\mathbb{Z}_{(\ell)}$ -submodule of all $\varphi : G_{\alpha} \rightarrow \mathbb{Z}_{\ell}$ satisfying

$\varphi(\tau_{v_\alpha}) \in \mathbb{Z}_{(\ell)}$ for all v_α . Notice that if $\mathfrak{T}' = \mathfrak{T}^\epsilon$ as above, then the corresponding lattice is $\mathcal{L}'_{\mathcal{G}_\alpha} = \epsilon^{-1} \cdot \mathcal{L}_{\mathcal{G}_\alpha}$. We call $\mathcal{L}_{\mathcal{G}_\alpha}$ a divisorial lattice for \mathcal{G}_α .

Definition/Remark 2.23. A divisorial morphism $\Phi_\alpha : \mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_\alpha$ is an open group homomorphism $\Phi_\alpha : \Pi_K \rightarrow G_\alpha$ which, inductively on $d := \text{td}(K|k)$, satisfies the following:

- a) Given the canonical system of inertia generators $(\tau_v)_v$ for the inertia groups $T_v \subset \Pi_K$ of the prime divisors v of $K|k$, there exists $\epsilon \in \mathbb{Z}_\ell^\times$ and $\epsilon_v \in \epsilon \cdot \mathbb{Z}_{(\ell)}$ satisfying:
 - i) For every v there exists v_α such that $\Phi_\alpha(\tau_v) = \tau_{v_\alpha}^{\epsilon_v}$, and $\Phi_\alpha(T_v) = \Phi_\alpha(Z_{v_\alpha})$ if $\epsilon_v \neq 0$.
 - ii) If D is a complete regular like and sufficiently large set of prime divisors of $K|k$, then $D_{v_\alpha} := \{v \in D \mid \Phi_\alpha(\tau_v) = \tau_{v_\alpha}^{\epsilon_v}, \epsilon_v \neq 0\}$ is finite and non-empty for every v_α .
- b) If $\Phi_\alpha(T_v) = 1$, then the induced homomorphism $\Phi_{\alpha,v} : \Pi_{Kv} \rightarrow G_\alpha$ has open image and defines a divisorial morphism $\Phi_{\alpha,v} : \mathcal{G}_{\mathcal{D}_{Kv}^{\text{tot}}} \rightarrow \mathcal{G}_\alpha$.

Notice that taking ℓ -adic duals, Φ_α gives rise to a Kummer homomorphism $\hat{\phi}_\alpha : \widehat{\mathcal{L}}_{\mathcal{G}_\alpha} \rightarrow \widehat{K}$, and by mere definitions, one has $\hat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha}) \subset \epsilon \cdot \mathcal{L}_K$ and $\hat{\phi}(\mathcal{L}_{\mathcal{G}_\alpha}) \cap \widehat{U}_K = 1$.

Recall that for every valuation v of K , we denote by $j^v : \widehat{K} \rightarrow \mathbb{Z}_\ell$ and $j_v : \widehat{U}_v \rightarrow \widehat{Kv}$ the ℓ -adic completions of $v : K^\times \rightarrow \mathbb{Z}$, respectively of the residue homomorphism $U_v \rightarrow Kv^\times$.

Definition 2.24. A divisorial morphism $\Phi_\alpha : \mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_\alpha$ is called an abstract rational quotient of $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$, if Φ_α is surjective, and setting $\widehat{\mathcal{L}}_\alpha := \hat{\phi}_\alpha(\widehat{\mathcal{L}}_{\mathcal{G}_\alpha}) \subset \widehat{K}$, the following hold:

- a) Every prime divisor v of $K|k$ satisfies: If $j_v(\widehat{\mathcal{L}}_\alpha \cap \widehat{U}_v) \neq 1$, then $\widehat{\mathcal{L}}_\alpha \subset \widehat{U}_v$, $\widehat{\mathcal{L}}_\alpha \cap \ker(j_v) = 1$.
- b) All sufficiently large complete regular like sets D of prime divisors of $K|k$ satisfy: Let $\Delta \subset \widehat{K}$ be a \mathbb{Z}_ℓ -module of finite co-rank containing \widehat{U}_K . Then there exist $v \in D$ such that $j^v(\widehat{\mathcal{L}}_\alpha) \neq 0$ and further, $\Delta \subset \widehat{U}_v$ and $\Delta \cap \ker(j_v) = \Delta \cap \widehat{\mathcal{L}}_\alpha$.

We recall that by POP [P3], Proposition 40, one has: If $x \in K$ is a generic element, i.e., $\kappa_x = k(x)$ is relatively algebraically closed in K , then the canonical projection of Galois groups $\Phi_{\kappa_x} : \Pi_K \rightarrow \Pi_{\kappa_x}$ defines an abstract rational quotient $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\kappa_x}$ in the sense above. We also notice that the canonical divisorial lattice for \mathcal{G}_{κ_x} is nothing but $\mathcal{L}_{\kappa_x} := j_{\kappa_x}(\kappa_x^\times) \subset \widehat{\kappa}_x$, and the Kummer homomorphism $\hat{\phi}_{\kappa_x} : \widehat{\kappa}_x \rightarrow \widehat{K}$ maps \mathcal{L}_{κ_x} isomorphically onto $j_K(\kappa_x^\times)$.

Using Proposition 2.22 above, we are now in the position to recover the geometric rational quotients of $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ in the case $\text{td}(K|k) > 2$ as follows:

Proposition 2.25. *Let k is an arbitrary algebraically closed field, and $\text{td}(K|k) > 2$. Then in the notations from above and of Proposition 2.22 the following hold:*

- 1) Let $\Phi_\alpha : \mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_\alpha$ be an abstract rational quotient. Then the following are equivalent:
 - i) Φ_α is geometric, i.e., there exists a generic element $x \in K$ and an isomorphism of decomposition graphs $\Phi_{\alpha, \kappa_x} : \mathcal{G}_\alpha \rightarrow \mathcal{G}_{\kappa_x}$ such that $\Phi_{\kappa_x} = \Phi_{\alpha, \kappa_x} \circ \Phi_\alpha$.
 - ii) There exists $\epsilon \in \mathbb{Z}_\ell^\times$ such that $\hat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha}) \subset \epsilon \cdot \mathcal{L}_K^0$.
- 2) Let $L|l$ be a function field over an algebraically closed field l , and $\Phi : \mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{D}_L^{\text{tot}}}$ be an abstract isomorphism of decomposition graphs. Then Φ is compatible with geometric rational quotients in the sense that if $\Phi_{\kappa_y} : \mathcal{G}_{\mathcal{D}_L^{\text{tot}}} \rightarrow \mathcal{G}_{\kappa_y}$ is a geometric rational quotient of $\mathcal{G}_{\mathcal{D}_L^{\text{tot}}}$, then $\Phi_\alpha := \Phi_{\kappa_y} \circ \Phi$ is a geometric rational quotient of $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$.

Proof. To 1): The implication i) \Rightarrow ii) is clear by the characterization of the geometric rational quotients and Proposition 2.22, 2), precisely the assertion that $J_K(K^\times) \subset \mathcal{L}_K^0$. For the implication ii) \Rightarrow i) we proceed as follows: Let $\mathcal{L}_{\mathcal{G}_\alpha}$ be a divisorial lattice for \mathcal{G}_α satisfying $\mathcal{L}_\alpha := \hat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha}) \subset \epsilon \cdot \mathcal{L}_K^0$. Then replacing $\mathcal{L}_{\mathcal{G}_\alpha}$ by its ϵ^{-1} multiple, without loss we can and will suppose that $\mathcal{L}_\alpha = \hat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha}) \subset \mathcal{L}_K^0$, thus $\mathcal{L}_\alpha \subset \mathcal{L}_K^0 \subseteq \mathcal{L}_K$. Recall that by POP [P3], Fact 32, (1), \mathcal{L}_α is $\mathbb{Z}_{(\ell)}$ -saturated in \mathcal{L}_K , i.e., $\mathcal{L}_K/\mathcal{L}_\alpha$ is $\mathbb{Z}_{(\ell)}$ -torsion free. Further, if $u \in K \setminus k$, and $\kappa_u \subset K$ is the corresponding relatively algebraically closed subfield, then $J_K(\kappa_u^\times)_{(\ell)}$ is $\mathbb{Z}_{(\ell)}$ -saturated in \mathcal{L}_K as well, i.e., $\mathcal{L}_K/J_K(\kappa_u^\times)_{(\ell)}$ is $\mathbb{Z}_{(\ell)}$ -torsion free. Finally, by Proposition 2.22, 2), for every $u \in \mathcal{L}_K^0$, there exists a multiple u^n such that $u^n \in \hat{U}_K \cdot J_K(K^\times)$.

Let $u \in \mathcal{L}_\alpha \subset \mathcal{L}_K^0$ be a non-trivial element, and $n_u > 0$ be the minimal positive integer with $u^{n_u} \in \hat{U}_K \cdot J_K(K^\times)$, say $u^{n_u} = \theta \cdot J_K(u)$ for some $u \in K^\times$ and $\theta \in \hat{U}_{\mathcal{D}_K}$, and notice that $J_K(u)$ and θ with the property above are unique, because $\hat{U}_{\mathcal{G}} \cap \mathcal{L}_\alpha$ is trivial, by POP [P3], Fact 32. Further, n_u is relatively prime to ℓ , because $\mathcal{L}_K/J_K(\kappa_u^\times)_{(\ell)}$ has no non-trivial $\mathbb{Z}_{(\ell)}$ -torsion. As in the proof of Proposition 5.3 from POP [P4], one concludes that actually $J_K(\kappa_u^\times)_{(\ell)} = \mathcal{L}_\alpha$. Note that in the proofs of the Claims 1, 2, 3, of loc.cit, and the conclusion of the proof of Proposition 5.3 of loc.cit., it was nowhere used that k is an algebraic closure of a finite field.

This concludes the proof of assertion 1).

To 2): Let $\Phi_{\kappa_y} : \mathcal{G}_{\mathcal{D}_L^{\text{tot}}} \rightarrow \mathcal{G}_{\kappa_y}$ be a geometric rational quotient of $\mathcal{G}_{\mathcal{D}_L^{\text{tot}}}$. Then as in the proof of assertion 2) of Proposition 5.3 of POP [P4], it follows that

$$\Phi_\alpha = \Phi_{\kappa_y} \circ \Phi : \mathcal{G}_{\mathcal{D}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\kappa_y} =: \mathcal{G}_\alpha$$

is an abstract rational quotient of $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ with Kummer homomorphism $\hat{\phi}_\alpha = \hat{\phi} \circ \hat{\phi}_{\kappa_y} : \hat{\kappa}_y \rightarrow \hat{K}$. We show that Φ_α satisfies hypothesis ii) from Proposition 2.25, 1). Indeed, $\mathcal{G}_\alpha := \mathcal{G}_{\kappa_y}$ has $\mathcal{L}_{\mathcal{G}_\alpha} := \mathcal{L}_{\kappa_y} = J_{\kappa_y}(\kappa_y^\times)$ as canonical divisorial lattice, and since $\Phi_{\kappa_y} : \mathcal{G}_{\mathcal{D}_L^{\text{tot}}} \rightarrow \mathcal{G}_{\kappa_y}$ is defined by the k -embedding $\kappa_y \hookrightarrow L$, one has by definitions that $\hat{\phi}_{\kappa_y}(\mathcal{L}_{\kappa_y}) = J_L(\kappa_y^\times)_{(\ell)} \subseteq J_L(L^\times)_{(\ell)}$. Thus finally, $\hat{\phi}_{\kappa_y}(\mathcal{L}_{\kappa_y}) \subseteq J_L(L^\times) \subseteq \mathcal{L}_L^0$ by Proposition 2.22, 2) applied to $L|l$. On the other hand, by Proposition 2.22, 1), one has that $\hat{\phi}(\mathcal{L}_L^0) = \epsilon \cdot \mathcal{L}_K^0$ for some $\epsilon \in \mathbb{Z}_\ell^\times$, and therefore:

$$\hat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha}) = \hat{\phi}(\hat{\phi}_{\kappa_y}(\mathcal{L}_{\kappa_y})) \subset \hat{\phi}(\mathcal{L}_L^0) = \epsilon \cdot \mathcal{L}_K^0.$$

Hence by applying assertion 1) we have: Φ_α is a geometric quotient of $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$, etc. \square

F) Concluding the proof of Theorem 1.1

First, by Proposition 2.25 above it follows that in the context of Theorem 1.1 one can recover the geometric rational quotients of $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$ from Π_K^c endowed with $\mathfrak{In.div}(K)$, and that the recipe to do so is invariant under isomorphisms $\Phi \in \text{Isom}^c(\Pi_K, \Pi_L)$ which map $\mathfrak{In.div}(K)$ onto $\mathfrak{In.div}(L)$. Conclude by applying Main Result from [P3], Introduction.

3. PROOF OF THEOREM 1.2

A) Recovering the nature of k

In this sub-section we prove the first assertion of Theorem 1.2. Precisely, for each non-negative integer $\delta \geq 0$, we will give inductively on δ a group theoretical recipe $\mathfrak{dim}(\delta)$ in terms of $\mathcal{G}_{\mathcal{D}_K^{\text{tot}}}$, thus in terms of Π_K^c , such that $\mathfrak{dim}(\delta)$ is true if and only if

$$(*) \quad \text{td}(K|k) > \dim(k) \quad \text{and} \quad \dim(k) = \delta.$$

The simplify language, recovering the above information about k will be called “recovering the nature of k .” Note that $\text{char}(k)$ is not part of “nature of k ” in the above sense. Moreover, $\mathfrak{dim}(\delta)$ will be invariant under isomorphisms as follows: If $L|l$ is a further function field over an algebraically closed field l , and $\mathcal{G}_{\mathcal{Q}_K^{\text{tot}}} \rightarrow \mathcal{G}_{\mathcal{Q}_L^{\text{tot}}}$ is an isomorphism, then $\mathfrak{dim}(\delta)$ holds for $K|k$ if and only if $\mathfrak{dim}(\delta)$ holds for $L|l$.

Before giving the recipes $\mathfrak{dim}(\delta)$, let us recall the following few facts about generalized quasi prime divisors, in particular, the facts from POP [P4], [P5], Section 4, and TOPAZ [To2].

First, letting $\mathcal{Q}(K|k)$ be the set of all the quasi prime divisors of $K|k$, let $\mathcal{T}^1(K) \subset \Pi_K$ the topological closure of $\cup_{\mathfrak{v} \in \mathcal{Q}(K|k)} T_{\mathfrak{v}}^1$ in Π_K . Further, for $l \subseteq k$ an algebraically closed subfield, let $\mathcal{Q}_l(K|k)$ be the set of all the quasi prime divisors \mathfrak{v} with $\mathfrak{v}|_l = \mathfrak{w}|_l$, and $\mathcal{T}_l^1(K) \subset \Pi_K$ be the topological closure of $\cup_{\mathfrak{v} \in \mathcal{Q}_l(K|k)} T_{\mathfrak{v}}^1$. Then by [P5], Theorem A, one has: $\mathcal{T}_l^1(K) \subseteq \mathcal{T}^1(K)$ consist of minimized inertia elements, and $\mathcal{T}_l(K)$ consists of minimized inertia elements at valuations v with $v|_l = \mathfrak{w}|_l$.

Second, for every generalized quasi prime r -divisor \mathfrak{w} , recall the canonical exact sequence

$$1 \rightarrow T_{\mathfrak{w}}^1 \rightarrow Z_{\mathfrak{w}}^1 \rightarrow Z_{\mathfrak{w}}^1/T_{\mathfrak{w}}^1 =: \Pi_{K\mathfrak{w}}^1 \rightarrow 1,$$

and recalling that for $\mathfrak{w} \leq \mathfrak{v}$, one has $T_{\mathfrak{w}}^1 \subset T_{\mathfrak{v}}^1$, $Z_{\mathfrak{w}}^1 \subset Z_{\mathfrak{v}}^1$, we endow $\Pi_{K\mathfrak{w}}^1 = Z_{\mathfrak{w}}^1/T_{\mathfrak{w}}^1$ with its minimized inertia/decomposition groups $T_{\mathfrak{v}/\mathfrak{w}}^1 := T_{\mathfrak{v}}^1/T_{\mathfrak{w}}^1 \subset Z_{\mathfrak{v}}^1/T_{\mathfrak{w}}^1 =: Z_{\mathfrak{v}/\mathfrak{w}}^1 \subset \Pi_{K\mathfrak{w}}^1$, which in turn, give rise in the usual way to the total quasi decomposition graph $\mathcal{G}_{\mathcal{Q}_{K\mathfrak{w}}^{\text{tot}}}$ for $K\mathfrak{w}|k\mathfrak{w}$.³

Further, let $\mathcal{T}_l(K\mathfrak{w}) \subseteq \mathcal{T}(K\mathfrak{w}) \subset \Pi_{K\mathfrak{w}}^1$ be the images of $\mathcal{T}_l^1(K) \cap Z_{\mathfrak{w}}^1 \subseteq \mathcal{T}^1(K) \cap Z_{\mathfrak{w}}^1$ under the canonical projection $Z_{\mathfrak{w}}^1 \rightarrow \Pi_{K\mathfrak{w}}^1$. Then $\mathcal{T}^1(\bullet)$ behaves functorially, in the sense that for $\mathfrak{w} \leq \mathfrak{v}$, one has: The image of $\mathcal{T}^1(K\mathfrak{w}) \cap Z_{\mathfrak{v}/\mathfrak{w}}^1$ under $Z_{\mathfrak{v}/\mathfrak{w}}^1 \rightarrow \Pi_{K\mathfrak{v}}^1$ equals $\mathcal{T}^1(K\mathfrak{v})$.

Finally, recall that a pro- ℓ abelian group G endowed with a system of procyclic subgroups $(T_{\alpha})_{\alpha}$ is called **complete curve like**, if there exists a system of generators $(\tau_{\alpha})_{\alpha}$ with $\tau_{\alpha} \in T_{\alpha}$ such that letting $T \subseteq G$ be the closed subgroup of G generated by $(\tau_{\alpha})_{\alpha}$, the following hold:

- i) $\prod_{\alpha} \tau_{\alpha} = 1$ and this is the only profinite relation satisfied by $(\tau_{\alpha})_{\alpha}$ in G .⁴
- ii) The quotient G/T is a finite \mathbb{Z}_{ℓ} -module.

A case of special interest is that of quasi prime r -divisors \mathfrak{w} , with $r = d - 1$, where $d = \text{td}(K|k)$, thus $\text{td}(K\mathfrak{w}|k\mathfrak{w}) = 1$. Then $\mathfrak{w}K/\mathfrak{w}k \cong \mathbb{Z}^{d-1}$ and the following hold: Let (t_2, \dots, t_d) be a system of elements of K such that $(\mathfrak{w}t_2, \dots, \mathfrak{w}t_d)$ define a basis of $\mathfrak{w}K/\mathfrak{w}k$, and $t_1 \in \mathcal{O}_{\mathfrak{w}}^{\times}$ be such that its residue $\bar{t}_1 \in K\mathfrak{w}$ is a separable transcendence basis of $K\mathfrak{w}|k\mathfrak{w}$. Then (t_1, \dots, t_d) is a separable transcendence basis of $K|k$, and letting $k_1 \subset K$ be the relative algebraic closure of $k(t_2, \dots, t_d)$ in K , it follows that $\text{td}(K|k_1) = 1$, and $k_1\mathfrak{w} = k\mathfrak{w}$. Hence we are in the situation of Section 4 from [P5], Theorem 4.1, thus we have:

Fact 3.1. In the above notations, the following hold:

- I) for every non-trivial element $\sigma \in \mathcal{T}^1(K\mathfrak{w})$ there exists a unique quasi prime divisor $\mathfrak{v}_{\sigma} > \mathfrak{w}$ such that $\sigma \in T_{\mathfrak{v}_{\sigma}}^1$, thus the image of σ in $\Pi_{K\mathfrak{w}}^1$ lies in $T_{\mathfrak{v}/\mathfrak{w}}^1$.
- II) Let $(T_{\alpha})_{\alpha}$ be a maximal system of distinct maximal cyclic subgroups of $\Pi_{K\mathfrak{w}}^1$ satisfying one of the following conditions:
 - i) $T_{\alpha} \subset \mathcal{T}^1(K\mathfrak{w})$ for each α .

³ Note that if $\text{char}(K\mathfrak{w}) = \ell$, the groups $T_{\mathfrak{v}/\mathfrak{w}}^1 \subset Z_{\mathfrak{v}/\mathfrak{w}}^1 \subset \Pi_{K\mathfrak{w}}^1$ are not Galois groups over $K\mathfrak{w}$.

⁴ This implies by definition that $\tau_{\alpha} \rightarrow 1$ in G , thus every open subgroup of G contains almost all T_{α} .

ii) $T_\alpha \subset \mathcal{T}_l^1(K\mathfrak{w})$ for each α .

Then $\Pi_{K\mathfrak{w}}^1$ endowed with $(T_\alpha)_\alpha$ is complete curve like if and only if:

- a) $k\mathfrak{w}$ is an algebraic closure of a finite field, provided i) is satisfied.
- b) $l\mathfrak{w} = k\mathfrak{w}$, provided ii) is satisfied.

Moreover, if either i), or ii), is satisfied, then $(T_\alpha)_\alpha$ is actually the set of the minimized inertia groups $T_\alpha = T_{v_\alpha}^1$ at all the prime divisors $v_\alpha := \mathfrak{v}_\alpha/\mathfrak{w}$ of $K\mathfrak{w}|k\mathfrak{w}$.

We are now prepared to give the recipes $\mathfrak{dim}(\delta)$.

- The recipe $\mathfrak{dim}(0)$:

We say that $K|k$ satisfies $\mathfrak{dim}(0)$ if and only if for every quasi prime divisor \mathfrak{w} of $K|k$ with $\text{td}(K\mathfrak{w}|k\mathfrak{w}) = \text{td}(K|k) - 1$, one has: $\Pi_{K\mathfrak{w}}^1$ endowed with any maximal system of maximal pro-cyclic subgroups $(T_\alpha)_\alpha$ satisfying condition i) above is complete curve like.

- Given $\delta > 0$, and the recipes $\mathfrak{dim}(0), \dots, \mathfrak{dim}(\delta - 1)$, we define $\mathfrak{dim}(\delta)$ as follows:

First, if $\text{td}(K|k) \leq \delta$, we say that $\mathfrak{dim}(\delta)$ does not hold for $K|k$. For the remaining discussion, we suppose that $\text{td}(K|k) > \delta$, and proceed as follows:

For $\delta' < \delta$, let $\mathcal{D}_{\delta'}(K)$ be the set of all the minimized quasi divisorial subgroups $T_{\mathfrak{v}}^1 \subset Z_{\mathfrak{v}}^1$ of Π_K such that $\Pi_{K\mathfrak{v}}^1$ endowed with the family $(T_{\mathfrak{v}'/\mathfrak{v}}^1)_{\mathfrak{v}' < \mathfrak{v}}$, does not satisfy $\mathfrak{dim}(\delta')$ for any $\delta' < \delta$. We set $\mathfrak{Jn}_{\delta}(K) := \cup_{\mathfrak{v} \in \mathcal{D}_{\delta}} T_{\mathfrak{v}}^1 \subset \Pi_K$, and notice that $\mathfrak{Jn}_{\delta}(K)$ consists of minimized inertia elements (by its mere definition), and it is closed under taking powers. Hence by POP [P5], Introduction, Theorem A, the topological closure $\overline{\mathfrak{Jn}_{\delta}}(K) \subset \Pi_K$ consists of minimized inertia elements in Π_K , and it is closed under taking powers, because $\mathfrak{Jn}_{\delta}(K) \subset \Pi_K$ was so. For $d := \text{td}(K|k)$ and every quasi $(d-1)$ -divisorial subgroup $T_{\mathfrak{v}}^1 \subset Z_{\mathfrak{v}}^1$ of Π_K , we set $\mathfrak{Jn}_{\mathfrak{v}} := \overline{\mathfrak{Jn}_{\delta}}(K) \cap Z_{\mathfrak{v}}^1$, and note that $\mathfrak{Jn}_{\mathfrak{v}}$ consists of minimized inertia elements of Π_K which are contained in $Z_{\mathfrak{v}}^1$, it is topologically closed, and closed under taking powers. Thus the image $\pi_{\mathfrak{v}}(\overline{\mathfrak{Jn}_{\mathfrak{v}}}) \subset \Pi_{K\mathfrak{v}}^1$ under the canonical projection $\pi_{\mathfrak{v}} : Z_{\mathfrak{v}}^1 \rightarrow Z_{\mathfrak{v}}^1/T_{\mathfrak{v}}^1 = \Pi_{K\mathfrak{v}}^1$ consists of minimized inertia elements, is topologically closed, and closed under taking powers. Finally, by Fact 3.1, I), above, it follows that for every $\sigma \in \mathfrak{Jn}_{\mathfrak{v}}^1$ which has a nontrivial image under $Z_{\mathfrak{v}}^1 \rightarrow \Pi_{K\mathfrak{v}}^1$, there exists a unique minimal valuation \mathfrak{v}_σ such that σ is a minimized inertia element at \mathfrak{v}_σ , and in particular, $\mathfrak{v}_\sigma > \mathfrak{v}$ by loc.cit. Hence $T_{\mathfrak{v}}^1 \subset T_{\mathfrak{v}_\sigma}^1$, and moreover, since $\text{td}(K\mathfrak{v}|k\mathfrak{v}) = 1$, it follows that \mathfrak{v}_σ is a quasi prime d -divisor of $K|k$. Thus we have:

Fact 3.2. Let $\mathfrak{v}_i, i \in I_{\mathfrak{v}}$, be the set of distinct quasi prime divisors of $K|k$ which satisfy: $\mathfrak{v}_i > \mathfrak{v}$ and $T_{\mathfrak{v}_i}^1$ contains some $\sigma \in \mathfrak{Jn}_{\mathfrak{v}}^1$ with $\sigma \notin T_{\mathfrak{v}_i}^1$. Then the following hold:

- i) \mathfrak{v}_i are quasi prime d -divisors of $K|k$.
- ii) Setting $T_{\mathfrak{v}_i}^1 := T_{\mathfrak{v}_i/\mathfrak{v}}^1 \subset \Pi_{K\mathfrak{v}}^1$, one has $T_{\mathfrak{v}_i}^1 \cap T_{\mathfrak{v}'_i}^1 = \{1\}$ for $\mathfrak{v}_i \neq \mathfrak{v}'_i$.
- iii) $\mathfrak{Jn}_{\mathfrak{v}}^1 \subset \cup_i T_{\mathfrak{v}_i}^1$, thus the image of $\mathfrak{Jn}_{\mathfrak{v}}^1$ under $Z_{\mathfrak{v}}^1 \rightarrow \Pi_{K\mathfrak{v}}^1$ equals $\cup_i T_{\mathfrak{v}_i}^1$.

Proposition 3.3. *In the above notations suppose that $d := \text{td}(K|k) > \delta$. The one has:*

- 1) *Suppose $\mathcal{D}_{\delta}(K)$ is empty. Then $\dim(k) < \delta$ and $\dim(k)$ is the maximal δ_k such that there exists some quasi prime divisor \mathfrak{v} whose $\Pi_{K\mathfrak{v}}^1$ endowed with $(T_{\mathfrak{v}_i}^1)_{\mathfrak{v}_i}$ satisfies $\mathfrak{dim}(\delta_k)$. Further, a quasi prime divisor \mathfrak{v} of $K|k$ is a prime divisor if and only if $\Pi_{K\mathfrak{v}}^1$ endowed with its $(T_{\mathfrak{v}_i}^1)_{\mathfrak{v}_i}$ satisfies $\mathfrak{dim}(\delta_k)$.*

2) Suppose $\mathcal{D}_\delta(K)$ is non-empty. Then $\dim(k) \geq \delta$, and in the notations from Fact 3.2 above one has: $\dim(k) = \delta$ iff the quasi prime $(d-1)$ -divisors \mathfrak{v} of $K|k$ satisfy:

i) There exist \mathfrak{v} such that $I_{\mathfrak{v}}^1$ is non-empty.

ii) If $I_{\mathfrak{v}}^1$ is non-empty, then $\Pi_{K\mathfrak{v}}^1$ endowed with $(T_{v_i}^1)_{v_i}$ is curve like.

If the above conditions are satisfied, a quasi prime divisor \mathfrak{v} of $K|k$ is a prime divisor of $K|k$ if and only if $\Pi_{K\mathfrak{v}}^1$ endowed with $(T_{v_i}^1)_{v_i}$ does not satisfy $\mathfrak{dim}(\delta')$ for any $\delta' < \delta$.

(*) In particular, there exists a group theoretical recipe to distinguish the divisorial groups $T_{\mathfrak{v}} \subset Z_{\mathfrak{v}}$ in Π_K among all the quasi divisorial subgroups, thus to recover the set of divisorial inertia $\mathfrak{In.Div}(K) \subset \Pi_K$. Further, the recipes to check $\text{td}(K|k) > \dim(k)$ and to recover $\mathfrak{In.Div}(K)$ from Π_K^c are invariant under isomorphisms $\Phi \in \text{Isom}^c(\Pi_K, \Pi_L)$.

Proof. To 1): First let $\dim(k) < \delta$. Then for all quasi prime divisors \mathfrak{v} of $K|k$ we have $\dim(k) \geq \dim(k\mathfrak{v})$, hence $\text{td}(K|k) > \delta > \dim(k) \geq \dim(k\mathfrak{v})$. Thus taking into account that $\text{td}(K\mathfrak{v}|k\mathfrak{v}) = \text{td}(K|k) - 1 \geq \delta$, we get: $\text{td}(K\mathfrak{v}|k\mathfrak{v}) > \dim(k\mathfrak{v})$, and $\Pi_{K\mathfrak{v}}^1$ endowed with $(T_{v_i}^1)_{v_i}$ satisfies the recipe $\mathfrak{dim}(\delta')$ with $\delta' = \dim(k\mathfrak{v}) < \delta$. Thus $\mathfrak{v} \notin \mathcal{D}_\delta(K)$, etc. Second, suppose that $\mathcal{D}_\delta(K)$ is empty. Equivalently, $\Pi_{K\mathfrak{v}}^1$ endowed with $(T_{v_i}^1)_{v_i}$ satisfies $\mathfrak{dim}(\delta')$ for some $\delta' < \delta$ for each quasi prime divisor \mathfrak{v} . Hence choosing \mathfrak{v} to be a prime divisor of $K|k$ we get: $k\mathfrak{v} = k$, and $K\mathfrak{v}|k$ satisfies $\mathfrak{dim}(\delta')$ for some $\delta' < \delta$. Thus $\dim(k) = \delta' < \delta$. The description of the divisors \mathfrak{v} of $K|k$ is clear, because \mathfrak{v} is a prime divisor of $K|k$ iff \mathfrak{v} is trivial on k iff $\dim(k\mathfrak{v}) = \dim(k)$.

To 2): First suppose that $\dim(k) = \delta$. Recall that a quasi prime divisor \mathfrak{v} is a prime divisor iff \mathfrak{v} is trivial on k iff $\dim(k) = \dim(k\mathfrak{v})$. Therefore, if \mathfrak{v} is a prime divisor, then $\mathfrak{dim}(\delta')$ is not satisfied by $\Pi_{K\mathfrak{v}}^1$ endowed with $(T_{v_i}^1)_{v_i}$ for any $\delta' < \delta$. Second, if \mathfrak{v} is not a prime divisor, then \mathfrak{v} is not trivial on k , and reasoning as above, one gets that $\mathfrak{dim}(\delta')$ holds for $\Pi_{K\mathfrak{v}}^1$ endowed with $(T_{v_i}^1)_{v_i}$ for $\delta' = \dim(k\mathfrak{v})$. Thus finally it follows that $\mathcal{D}_\delta(K)$ consists of exactly all the prime divisors of $K|k$. Hence by POP [P2], Introduction, Theorem B, it follows that $\overline{\mathfrak{In}}_\delta(K)$ consists of all the tame inertia at all the k -valuations of $K|k$. Thus reasoning as in the proof of Proposition 3.5 of POP [P4], but using all the divisorial subgroups instead of the quasi divisorial subgroup, and $\overline{\mathfrak{In}}_\delta(K)$ instead of $\mathfrak{In.tm}(K)$ as in loc.cit., it follows that the *flags of divisorial subgroups* –as introduced in Definition 3.4 of loc.cit.– can be recovered by a group theoretical recipe from Π_K endowed with $\overline{\mathfrak{In}}_\delta(K)$. Hence in the notations from Fact 3.2 above, one has the following: Let \mathfrak{v} be a quasi $(d-1)$ -prime divisor of $K|k$.

a) Suppose that \mathfrak{v} is trivial on k . Then by the discussion above, $(T_{v_i}^1)_{v_i}$ is exactly the set of all the divisorial inertia subgroups in $\Pi_{K\mathfrak{v}}^1$. By applying Fact 3.1 above, one concludes that $\Pi_{K\mathfrak{v}}^1$ endowed with $(T_{v_i}^1)_{v_i}$ is curve like, etc.

b) Suppose that \mathfrak{v} is non-trivial on k . We claim that the set $I_{\mathfrak{v}}$ from Fact 3.2 above is empty. Indeed, since each non-trivial $\sigma \in \overline{\mathfrak{In}}_\delta(K)$ is tame inertia element at some k -valuations of $K|k$, it follows that \mathfrak{v}_σ is a k -valuation. Thus one cannot have $\mathfrak{v}_\sigma > \mathfrak{v}$, i.e., $\mathfrak{v}_\sigma \notin I_{\mathfrak{v}}$.

Now suppose that $\dim(k) > \delta$. We show that there exist prime $(d-1)$ -divisors \mathfrak{v} such that $I_{\mathfrak{v}}$ is not empty, but $\Pi_{K\mathfrak{v}}^1$ endowed with $(T_{v_i}^1)_{v_i}$ is not curve like. Indeed, let $l \subset k$ be an algebraically closed subfield with $\text{td}(k|l) = 1$. Then $\dim(l) \geq \delta$, and therefore, if \mathfrak{v} is a quasi divisor of $K|k$ which is trivial on l , we have $\dim(k\mathfrak{v}) \geq \dim(l) \geq \delta$. Hence $\mathfrak{dim}(\delta')$ does not hold for $K\mathfrak{v}|k\mathfrak{v}$ for all $\delta' < \delta$. Therefore, $\mathfrak{v} \in \mathcal{D}_\delta$. We then can proceed as in the proof of Proposition 4.4 of POP [P4], but using all the quasi prime divisors of $K|k$ which are trivial

on l instead of using all the (maximal) quasi prime divisors of $K|k$. Namely let $\mathfrak{In.tm}_l(K)$ be the set of the inertia elements at valuations which are trivial on l . Then by POP [P2], Introduction, Theorem A, the set $\mathfrak{In.tm}_l(K)$ is closed in Π_K , and by loc.cit. Theorem B, the set $\mathfrak{In.tm.q.div}_l(K)$ of tame inertia at the quasi divisors \mathfrak{v} which are trivial on l is dense $\mathfrak{In.tm}_l(K)$. On the other hand, by the discussion above, every quasi prime divisor \mathfrak{v} which is trivial on l lies in $\mathcal{D}_\delta(K)$. Hence we conclude that $\mathfrak{In.tm.q.div}_l(K) \subseteq \mathfrak{In}_\delta(K)$, and so, $\mathfrak{In.tm}_l(K)$ is contained in the closure $\overline{\mathfrak{In}_\delta(K)}$. In other words, if \mathfrak{v} is any prime $(d-1)$ -divisor of $K|k$, and $\pi_{\mathfrak{v}} : Z_{\mathfrak{v}} \rightarrow \Pi_{K_{\mathfrak{v}}}$ is the canonical projection, it follows that $\pi_{\mathfrak{v}}(\overline{\mathfrak{In}_\delta(K)})$ equals all the tame inertia at valuations which are trivial on l . Therefore, in the notations from Fact 3.2 above, the set $I_{\mathfrak{v}}$ is non-empty, and $(T_{v_i})_{v_i}$ consists of the inertia groups of all the quasi prime divisors of $K_{\mathfrak{v}}|k$ which are trivial on l . Since $l \subset k$ strictly, by Fact 3.1 it follows that $\Pi_{K_{\mathfrak{v}}}$ endowed with all $(T_{v_i})_{v_i}$ is not curve like.

Finally, the last assertion (*) of the Proposition is an obvious consequence of the recipes described at 1) and 2). \square

B) Concluding the proof of Theorem 1.2

By Proposition 3.3, there are group theoretical recipes to check whether $\text{td}(K|k) > \dim(k)$ and if so, the group theoretical recipes recover the divisorial inertia $\mathfrak{In.div}(K)$ from the group theoretical information encoded in Π_K^c . Further, the recipes to do so are invariant under isomorphisms $\Phi \in \text{Isom}^c(\Pi_K, \Pi_L)$. Conclude by applying Theorem 1.1.

Appendix:

ON THE ORDER OF THE REDUCTION OF POINTS ON ABELIAN SCHEMES

PETER JOSSEN

We fix an integral scheme S of finite type over $\text{Spec } \mathbb{Z}$, and a prime number ℓ different from the characteristic of the function field of S . Our goal is to establish the following

Theorem 1. *Let A be an abelian S -scheme, and $P \in A(S)$ be a point of infinite order. Then the set of closed points $s : \text{Spec}(\kappa) \rightarrow S$ such that ℓ divides the order of the image $P_s \in A(\kappa)$ of P has positive Dirichlet density in S , thus it is Zariski dense as well.*

In the special case where $S \subseteq \text{Spec } \mathcal{O}_k$, with \mathcal{O}_k the ring of integers of a number field k , the Theorem 1 above was proven by PINK, see [Pk], Theorem 2.8 (and some arguments in our strategy are similar to his). The proof rests on two classical theorems: These are the general version of the Mordell–Weil theorem, stating that $A(S)$ is a finitely generated (abelian) group, and a generalization of Chebotarev’s density theorem.

Remarks/Basic Facts. Since ℓ is different from the characteristic, S has an open dense subsets on which ℓ is invertible. Thus replacing S by such an open dense subset, without loss of generality, we can suppose that ℓ is invertible on S . Let $\pi_1 := \pi_1^{\text{ét}}(S, \bar{\eta})$ be the étale

fundamental group of S . We view \mathbb{Z} as the constant group S -scheme, and for every abelian S -scheme A , we let $\mathbb{T}_\ell A$ be the ℓ -adic Tate module of A viewed as a π_1 -module.

a) Consider the complex M of group S -schemes, viewed as **1-motive** in the sense of Deligne:

$$M = [u : \mathbb{Z} \rightarrow A], \quad u(1) = P$$

b) One associates with M , in a functorial way, a finitely generated free \mathbb{Z}_ℓ -module $\mathbb{T}_\ell M$ with continuous π_1 -action, by setting $\mathbb{T}_\ell M =: \varprojlim_e M_{\ell^e}(\bar{k})$, where

$$M_{\ell^e}(\bar{k}) := \{(Q, n) \in A(\bar{k}) \times \mathbb{Z} \mid \ell^e Q = nP\} / \{(nP, \ell^e n) \in A(\bar{k}) \times \mathbb{Z} \mid n \in \mathbb{Z}\}$$

Hence $M_{\ell^e}(\bar{k})$ is the group of \bar{k} -points of a finite étale group S -scheme M_{ℓ^e} killed by ℓ^e .

c) For later use, we notice that the elements of $\mathbb{T}_\ell M$ are represented by sequences $(P_i, n_i)_{i \in \mathbb{N}}$, with $P_i \in A(\bar{k})$ and $n_i \in \mathbb{Z}$, satisfying the three relations: $\ell^i P_i = n_i P$, $\ell P_i - P_{i-1} = m_i P$, $n_i - n_{i-1} = \ell^{i-1} m_i$, for suitable $m_i \in \mathbb{Z}$. Two such sequences $(P_i, n_i)_i$, $(P'_i, n'_i)_i$ represent the same element of $\mathbb{T}_\ell M$ if there exists a sequence $(m_i)_i$ in \mathbb{Z} such that

$$\ell^i m_i = n_i - n'_i, \quad m_i P = P_i - P'_i \quad \text{for all } i \in \mathbb{N}.$$

d) Letting π_1 act trivially on \mathbb{Z}_ℓ , one gets canonically an exact sequence of π_1 -modules

$$0 \rightarrow \mathbb{T}_\ell A \rightarrow \mathbb{T}_\ell M \rightarrow \mathbb{Z}_\ell \rightarrow 0.$$

In particular, $\mathbb{T}_\ell M$ is a free \mathbb{Z}_ℓ -module of rank $2g + 1$, where $g = \dim_S(A)$.

e) Let $\pi_1 \rightarrow \mathrm{GL}(\mathbb{T}_\ell M)$ define $\mathbb{T}_\ell M$ as a π_1 -module, and consider the ℓ -adic Lie groups

$$L^M := \mathrm{im}(\pi_1 \rightarrow \mathrm{GL}(\mathbb{T}_\ell M)), \quad L^A := \mathrm{im}(\pi_1 \rightarrow \mathrm{GL}(\mathbb{T}_\ell A)).$$

By restriction, $L^M \twoheadrightarrow L^A$, and set $L_A^M := \ker(L^M \twoheadrightarrow L^A) = \{\sigma \in L^M \mid \sigma|_{\mathbb{T}_\ell A} = \mathrm{id}\}$.

f) Finally, we notice that if $P \in A(S)$ is a torsion point, then the sequence of π_1 -modules $0 \rightarrow \mathbb{T}_\ell A \rightarrow \mathbb{T}_\ell M \rightarrow \mathbb{Z}_\ell \rightarrow 0$ splits after passing to \mathbb{Q}_ℓ -coefficients, and it follows that the group L_A^M is trivial in that case. The point is that the converse holds as well:

Proposition 2. *If $P \in A(S)$ has infinite order, then the group L_A^M is not trivial.*

Proof. This is a special case of JOSSEN [Jo], Theorem 2. □

Let $s \hookrightarrow S$ be a closed point, and $G_{\kappa(s)} \rightarrow \pi_1$ be the embedding defined by some path $\bar{s} \rightarrow \bar{\eta}$. The image $F_s \in \pi_1$ of the Frobenius element of $G_{\kappa(s)}$ is called a **Frobenius element** over S . We shall make use of the following version of Chebotarev's Density Theorem:

Theorem 3 (Artin–Chebotarev). *The set of all Frobenius elements is dense in π_1 .*

This follows in principle from Theorem 7 in [Se1], but see rather HOLSCHBACH [Ho] for a complete proof (of this and other related assertions). The deduction of our Theorem 3 from SERRE's Theorem 7 in [Se1] goes as follows: Let $\pi_1 \rightarrow G$ be a finite quotient of π_1 , corresponding via the defining property of the fundamental group to a finite étale Galois cover X of S . Let $R \subseteq G$ be any subset stable under conjugation. Then the Artin–Chebotarev Density Theorem for the scheme S states that the set of closed points in S whose Frobenius conjugacy class in G is contained in R has Dirichlet density $|R|/|G|$, and in particular it is Zariski dense in S . Moreover, every element of G lies in a Frobenius conjugacy class. This being true for all finite quotients of π_1 , the statement of Theorem 3 follows.

Lemma 4. *Let $s = \text{Spec } \kappa \rightarrow S$ be a closed point and let $F_s \in \pi_1$ be a Frobenius element over s . The order of the image of P in $A(\kappa)$ is prime to ℓ if and only if the homomorphism $(\mathbb{T}_\ell M)^{\langle F_s \rangle} \rightarrow \mathbb{Z}_\ell$ is surjective.*

Proof. The order of P in the finite group $A(\kappa)$ is prime to ℓ if and only if P is ℓ -divisible in $A(\kappa)$. From the description of elements of $\mathbb{T}_\ell M$ by sequences as indicated at Remarks and Basic Facts, c), above, it follows that this is the case if and only if $\mathbb{T}_\ell M$ contains an element fixed by F_s which is mapped to $1 \in \mathbb{Z}_\ell$ by the canonical projection $\mathbb{T}_\ell M \rightarrow \mathbb{Z}_\ell$. \square

We are done once we have shown that $\Sigma_P := \{\sigma \in \pi_1 \mid (\mathbb{T}_\ell M)^{\langle \sigma \rangle} \rightarrow \mathbb{Z}_\ell \text{ is not surjective}\}$ contains a nonempty open subset of π_1 , provided $P \in A(S)$ has infinite order. We can also work with the image of π_1 in $\text{GL}(\mathbb{T}_\ell M)$ in place of π_1 , which we denoted by L^M .

Proposition 5. *If $P \in A(S)$ has infinite order, Σ_P contains a nonempty open subset of L^M .*

Proof. Let G be the subgroup of $\text{GL}(\mathbb{T}_\ell M)$ consisting of those elements which leave invariant the subspace $\mathbb{T}_\ell A$ of $\mathbb{T}_\ell M$ and act trivially on the quotient $(\mathbb{T}_\ell M/\mathbb{T}_\ell A) = \mathbb{Z}_\ell$. Relative to an appropriate \mathbb{Z}_ℓ -basis of $\mathbb{T}_\ell M$, the group G consists of matrices of the form

$$\left(\begin{array}{c|c} U & v \\ \hline 0 & 1 \end{array} \right), \quad U \in \text{GL}(2g, \mathbb{Z}_\ell), \quad v \in \mathbb{Z}^{2g}$$

where g is the relative dimension of A over S . The group L^M is a closed subgroup of G . By Proposition 2, there exist $\sigma \in L^M$, $\sigma \neq 1$. As matrices, elements $\sigma \neq 1$ are of the form:

$$\sigma = \left(\begin{array}{c|c} \text{id}_{2g} & w \\ \hline 0 & 1 \end{array} \right), \quad w \in \mathbb{Z}^{2g}, \quad w \neq 0$$

Let $N \geq 0$ be an integer such that $\ell^{-N} = |\ell^N|_\ell < \max_i |w_i|_\ell$, where w_i are the coefficients of w . We consider the subset X of G consisting of the matrices of the form

$$\left(\begin{array}{c|c} \text{id}_{2g} + \ell^N M & w + \ell^N v \\ \hline 0 & 1 \end{array} \right), \quad M \in \text{M}(2g \times 2g, \mathbb{Z}_\ell), \quad v \in \mathbb{Z}^{2g}$$

This set is open and closed in G , hence $X \cap L^M$ is open and closed in L^M . Moreover, the intersection $X \cap L^M$ is not empty because it contains σ . We are done if we show that $X \cap L^M$ is contained in Σ , so let us show that for all $x \in X$, the map $(\mathbb{T}_\ell M)^{\langle x \rangle} \rightarrow \mathbb{Z}_\ell$ is not surjective. Let $t \in (\mathbb{T}_\ell M)^{\langle x \rangle}$. We claim that the image of t in \mathbb{Z}_ℓ lies in $\ell \mathbb{Z}_\ell$. Indeed, set

$$x = \left(\begin{array}{c|c} \text{id}_{2g} + \ell^N M & w + \ell^N v \\ \hline 0 & 1 \end{array} \right), \quad t = \left(\begin{array}{c} t' \\ \lambda \end{array} \right), \quad t' \in \mathbb{Z}_\ell^{2g}, \quad \lambda \in \mathbb{Z}_\ell$$

We assume that $xt = x$, hence

$$0 = (x - \text{id}_{2g+1})t = \left(\begin{array}{c} \ell^N(Mt' + \lambda v) + \lambda w \\ 0 \end{array} \right)$$

However, the equality $\ell^N(Mt' + \lambda v) + \lambda w = 0$ can only hold if we have $|\lambda|_\ell < 1$, because of our choice of N . The image of t in \mathbb{Z}_ℓ is λ , and the last inequality shows $\lambda \in \ell \mathbb{Z}_\ell$. \square

Proof of Theorem 1. We just have to put the pieces together: If $P \in A(S)$ has infinite order, there exists by Proposition 5 and Theorem 3 a closed point $s : \text{Spec}(\kappa) \rightarrow S$ and a Frobenius element $F_s \in \pi_1$ over s , such that $(\mathbb{T}_\ell M)^{\langle F_s \rangle} \rightarrow \mathbb{Z}_\ell$ is not surjective. By Lemma 4 that means that the order of the image of P in $A(\kappa)$ is divisible by ℓ , so we are done. In fact,

the above arguments show that the set Σ_P is precisely the set of all the $s \in S$ such that the specialization P_s of P at s has order divisible by ℓ . On the other hand, with the notion of Dirichlet density (of closed points) as introduced in SERRE [Se1], it follows that Σ_P has positive Dirichlet density (which in principle can be explicitly given). \square

REFERENCES

- [Ar] Artin, E., *Geometric Algebra*, Interscience Publishers Inc., New York 1957
- [Bo] Bogomolov, F. A., *On two conjectures in birational algebraic geometry*, in: Algebraic Geometry and Analytic Geometry, ICM-90 Satellite Conference Proceedings, ed. A. Fujiki et al, Springer Verlag Tokyo 1991
- [B–T] Bogomolov, F. A. and Tschinkel, Y., *Reconstruction of function fields*, Geometric And Functional Analysis, Vol **18** (2008), 400–462.
- [BOU] Bourbaki, N., *Algèbre commutative*, Hermann Paris 1964.
- [GGA] Geometric Galois Actions I, LMS LNS Vol **242**, eds. Schneps, Lochak; Cambridge Univ. Press 1998.
- [G1] Grothendieck, A., Letter to Faltings, June 1983. See [GGA].
- [G2] Grothendieck, A., Esquisse d’un programme, 1984. See [GGA].
- [G–M] Grothendieck, A. and Murre, J., *The Tame Fundamental Group of a Formal Neighbourhood of a Divisor with Normal Crossings on a Scheme*, Springer 1971. ISBN-13: 9783540054993.
- [Ha] Harbater, D., *Fundamental groups and embedding problems in characteristic p* , in: Recent Developments in the Inverse Galois Problem, M. Fried et al, eds. AMS Contemporary Mathematics Series, Vol. **186** (1995), pp. 353–369.
- [Ho] Holschbach, A., *Diplomarbeit* (in German), Bonn 2004.
- [I1] Ihara, Y., *On Galois represent. arising from towers of covers of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$* , Invent. Math. **86** (1986), 427–459.
- [ILO] Illusie, L. and Laszlo, Y. and Orgogozo, F., *Travaux de Gabber sur l’uniformisation locale et la cohomologie étale des schémas quasi-excellents*, Séminaire à l’École Polytechnique 2006–08, See [arXiv:1207.3648v1 \[math.AG\]](https://arxiv.org/abs/1207.3648v1), July 2012
- [IM] Ihara, Y. and Matsumoto, M., *On Galois actions on profinite completions of braid groups*, in: Recent developments in the inverse Galois problem (Seattle, WA, 1993), 173–200, Contemp. Math. **186** AMS Providence, RI, 1995.
- [Jo] Jossen, P., *On the Mumford–Tate conjecture for 1-motives*, Invent. Math. **195** (2014), 393–439.
- [Kl] Kleiman, S. L., *The Picard Scheme. Fundamental algebraic geometry*, Math. Surveys Monogr. **123**, 235–321.
- [L1] Lang, S., *Algebra*, Revised third edition. Graduate Texts in Mathematics 211. Springer-Verlag, New York, 2002.
- [L2] Lang, S., *Introduction to Algebraic geometry*, Third printing, with corrections. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1972.
- [L3] Lang, S., *Diophantine Geometry*, Interscience, New-York 1962.
- [Mu] Mumford, D., *The Red Book of Varieties and Scheme*, LNM 1358, 2nd expanded edition, Springer-Verlag 1999.
- [Pk] Pink, R., *On the order of the reduction of a point on an abelian variety*, Math. Ann. **330** (2004), 275–291.
- [P0] F. Pop, *Étale Galois covers of affine smooth curves*, Invent. Math. **120** (1995), 555–578.
- [P1] Pop, F., *Pro- ℓ abelian-by-central Galois theory of Zariski prime divisors*, Israel J. Math. **180** (2010), 43–68.
- [P2] Pop, F., *Inertia elements versus Frobenius elements*, Math. Annalen **438** (2010), 1005–1017.
- [P3] Pop, F., *Recovering fields from their decomposition graphs*, in: Number theory, Analysis and Geometry –In memory of Serge Lang, eds: Goldfeld, Jorgenson, Jones, Ramakrishnan, Ribet, Tate; Springer 2011, pp. 519–594.
- [P4] Pop, F., *On the birational anabelian program initiated by Bogomolov I*, Invent. Math. **187** (2012), 511–533.

- [P5] Pop, F., *On the minimized decomposition theory of valuation*, Bull. Math. Soc. Sci. Math. Roumanie **58 (106)** (2015), 331–357.
- [P6] Pop, F., *On I/OM*, Manuscript 2014. See <http://www.math.upenn.edu/~pop/Research/Papers.html>.
- [R1] Roquette, P., *Zur Theorie der Konstantenreduktion algebraischer Mannigfaltigkeiten*, J. reine angew. Math. **200** (1958), 1–44.
- [R2] Roquette, P., *Nonstandard aspects of Hilbert’s irreducibility theorem*, in: Model theory and algebra (A memorial tribute to Abraham Robinson), LNM Vol. **498**, Springer, Berlin, 1975. 231–275.
- [S-T] Saidi, M. and Tamagawa, T., *Prime to p version of Grothendieck’s anabelian conjecture for Hyperbolic Curves over Finite Fields of Characteristic $p > 0$* , Publ. RIMS Kyoto University **45**, no. 1 (2009), 135–186.
- [Sb] Silberstein, A., *Anabelian Intersection Theory I: Conjecture of Bogomolov–Pop and Applications*, See [arXiv:1211.4608](https://arxiv.org/abs/1211.4608) [math.AG]
- [Se1] J-P. Serre, *Zeta and L -Functions*, in *Arithmetical Algebraic Geometry*, Proceedings of a Conference held at Purdue University, December 5-7, 1963, Harper & Row.
- [Se2] Serre, J.-P., *Cohomologie Galoisienne*, LNM 5, Springer 1965.
- [Sz] Szamuely, T., *Groupes de Galois de corps de type fini (d’après Pop)*, Astérisque **294** (2004), 403–431.
- [To1] Topaz, A., *Commuting-Liftable Subgroups of Galois Groups II*, J. reine angew. Math. (to appear) See [arXiv:1208.0583](https://arxiv.org/abs/1208.0583) [math.NT]
- [To2] Topaz, A., *Minimized decomposition theory for function fields of curves*, See the *Appendix* of [P5].
- [Z-S] Zariski, O. and Samuel, P., *Commutative Algebra*, Vol. II, Springer-Verlag, New York, 1975.

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