ON A CONJECTURE OF COLLIOT-THÉLÈNE

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ABSTRACT. The aim of this short note is to extend results by Denef and Loughran–Skorobogatov–Smeets concerning refinements of a conjecture of Colliot-Thélène. The problem is about giving necessary and sufficient conditions for morphisms of varieties to be surjective on local points for almost all localizations.

1. Introduction/Motivation

The aim of this note is to shed new light on a conjecture by Colliot-Thélène, cf. [CT], concerning the image of local rational points under dominant morphisms of varieties over global fields. The precise context is as follows:

- Let \( k \) be a global field, \( \mathbb{P}(k) \) be the places of \( k \), and \( k_v \) be the completion of \( k \) at \( v \in \mathbb{P}(k) \).
- Let \( f : X \to Y \) be a morphism of \( k \)-varieties.

For every \( v \in \mathbb{P}(k) \), the \( k \)-morphism \( f \) gives rise to a canonical map \( f^{k_v} : X(k_v) \to Y(k_v) \).

There are obvious examples showing that, in general, \( f^{k_v} \) is not surjective, e.g. \( f : \mathbb{P}_Q^1 \to \mathbb{P}_Q^1 \) of degree two. Therefore, for \( f : X \to Y \) as above, it is natural to consider the basic property:

\[(Srj) \quad f^{k_v} : X(k_v) \to Y(k_v) \text{ is surjective for almost all } v \in \mathbb{P}(k).\]

and to ask the following fundamental:

**Question:** Give necessary and sufficient conditions for \( f : X \to Y \) to have property \((Srj)\).

This problem was considered in a systematic way by Colliot-Thélène [CT], under the following restrictive but to some extent natural hypothesis:

\[(*) \quad k \text{ is a number field, } X, Y \text{ are projective smooth integral } k \text{-varieties, and } f : X \to Y \text{ is a dominant morphism with geometrically integral generic fiber.}\]

In particular, if \( L := k(Y) \) is the function field of \( Y \), then the generic fiber \( X_L \) of \( f \) can be viewed as an \( L \)-variety. In this notation, for morphisms \( f : X \to Y \) satisfying \((*)\), Colliot-Thélène considered the hypothesis \((CT)\) and made the conjecture \((CCT)\) below:

\[(CT) \quad \text{For each discrete valuation } k\text{-ring } R \subseteq L, \text{ and its residue field } \kappa_R, \]

\[\text{there is a regular flat } R\text{-model } X_R \text{ of } X_L \text{ whose special fiber } X_{R,R} \]

\[\text{has an irreducible component } X_\alpha \text{ which is } \kappa_R\text{-geometrically integral.}\]
Conjecture of Colliot-Thélène (CCT). Let $f : X \to Y$ be a dominant morphism of proper smooth geometrically integral varieties over a number field $k$, and suppose that hypotheses $(*)$ and (CT) are satisfied. Then $f : X \to Y$ has the property (Srj).

In a recent paper, Denef [D2] proved a much stronger form of the conjecture (CCT), by replacing the hypothesis (CT) by the weaker hypothesis (D) below. In order to explain Denef’s result, recall the following terminology: Let $f : X \to Y$ be a morphism satisfying hypothesis $(*)$. A modification of $f$ is any morphism $f' : X' \to Y'$ satisfying hypothesis $(*)$ such that there exist modifications (i.e., birational morphisms) $p : X' \to X$, $q : Y' \to Y$ satisfying $q \circ f' = f \circ p$. Given a modification $f' : X' \to Y'$ of $f$, for every Weil prime divisor $E' \subset Y'$, and the Weil prime divisors $D'$ of $X'$ above $E'$, consider the hypothesis: Theorem (Denef [D2], Main Theorem 1.2).

Let $f : X \to Y$ satisfy the hypotheses $(*)$ and (D). Then $f$ has the property (Srj).

Finally we recall the very recent results by Loughran–Skorobogatov–Smeets [LSS] which, for morphisms $f : X \to Y$ satisfying the hypothesis $(*)$ above, give necessary and sufficient conditions such that $f : X \to Y$ has property (Srj). Namely, following [LSS], in the notation introduced above, let $f' : X' \to Y'$ be a modification of $f : X \to Y$. For a Weil prime divisor $E'$ of $Y'$ and a Weil prime divisor $D'$ of $X'$ above $E'$, let $k(D')|k(E')$ be the function field extension defined by the dominant map $f'_D : D' \to E'$. One says that $E'$ is pseudo-split under $f' : X' \to Y'$, if for every element of the absolute Galois group $\sigma \in G_{k(E')}$, there is some Weil prime divisor $D'$ of $X'$ above $E'$ satisfying:

$$e(D'|E') = 1 \quad \text{and} \quad k(D') \otimes_{k(E')} \overline{k(E')} \text{ has a factor fixed by } \sigma.$$ 

Following Loughran–Skorobogatov–Smeets [LSS], consider the hypothesis:

(LSS) All Weil prime divisors $E' \subset Y'$ are pseudo-split under all modification $f'$ of $f$.

Note that if $D', E'$ satisfy hypothesis (D), then $k(D')|k(E')$ is a regular field extension, hence $k(D') \otimes_{k(E')} \overline{k(E')} \text{ is a field stabilized by all } \sigma \in G_{k(E')} \text{ (and } E' \text{ is called split). Hence hypothesis (D) implies (LSS), leading to the following sharpening of Denef’s result above:}

Theorem (Loughran–Skorobogatov–Smeets [LSS], Theorem 1.4).

Let $f : X \to Y$ satisfy $(*)$. Then $f$ satisfies hypothesis (LSS) iff $f$ has property (Srj).

The aim of this note is to provide a different approach to the basic problem and (CCT) considered above, which among other things allows the following:

- There are no smoothness/properness/irreducibility hypotheses on the $k$-varieties $X, Y$.
- Hypotheses (D), (LSS) can be replaced by the weaker hypotheses $(*)$, $(*)_{\Sigma_k}$ below.
- $k$ can be more general, e.g. a finitely generated field of characteristic zero, or finitely generated over a PAC field of characteristic zero.
- Finally, in positive characteristic $p > 0$, we give sufficient condition for (Srj) to hold, e.g. in the case $k$ is finitely generated, or finitely generated over a PAC field.
We begin by introducing notation and making a few remarks concerning basics about valuation theory to be used throughout the presentation.

Notations/Remarks 1.1.

- **Field extensions**: For a field $N$, let $\overline{N}|N$ denote an algebraic closure, and $N'|N \hookrightarrow \overline{N}|N$ be a subextension. A field extension $M|N$ is called $N'$-split, or split above $N'$, if $M \otimes_{N} N'$ has a field $M'|N'$ as a factor. If $N'$ is perfect, the following are equivalent:
  
  i) $M|N$ is $N'$-split (and if so, $M \otimes_{N} N'$ has regular field extension $M'|N'$ as a factor).
  
  ii) $M|N$ is separably generated, and $\overline{N} \cap M$ is embeddable in $N'$.

Let $M|N$ be $N'$-split, and $L|M$ be $M'$-split. One has the following **transitivity of splitness**:

a) If $N' = M' \cap N$, then $L|N$ is split above $N'$.

b) If $\tilde{M}|N \hookrightarrow M|N$ and $\tilde{N'}|N \hookrightarrow N'|N$ are subextensions, then $\tilde{M}|N$ is $\tilde{N'}$-split.

Finally, a field subextension $N'|N$ of $\overline{N}|N$ is called **co-procyclic**, if $\overline{N}|N'$ is procyclic.

- **$k$-Valuations/Prime divisors**: An (integral) $k$-variety is any separated reduced (integral) scheme of finite type over $k$. For a function field $N|k$, a **model** of $N|k$ is any integral $k$-variety $Z$ with function field $k(Z) = N$. For a valuation $v$ of $N$, let $m_v \subset O_v$ be its valuation ideal/ring, $vN$ be its value group, and $\eta v := \kappa(v)$ be the residue field of $v$.

A valuation $v$ of $N$ is called a **$k$-valuation** if $k \subset O_v$. Recall that a model $Z$ of $N|k$ is proper iff every $k$ valuation $v$ of $N$ has a center $z_v \in Z$, i.e., $O_{z_v} \subset O_v$, $m_{z_v} = O_{z_v} \cap m_v$.

The space of $k$-valuations $\text{Val}_k(N)$ of $N|k$, called the **Riemann–Zariski space** of $N|k$, carries naturally the **Zariski topology** via the following geometric interpretation:

Let $(Z_\alpha)_\alpha$ be any cofinal family of proper $k$-models of $Z_\alpha$ w.r.t. the domination relation. For $v \in \text{Val}_k(N)$, let $z_{\alpha,v} \in Z_\alpha$ be the center of $v$ on $Z_\alpha$. Then one has:

$$\text{Val}_k(N) = \lim_{\alpha} Z_\alpha, \quad v = (z_{\alpha,v})_\alpha, \quad O_v = \lim_{\alpha} O_{z_{\alpha,v}}, \quad m_v = \lim_{\alpha} m_{z_{\alpha,v}}.$$

A valuation $v$ of $N$ is called a **prime divisor** of $N|k$ if $v$ is defined by a Weil prime divisor $D \subset Z$ of some normal model $Z$ of $N|k$, i.e., $O_v = O_{\eta D}$ with $\eta D$ the generic point of $D$. In particular, $\eta v|k$ is a function field. For $v \in \text{Val}_k(N)$ the following are equivalent:

i) $v$ is a prime divisor of $N|k$.

ii) In the above notation, the center $z_{\alpha,v}$ of $v$ on some $Z_\alpha$ has codim$Z_{\alpha}(z_{\alpha,v}) = 1$.

iii) $\text{td}(N|k) = \text{td}(Nv|k) + 1$.

Let $D(N|k)$ denote the **set of prime divisors** of $N|k$ together with the trivial valuation.

- **Morphisms**: For extensions of function fields $M|N$ over $k$, the canonical restriction map $\text{Val}_k(M) \to \text{Val}_k(N)$, $v \mapsto v|_N$ is surjective, and defines a surjective map $D(M|k) \to D(N|k)$.

Let $D_w(M|k) := \{v \in D(N|k)|v \mapsto w\}$ be the fiber at $w$. For $v \in D_w(M|k)$ one has:

a) $O_w \hookrightarrow O_v$ defines a $k$-embedding of (residue) function fields $Nw \hookrightarrow Nv$.

b) $e(v|w) := (v M : wN)$ is finite iff either $w$ is non-trivial or $v$ is trivial.

**Definition 1.2.** Let $K|L$ be an extension of function fields over $k$.

1) $w \in D(L|k)$ is called **pseudo-split** in $D(K|k)$, if for every co-procyclic extension $l'|Lw$ there is $v \in D_w(K|k)$, with $Kv|Lw$ split above $l'$, and $e(v|w) = 1$ for $w$ non-trivial.

2) We call $D(L|k)$ **pseudo-split** in $D(K|k)$, if all $w \in D(L|k)$ are pseudo-split in $D(K|k)$.
We notice the following: Let \( f : X \to Y \) be a dominant morphism of projective smooth varieties over a number field \( k \), and \( K|L \) be the corresponding extension of function fields \( K = k(X), L = k(Y) \) over \( k \). Let \( (f_\alpha : X_\alpha \to Y_\alpha)_{\alpha} \) be the system of all the modifications of \( f \) satisfying the hypothesis (\(*\)). Then by Hironaka’s Desingularization Theorem, it follows that \( (X_\alpha, Y_\alpha)_{\alpha} \) are cofinal in the system of all the proper models of \( K \), respectively \( L \). Hence by the discussion at Notation/Remarks 1.1 above under Morphisms, it follows that the canonical restriction map \( D(K|k) \to D(L|k) \) is defined by the system \( (f_\alpha)_{\alpha} \). One has:

**Fact 1.3.** Let \( f : X \to Y \) be a morphism which satisfies hypothesis (LSS), and \( K|L \) be the corresponding extension of function fields over \( k \). Then \( D(L|k) \) is pseudo-split in \( D(K|k) \).

**Proof.** Klar, by mere definitions.

Next let \( Z \) be an integral \( k \)-variety, and \( N = k(Z) \) be its function field. A point \( z \in Z \) is called **valuation regular-like** (v.r.l.), if there exist \( \tilde{v} \in \text{Val}_k(N) \) and \( v \in D(N|k) \) both having center \( z \in Z \) such that \( N\tilde{v} = \kappa(z) \), \( Nv|k \) is a regular field extension, and \( v(u) = 1 \) for all \( u \in m_z \setminus m_z^2 \). Notice that **regular points** \( z \in Z \) are v.r.l.: Indeed, if \( (t_1, \ldots, t_d) \) is a system of regular parameters of \( \mathcal{O}_z \), the canonical k-embedding \( N \hookrightarrow \kappa(z)((t_1)) \cdots ((t_d)) \) defines a valuation \( \tilde{v} \in \text{Val}_k(N) \) with \( N\tilde{v} = \kappa(z) \). Further, the **degree valuation** \( v \) defined by \( (m_z^t)_i \), has as residue field the rational function field \( N\tilde{v} = \kappa(z)(t_i/t_d)^{i<d} \) and satisfies \( v(u) = 1 \) for all \( u \in m_z \setminus m_z^2 \). Finally, we say that \( Z \) is v.r.l., if all points \( z \in Z \) are v.r.l. points. In particular, regular \( k \)-varieties are v.r.l., but the converse does not hold: Indeed, rational double points and/or rational cusps of curves are v.r.l. points, but not regular points.

This being said, a first result extending/generalizing and shedding new light on the above mentioned [Df2], Main Theorem 1.2, and [LSS], Theorem 1.4, is as follows:

**Theorem 1.4.** Let \( K|L \) be function fields over a number field \( k \), and \( f : X \to Y \) be a dominant morphism of proper v.r.l. \( k \)-varieties defining \( K|L \). The following are equivalent:

i) The morphism \( f : X \to Y \) has property (Srj).

ii) \( D(L|k) \) is pseudo-split in \( D(K|k) \).

Hence the property (Srj) for dominant morphisms of proper v.r.l. \( k \)-varieties is **birational**.

The above Theorem 1.4 is proved in section 4, as a consequence of Theorem 4.1, and the more general Theorem 1.6 below, which considers the property (Srj) for morphisms of general varieties over number fields. In order to announce the latter result, we introduce notation and terminology as follows: Let \( f : X \to Y \) be a morphism of arbitrary varieties over an arbitrary base field \( k \), and let \( X_y \) be the reduced fiber of \( f \) at \( y \in Y \). For \( y \in Y \) and \( x \in X_y \), we denote \( L_y := \kappa(y), K_x := \kappa(x) \), hence \( f \) defines canonically a k-embedding of function fields \( K_x|L_y \). In particular, one has the canonical restriction map \( D(K_x|k) \to D(L_y|k) \), \( v_x \mapsto w_y := v_x|_{L_y} \), and too simplify notation, we set \( l_y := L_yw_y \) and \( k_x := K_xv_x \), hence \( \mathcal{O}_{v_y} \hookrightarrow \mathcal{O}_{v_x} \) gives rise to the canonical residue field k-embedding \( k_x|l_y \).

**Definition 1.5.** Let \( f : X \to Y \) be a morphism of \( k \)-varieties. In the above notation, define:

1) \( w_y \in D(L_y|k) \) is **pseudo-split** under \( f \), if for every co-procyclic extension \( l'|l_y \) there is \( x \in X_y \) and \( v_x \in D(K_x|k) \) such that \( k_x|l_y \) is \( l' \)-split, and \( e(v_x|w_y) = 1 \) if \( w_y \) is non-trivial.

2) We say that \( f \) is **pseudo-split**, if all \( w_y \in D(L_y|k), y \in Y \), are pseudo-split under \( f \).
Finally consider the following hypothesis:

\[ f : X \to Y \text{ is a pseudo-split morphism of } k\text{-varieties.} \]

**Theorem 1.6.** Let \( f : X \to Y \) be a morphism of arbitrary varieties over a number field \( k \). Then \( f : X \to Y \) satisfies hypothesis \((*)\) iff \( f : X \to Y \) has property \((\text{Srj})\).

We will prove actually a more general result, see Theorem 3.2 in section 3. The main point in our approach is to use Ax–Kochen–Ershov Principle (AKE) type results (together with some general model-theoretical principles about rational points and ultraproducts of local fields), as originating from \([Ax, A-K1, A-K2]\), see e.g. \([P-R]\) for details on AKE. Moreover, a weak form of AKE in positive characteristic, see hypothesis \((\text{qAKE})_{X_k}\) after Fact 2.6 below, implies that \((*)\) suffices for \((\text{Srj})\) to hold. To the contrary, \([Df2]\) and \([LSS]\) are based on quite deep desingularization results, building on previous results and ideas, see e.g. \([Df1, L-S, Sk]\) aimed—among other things—at giving arithmetic geometry proofs of AKE.

Here is an enlightening example—pointed out to me by DANIEL Loughran, where the above Theorem 1.6 applies, but the situation it is not covered by the previous methods.

**Example 1.7.** Let \( k = \mathbb{Q}, \ R = k[t] \), \( X = \text{Proj } R[T_0, T_1, T_2]/(T_0^2 + T_1^3 - t^2T_2^3) \), \( Y = \text{Spec } R \). One checks directly that the canonical projection \( f : X \to Y \) has the property \((\text{Srj})\), and \( f \) is smooth and split above \( y \in Y \) for \( y \neq (t) \). But the point \( x = (t_0, t_1, t) \in X \) above \( y = (t) \in Y \) is not v.r.l., hence this situation is not covered by previous work. On the other hand, \( f \) satisfies hypothesis \((*)\): Namely \( f \) is split above \( y \neq (t) \), and for \( y = (t) \) one has \( X_y \ni x \ni (t_0, t_1, t) \mapsto (t) \in Y \). \( K_x = k = L_y \), and \( D(K_x|k) = D(L_y|k) \) consist of the trivial valuation \( v_x^0 = v_y^0 \) only. Hence \( f \) is pseudo-split above \( y \) in the sense of Definition 1.5.

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2. Basic facts

2.1. Abstract approximation results for points.

We begin by recalling a few facts, which are/might be well known to experts. See e.g. \([B-S], [Ch], [F-J], \) Ch.7, for details on ultraproducts and other model theoretical facts.

**Fact 2.1.** Let \((k_i|k)_i \in \mathcal{I}\) be a family of field extensions, \(\mathcal{P}_I \) be a fixed prefilter on \( I \), and for every ultrafilter \( U \) on \( I \) with \( \mathcal{P}_I \subset U \), let \( k\mathcal{U} := \prod_{i \in I} k_i/U \) be the corresponding ultraproduct. Then for every morphism \( f : X \to Y \) of \( k\)-varieties, the following are equivalent:

i) There is \( I_0 \in \mathcal{P}_I \) such that the map \( f^{k\mathcal{U}} : X(k\mathcal{U}) \to Y(k_i) \) is surjective for all \( i \in I_0 \).

ii) The map \( f^{k\mathcal{U}} : X(k^{\mathcal{U}}) \to Y(k^{\mathcal{U}}) \) is surjective for all ultrafilters \( U \).

In particular, if \( I \) is infinite, then \( f^{k\mathcal{U}} : X(k\mathcal{U}) \to Y(k_i) \) is surjective for almost all \( i \in I \) if and only if \( f^{k\mathcal{U}} : X(k^{\mathcal{U}}) \to Y(k^{\mathcal{U}}) \) is surjective for all non-principal ultrafilters \( U \) in \( I \).

**Proof.** To i) \( \Rightarrow \) ii): To simplify notation, we can suppose that \( I = I_0 \), or equivalently, \( f^{k\mathcal{U}} : X(k\mathcal{U}) \to Y(k_i) \) is surjective for every \( i \in I \). Let \( U \) be an ultrafilter on \( I \) with \( \mathcal{P}_I \subset U \), and \( \kappa(y) \in k^{\mathcal{U}} \) be defined by \( \kappa(y) \mapsto \kappa_i \) for some \( y \in Y \). Let \( V \subset Y \) be an affine open neighborhood of \( y \), say \( k[V] = k[u] \).
the $k$-algebra $S$. Then by mere definitions, there is a system $u_i$ of $n$ elements of $k_i$ such that $y_i$ is defined by the morphism of $k$-algebras

$^*\psi_i : S \to S/y \hookrightarrow k_i, \quad u \mapsto u_i,$

Hence, $U$-locally, there exist systems $u_i$ of $n$ elements of $k_i$ and morphisms of $k$-algebras

$\psi_i : S \to S/y \hookrightarrow k_i, \quad u \mapsto u_i,$

defining $\psi_i$, i.e., $u_i = (u_i)_{i} \in U$, and let $y_i \in Y(k_i)$ be the $k_i$-rational point defined by $\psi_i$.

Finally, let $(U_a)_0, U_a = \text{Spec } R_a,$ be a finite open affine covering of $f^{-1}(V) \subset X$. Then $X(k_i) = \bigcup O_i U_a(k_i)$ for all $k_i$, and $y_i \in \bigcup O_i f(U_a(k_i))$ for every $i \in I$. Since $(U_a)_0$ is finite, there exists some $U := U_{ao}$ such that $U$-locally one has: $y_i \in f(U(k_i))$. Equivalently, $U$-locally, there exists $x_i \in U(k_i)$ such that $f^k(x_i) = y_i$. Let $R := k[U]$ be the $k$-algebra of finite type with $U = \text{Spec } R$. Then $f|_U : U \to V$ is defined by a unique morphism $f^\#_U : S \to R$ of $k$-algebras, and there is a unique $k$-morphism

$\phi_i : R \to R/x_i \hookrightarrow k_i$

defining $x_i \in U(k_i)$. Further, the fact that $f^k(x_i) = y_i$ is equivalent to $f^\#_U \circ \psi_i = \phi_i$. Hence if $\phi_i : R \to k_i$ is the $k$-morphism having $U$-local representatives $\phi_i : R \to k_i$, then one has

$\psi_i \circ f^\#_U = \phi_i.
$

Hence if $x_{u_i} \in X(k_{u_i})$ is the $k_{u_i}$-rational point of $X$, it follows that $f^\#_{U}(x_{u_i}) = y_{u_i}$.

To ii) $\Rightarrow$ i): By contradiction, suppose that for every $J \in P_I$ there exists $j \in J$ such that $f^k_j : X(k_j) \to Y(k_j)$ is not surjective. Then setting $I' := \{i \in I \mid f^k_i$ is not surjective$, one has: $P'_I := \{j \cap I' \mid J \in P_I\}$ is a prefilter on $I'$, and since $P_I \prec P'_I$, every ultrafilter $\mathcal{U}$ on $I'$ containing $P'_I$ is the restriction $\mathcal{U}' = \mathcal{U}|_{P'_I}$ of an ultrafilter $\mathcal{U}$ on $I$ containing $P_I$. Hence mutatis mutandis, w.l.o.g., we can suppose that there is an ultrafilter $\mathcal{U}$ continuing $P_I$ and a set $J \in \mathcal{U}$ such that $f^k_J$ is not surjective for all $i \in J$. Let $(V_{\beta})_\beta$ be a finite open affine covering of $Y$. Then reasoning as above, there exists some $V := V_{\beta_0}$ such that $U$-locally one has: $V(k_i) \not\subseteq f^k_i(X(k_i))$. Equivalently, $U$-locally, there exists $y_i \in V(k_i)$ such that $y_i \not\in f^k_i(X(k_i))$. That being said, let $\psi_i : S := k[V] \hookrightarrow k_i$ be the $k$-algebra of $k$-algebras defining $y_i \in V(k_i)$, and $^*\psi_i : S \to k_i$ be the $k$-morphism defined by $(\psi_i)_i$. Then $^*\psi_i : S \to k_i$ defines a $k_i$-rational point $y_i \in V(k_{u_i}) \subset Y(k_{u_i})$. Hence by the hypothesis, there is $x_{u_i} \in X(k_{u_i})$ such that $f^k_{u_i}(x_{u_i}) = y_{u_i}$. Let $y \in V$ and $x \in X$ be such that $^*y_i$ and $x_{u_i}$ are defined by $k$-embeddings $\kappa(y) \mapsto k_i$, respectively $\kappa(x) \mapsto k_i$. Then choosing $U \subset X$ affine open with $x \in U$ and $f(U) \subset V$, and setting $R := k[U]$, the following hold:

a) $f|_U : U \to V$ is defined by a unique morphism of $k$-algebras $f^\#_U : S \to R$.

b) $x_{u_i}$ is defined by a unique morphism of $k$-algebras $\phi_i : R \to R/x \hookrightarrow k_i$.

c) One has that $\phi_i = \psi_i \circ f^\#_U$.

Therefore, letting $\phi_i : R \to k_i$ be $U$-local representatives for $\phi_i$, by the general nonsense of ultraproducts, $U$-locally one has:

$\phi_i = \psi_i \circ f^\#_U$.

Hence if $x_i$ is the $k_i$-rational point of $X$ defined by $\phi_i : R \to k_i$, it follows that $f^k_i(x_i) = y_i$. Therefore, $U$-locally, one must have that $y_i \in f(X(k_i))$, contradiction!
Finally, for the last assertion of Fact 2.1, we notice: First, the set $\mathcal{P}_I$ of all the cofinite subsets of $I$ is a prefilter on $I$, and $I' \in \mathcal{P}_I$ iff $I \setminus I'$ is finite. Second, an ultrafilter $\mathcal{U}$ on $I$ is non-principal iff $\mathcal{P}_I \subset \mathcal{U}$. Conclude by applying the equivalence i) $\iff$ ii) to this situation. □

**Definition 2.2.** A field $k$-embedding $k' \rightarrow l'$ is called quasi-elementary, if there are field $k$-embeddings $k' \rightarrow l' \rightarrow k'' \rightarrow l''$ with $k'' | k'$ and $l'' | l'$ elementary $k$-embeddings.

**Fact 2.3.** Let $f : X \rightarrow Y$ be a morphism of varieties over an arbitrary base field $k$, and let $\mathcal{C}_f$ be the class of all the field extensions $k'|k$ with $f^{k'} : X(k') \rightarrow Y(k')$ surjective, One has:

1. $\mathcal{C}_f$ is an elementary class, i.e., $\mathcal{C}_f$ is closed w.r.t. ultraproducts and sub-ultrapowers.

2. Let $k' \hookrightarrow l'$ be a quasi-elementary $k$-field extension. Then $k' \in \mathcal{C}_f$ iff $l' \in \mathcal{C}_f$.

**Proof.** Assertion 1) follows from Fact 2.1 by mere definition. To 2): We begin by noticing that $k_1 \subseteq l_1$ implies $X(k_1) \subseteq X(l_1)$ for all field extensions $k_1 \subseteq l_1$. First, suppose that $l' \in \mathcal{C}_f$. Then $Y(k') \subseteq Y(l') = f^{l'}(X(l')) = f^{k''}(X(k''))$, thus $Y(k') \subseteq f^{k'}(X(k'))$, because $k'$ is existentially closed in $k''$. Hence finally $Y(k') = f^{k'}(X(k'))$. Second, let $k' \in \mathcal{C}_f$. Embeddings $k' \hookrightarrow l' \hookrightarrow k'' \hookrightarrow l''$ as in Definition 2.10 imply both: $k'' \in \mathcal{C}_f$, by assertion 1) above; and $l'$ is existentially closed in $l''$. Hence reasoning as in the first case, one gets $l' \in \mathcal{C}_f$. □

2.2. **Ultraproducts of localizations of arithmetically significant fields.**

We introduce notation and recall well known facts. We generalize the context in which the conclusion of Theorem 1.6 holds, finally allowing to announce Theorem 3.2 below.

**Notations/Remarks 2.4.** For arbitrary fields $k$, consider sets $\Sigma_k$ of (equivalence classes of) discrete valuations $v$ of $k$ such that for all finite non-empty subsets $A \subseteq k$ one has:

(P) $U_A := \{v \in \Sigma_k | A \subseteq O_k^v \} \neq \emptyset$, hence $\mathcal{P}_{\Sigma_k} := \{U_A | A \subseteq k^\times \text{finite} \}$ is a prefilter on $\Sigma_k$.

**Example 2.5.** Let $X$ be an integral $S$-variety, where $S$ is either $\mathbb{Z}$ or a field, $k := \kappa(X)$ be its function field, and $X_0 \subseteq X$ be the set of regular closed points in $X$.

a) $\Sigma_k$ satisfies (P) iff $X_{\Sigma_k} := \{x_v \in X | x_v \text{ is the center of } v \in \Sigma_k \}$ is Zariski dense in $X$.

b) If $X_0$ is Zariski dense, there are $\Sigma_k$ with $X_{\Sigma_k} = X_0$, and satisfying the following:

i) If $X$ is an integral $\mathbb{Z}$-variety, then $kv = \kappa(x_v)$ for all $v \in \Sigma_k$.

ii) If $X$ is an integral variety over a field $k_0$, then $kv = \kappa(x_v)$ for all $v \in \Sigma_k$.

**Convention.** From now on, let $U$ always denote ultrafilters on $\Sigma_k$ with $\mathcal{P}_{\Sigma_k} \subseteq U$. Further, let $k_v$ be the completion of $k$ at $v \in \Sigma_k$, and consider the ultraproducts:

$^\ast k_v := \prod_v k_v / U$, $^\ast O_v := \prod_v O_v / U$, $^\ast m_v := \prod_v m_v / U$, $^\ast \kappa_v := \prod_v k_v / U$.

Then $^\ast O_v$ is the valuation ring of $^\ast k_v$, say $^\ast O_v = O_{^\ast k_v}$ of the valuation $^\ast \kappa_v$, with valuation ideal $^\ast m_v = m_v$, residue field $^\ast k_v / ^\ast m_v = \kappa_v$, and value group $^\ast v_v k_v = \prod_v k_v / U = \mathbb{Z}^\Sigma_k / U$.

1) One has the (canonical) diagonal field embedding $^\ast \kappa_v \hookrightarrow ^\ast k_v$ and $^\ast \kappa_v$ is trivial on $k$ (by the fact that $\mathcal{P}_{\Sigma_k} \subseteq U$).

2) If $\omega_v \subseteq O_v$ is a set of representatives of $k_v$, then $^\ast \omega_v := \prod_v \omega_v \subseteq ^\ast O_v$ is a system of representatives of $^\ast k_v \subseteq ^\ast O_v$ inside $^\ast O_v$. In particular, if $\omega_v$ are multiplicative, so is $^\ast \omega_v$.

3) The value group $^\ast v_v k_v$ is a $\mathbb{Z}$-group. Further, if $\pi_v \in k_v$ is a uniformizing parameter for $v \in \Sigma_k$, then $^\ast \pi_v = (^\ast \pi_v) / U$ is an element of minimal value in $^\ast v_v k_v$.  


4) The field $^*k_u\!$ is Henselian with respect to $^*v_{u\!}$, and one has:
   a) Suppose that $\text{char}(k) = 0$. Then $^*v_{u\!}$ is trivial on $\mathbb{Q} \subset k_{u\!}$, and if $\mathcal{T} \subset ^*\mathcal{O}_{u\!}$ is any lifting of a transcendence basis of $k_{u\!} / \mathbb{Q}$, by Hensel Lemma one has: The relative algebraic closure $k_{u\!'\!} \subset ^*\mathcal{O}_{u\!}$ of $\mathbb{Q}(\mathcal{T})$ in $^*k_{u\!}$ is a field of representatives for $k_{u\!}$.
   b) The fields of representatives $k_{u\!'\!} \subset ^*\mathcal{O}_{u\!}$ for $k_{u\!}$ are relatively algebraically closed in $^*k_{u\!}$.
   c) For $k_{u\!'\!} \subset k_{u\!}$ as above, let $k_{u\!} := k_{u\!'\!}(\pi_{u\!})^h$ be the Henselization of $k_{u\!'\!}(\pi_{u\!})$ w.r.t. the $\pi_{u\!}$-adic valuation, and set $v_{u\!} := ^*v_{u\!}|_{k_{u\!}}$. Then one has $k$-embeddings:
   
   $$k_{u\!} := k_{u\!'\!}(\pi_{u\!})^h \hookrightarrow ^*k_{u\!} \text{ as valued fields, and } \kappa(u\!_{u\!}) = k_{u\!} = \kappa(^*v_{u\!}|_{k_{u\!}}),$$

   and further, $v_{u\!}k_{u\!} = \mathbb{Z} \hookrightarrow \mathbb{Z}^{\mathbb{Z}_{p \neq q}} / U = \hat{v_{u\!}}^*k_{u\!}$ are $\mathbb{Z}$-groups having $\pi_{u\!}$ as the element of minimal positive value. Hence the Ax–Kochen–Ershov Principle (AKE) implies:

   **Fact 2.6.** If $\text{char}(k) = 0$, then $k_{u\!} \hookrightarrow ^*k_{u\!}$ is an elementary $k$-embedding of (valued) fields.

Unfortunately, if $\text{char}(k) = p > 0$, it is not known whether the conclusion of Fact 2.6 holds. Therefore, for $U$ on $\Sigma_{\!k}$ as in Remarks/Notations 2.4 above, and the corresponding $k$-embeddings $k_{u\!} = \kappa_{u\!}(\pi_{u\!})^h \hookrightarrow ^*k_{u\!}$, consider the following hypothesis—which is weaker than the Ax–Kochen–Ershov Principle (AKE), holding if $\text{char}(k)$ is zero:

   $$(qAKE)_{\Sigma_{\!k}} \quad k_{u\!} \hookrightarrow ^*k_{u\!} \text{ are quasi-elementary } k\text{-embeddings for all } U.$$  

In the above notation, one has (see CHATZIDAKIS [Ch], and FRIED–JARDEN [F-J], Ch. 11):

   **Fact 2.7 (Residue fields).** Let $k$ be as in Example 2.5, b).
   
   1) In case i), $\kappa_{u\!}$ is a perfect $\mathfrak{N}_1$-saturated PAC quasi-finite field.\(^1\)
   2) In case ii), let $k_0$ be PAC (perfect). Then $\kappa_{u\!}$ is PAC (perfect) and $\mathfrak{N}_1$-saturated.

Note that if $\text{char}(k) = p > 0$, then in case 1) above one has: If $F_v := \mathbb{F}_p \cap k_v$, then $k_v = F_v((\pi_v))$ for any $\pi_v \in k$ with $v(\pi_v) = 1$. Hence $F_U := \prod_v F_v / U \subset ^*\mathcal{O}_{u\!}$ is a perfect field and a system of representatives for $k_{u\!}$. Further, if $t \in k$ is part of a separable transcendence basis for $k|\mathbb{F}_p$, then $U_t := \{v|v(t) = 0\}$ lies in $U$, and if $a_v = tv \in k_v v = F_v$ is the residue of $t$ at $v$, then $\pi_v := t - a_v \in k_v$ satisfies $v(\pi_v) = 1$, and one has a canonical embedding:

   $$k \hookrightarrow F_v((\pi_v)) = k_v.$$

Thus setting $a_v := (a_v)v / U \in \kappa_{u\!}$, one has $\pi_{u\!} = t - a_{u\!} \in kF_{u\!}$. But despite of these special/particular facts, it is unknown whether the conclusion of Fact 2.6 holds in this case.

### 2.3. Generalized pseudo-split extensions.

We begin by discussing the case of fields $k$ as in Example 2.5, b), i). Precisely, $k = \kappa(X)$ is the function field of an integral $\mathbb{Z}$-variety $X$, and further: $X_0 \subset X$ is the set of closed regular points (which is Zariski open in the set of all closed points), and $X_{\Sigma_{\!k}} \subset X_0$ is Zariski dense.

We say that $\sigma \in G_k$ and the co-procid extension $\mathbb{F}_l^\sigma | k$ of $k$ are $\Sigma_{\!k}$-definable, if for all finite Galois extensions $l|k$, and all $U_A \in \mathcal{P}_{\Sigma_{\!k}}$, one has:

$$U_{A,l|k}(\sigma) := \{v \in U_A|v \text{ unramified in } l|k \text{ and Frob}(v) = \sigma|_l \neq \emptyset\}.$$

Notice that in the case $X_{\Sigma_{\!k}} \subset X_0$ and $kv = \kappa(x_v)$ for all $v \in \Sigma_{\!k}$ one has:

\(^1\)That is, the absolute Galois group $G_{k_{u\!}}$ of $k_{u\!}$ is $G_{k_{u\!}} \cong \hat{\mathbb{Z}} \cong G_F$, where $F$ is any finite field.
- If $X_{\Sigma}$ has Dirichlet density $\delta(X_{\Sigma}) = 1$, e.g. if $X_{\Sigma} \subset X_0$ is Zariski open, it follows by the Chebotarev Density Theorem, see e.g. Serre [Se1], that all $\sigma \in G_k$ are $\Sigma_k$-definable.
- If $\Sigma_k$ is Frobenian in the sense of Serre [Se2], 3.3, say defined by a finite Galois extension $l|k$ and a set of conjugacy classes $\Phi \subset \text{Gal}(l|k)$, then $\sigma \in G_k$ is $\Sigma_k$-definable iff $\sigma|_l \in \Phi$.

**Fact 2.8.** In the above notation, $\sigma \in G_k$ is $\Sigma_k$-definable iff $\overline{\kappa} = \kappa_u \cap \overline{k}$ for some $U$.

**Proof.** For the direct implication, notice that $\mathcal{P}_{\Sigma_k}(\sigma) := \{U_{A,l|_l k}\} A,l|_l k$ is a prefilter on $\Sigma_k$ such that any ultrafilter $U$ containing $\mathcal{P}_{\Sigma_k}(\sigma)$ contains $\mathcal{P}_{\Sigma_k}$. Let $l|k$ be a finite Galois extension. Then for $v \in U_{A,l|_l k}(\sigma) \in U$, setting $l_v := l_k^v$ one has: $l_v|k_v$ is unramified and $l^v = l \cap k_v$.

Hence $l^v = l \cap \kappa_u$, and finally $\overline{k} = \overline{k} \cap \kappa_u$.

Conversely, let $U$ be such that $\overline{\kappa} = \kappa_u \cap \overline{k}$. To show that $\sigma$ is $\Sigma_k$-definable, we have to show that all the sets $U_{A,l|_l k}(\sigma)$ are non-empty. First, since $\overline{k} = \kappa_u \cap \overline{k}$, it follows that for every finite Galois extension $l|k$, one has $l^v = \kappa_u \cap l$. Hence for every $l|k$ there exists a set $V_l \in U$ such that for all $v \in V_l$ one has $l^v = k_v \cap l$. Further, let $U_A \subset \Sigma_k$ be given. Since $\mathcal{P}_{\Sigma_k} \subset U$, hence $U_{A} \in U$, w.l.o.g., we can suppose that $V_l \subset U_A$. Second, let $B \subset k^x$ be a finite set such that all discrete valuations $w$ of $k$ with $w(B) = 0$ are unramified in $l|k$. (Note that such sets $B$ exist: If $X_l \to X$ is the normalization of $X$ in the finite Galois extension $l|k$, then there exists an affine open dense subset $X' \subset U$ such that $X_l$ is étale above $X'$. Hence if $w$ has its center in $X'$, then is unramified in $l|k$, etc.) Then setting $A_l := A \cup B$, one has: $V_l \cap U_{A_l} \in U$, and all $v \in V_l \cap U_{A_l}$ are unramified in $l|k$. Hence $U_{A_l,l|_l k} \neq \emptyset$, thus $U_{A,l|_l k} \cap U_{A,l|_l k}$ is non-empty as well, thus concluding that $\sigma$ is $\Sigma_k$-definable. $\square$

Finally, in the above notation, let $N|k$ be a field extension. We say that $\sigma|_N \in G_N$ and the co-cyclic field extension $N' := \overline{N}^{\sigma_N}$ of $N$ is $\Sigma_k$-definable, if $\sigma|_N|_k =: \sigma \in G_k$ is $\Sigma_k$-definable.

**Proposition 2.9.** Let $M|N$ be function fields over $k$ as in Example 2.5, b), i). One has:

1) An algebraic extension $N'|N$ is co-procyclic $\Sigma_k$-definable if and only if there is $U$ and a $k$-embedding $N \hookrightarrow \kappa_u$ such that $N' = \overline{N} \cap \kappa_u$.

2) Let $N' = \overline{N} \cap \kappa_u$ as above be given. Then $M|N$ is split above $N'$ iff $M|N$ is separably generated and $N \hookrightarrow \kappa_u$ prolongs to a field embedding $M \hookrightarrow \kappa_u$.

**Proof.** To 1): To the direct implication: Since $\kappa_u$ is a perfect pseudo-finite field, $k \hookrightarrow N \hookrightarrow \kappa_u$ gives rise to embedding of perfect fields $k' = \overline{k} \cap \kappa_u \hookrightarrow N' = \overline{N} \cap \kappa_u \hookrightarrow \kappa_u$, and to surjective projections $\overline{\mathbb{Z}} \cong G_{\kappa_u} \to G_{N'} \to G_K$. Hence $N'|N$ is by mere definitions co-procyclic and $\Sigma_k$-definable. For the converse implication, let $N'|N$ be co-procyclic and $\Sigma_k$-definable. Then $k' := \overline{k} \cap N'$ is obviously co-procyclic and $\Sigma_k$-definable. Hence, there is some $U$ such that $k' = \overline{k} \cap \kappa_u$, and obviously, $N'|k'$ is a regular field extension. We claim that there is a $k$-embedding $N \hookrightarrow \kappa_u$ such that $N' = \overline{N} \cap \kappa_u$, hence $k' \subset N'$. First, $N_0' := N k' \subset N'$ is a regular function field over $k'$, and setting $\tilde{N}_0 = N_0'$, there is an increasing sequence of cyclic field subextensions $(\tilde{N}_i|N_i')_{i \in \mathbb{N}}$ of $\overline{N}|N'$ such that $N' = \cup_{i \in \mathbb{N}} N_i'$, $\overline{N} = \cup_{i \in \mathbb{N}} \tilde{N}_i$, and $\tilde{N}_i|N_i'$ is the maximal subextension of $\overline{N}|N'$ of degree $\leq i$. By algebra general non-sense, the sequence $(\tilde{N}_i|N_i')_{i}$ and the conditions it satisfies are expressible by a type $p(\mathbf{t})$ over $k'$, where $\mathbf{t}$ is a transcendence basis of $N_0|k'$; and since $\kappa_u$ is a perfect PAC quasi-finite field, the type $p(\mathbf{t})$ is finitely satisfiable. Thus $\kappa_u$ being $\kappa_1$-saturated, the type $p(\mathbf{t})$ is satisfiable in $\kappa_u$, thus $N = N_0$ has a $k'$-embedding $N \hookrightarrow \kappa_u$ such that $N' = \overline{N} \cap \kappa_u$. 9
To 2): For the direct implication, let $M'$ be a factor of $M \otimes_N N'$ such that $M'|N'$ is a regular field extension. Since $N'|N$ contains the prefect closure of $N$, it follows that $M|N$ must be separably generated (because otherwise all the factors of $M \otimes_N N'$ have non-trivial nilpotent elements). Hence $M = N(Z_N)$ for integral $N$-variety $Z_N$ such that $Z_N \times_N N'$ has a geometrically integral irreducible component $Z_{N'}$ of multiplicity one with $M' = N'(Z_{N'})$. Since $\kappa_{\mathfrak{N}}$ is a $\mathfrak{N}$-saturated PAC (quasi-finite) field, $Z_{N'}(\kappa_{\mathfrak{N}})$ contains “generic points” of $X_{N'}$, that is, $M'$ is $N'$-embeddable into $\kappa_{\mathfrak{N}}$.

For the reverse implication, since $M|N$ is separably generated, it follows that $M \otimes_N N'$ is a product of fields. Let $M \hookrightarrow \kappa_{\mathfrak{N}}$ be a prolongation of $N \hookrightarrow \kappa_{\mathfrak{N}}$. Then

$$N' := N \cap \kappa_{\mathfrak{N}} \hookrightarrow \overline{M} \cap \kappa_{\mathfrak{N}} =: M' \hookrightarrow \kappa_{\mathfrak{N}}$$

are co-procyclic extensions, and $M \otimes_N N'$ has a factor $M_{N'}$ which is $N'$-embeddable in $M'$. Since $N'$ is perfect, $N' = \overline{N} \cap M' \hookrightarrow M'$ is regular, hence $M_{N'}|N'$ is regular. 

The Proposition 2.9 above hints at the following generalization of the pseudo-splitness:

**Definition 2.10.** In Notations/Remarks 2.4, let $M|N$ be a $k$-field extension, and $j : N \to \kappa_{\mathfrak{N}}$ be a field $k$-embedding.

1) An algebraic $k$-field extension $N'|N$ is called $j$-definable, if $N'$ is isomorphic to $\overline{N} \cap \kappa_{\mathfrak{N}}$ as $N$-field extensions. To simplify notation, we write $N' = \overline{N} \cap \kappa_{\mathfrak{N}}$.

2) For $j : N \hookrightarrow \kappa_{\mathfrak{N}}$ defining $N'|N$ as above, $M|N$ is called $j$-split, if $M|N$ is separably generated, and $j$ prolongs to $M' \hookrightarrow \kappa_{\mathfrak{N}}$.

**Proposition 2.11.** Let $M|N$ be an extension of $k$-function fields over $k$. Let $j : N \hookrightarrow \kappa_{\mathfrak{N}}$ be a $k$-embedding, and $N' = \overline{N} \cap \kappa_{\mathfrak{N}}$ be a $j$-definable extension of $N$. One has:

1) Let $M = N(Z_N)$ with $Z_N$ an integral $N$-variety. Then $M|N$ is $j$-split iff $Z_N \times_N N'$ is geometrically reduced and $Z_N(\kappa_{\mathfrak{N}})$ is Zariski dense.

2) In particular, for a $N' = \overline{N} \cap \kappa_{\mathfrak{N}}$ as above, the following hold:
   a) If $\kappa_{\mathfrak{N}}$ is PAC, $M|N$ is $N'$-split iff $M \otimes_N N'$ factor $M'$ with $M'|N'$ regular.
   b) If $\text{char}(k) = 0$, $M|N$ is $N'$-split iff $j: N \hookrightarrow \kappa_{\mathfrak{N}}$ has a prolongation $M \hookrightarrow \kappa_{\mathfrak{N}}$.

**Proof.** To 1): The implication $\Rightarrow$ is simply a reformulation in terms of algebraic geometry of the fact that $M|N$ is $N'$-split. For the converse implication, one has: First, $Z_{N'} := Z_N \times_N N'$ being reduced, its ring of rational functions is the product of the function fields $M'_\alpha := N'(Z'_{\alpha})$ of the irreducible components $Z'_{\alpha}$ of $Z_{N'}$. Second, since $Z_N(\kappa_{\mathfrak{N}})$ is Zariski dense, $Z'_{\alpha}(\kappa_{\mathfrak{N}})$ is Zariski dense for some $\alpha$. And since $\kappa_{\mathfrak{N}}$ is $\mathfrak{N}$-saturated, by general model theoretical nonsense, $Z'_{\alpha}(\kappa_{\mathfrak{N}})$ contains “generic points” of the $N'$-variety $Z'_{\alpha}$. Finally, each such point defines an $N'$-embedding $M'_\alpha = N'(Z'_{\alpha}) \hookrightarrow \kappa_{\mathfrak{N}}$, which prolongs $j : N \hookrightarrow \kappa_{\mathfrak{N}}$.

To 2): First, the implication $\Rightarrow$ is the same as in assertion 1. The converse implication in case b) is clear, and in case a) it follows from assertion 1): Since $\kappa_{\mathfrak{N}}$ is a PAC field, and $Z_{N'}$ is a geometrically integral $N'$-variety, it follows that $Z_{N'}(\kappa_{\mathfrak{N}})$ is Zariski dense, etc. 

**Corollary 2.12 (Example 2.5 revisited).** Let $k$ and $\Sigma_k$ be as in Example 2.5, b). Let $N|k$ be a function field over $k$, and $N'|N$ be $j$-definable. A $k$-extension of function fields $M|N$ is $N'$-split iff $M \otimes_N N'$ has a factor $M'$ such that $M'|N'$ is a regular field extension.
3. Proof of (Generalizations of) Theorem 1.6

3.1. Setup for a generalization of Theorem 1.6.

The generalization of Theorem 1.6 we aim at is based on generalizing hypothesis \((\star)\), i.e., the notion of pseudo-split morphism \(f : X \to Y\), as already hinted at in Definition 2.10. Recall Notations/Remarks 1.1, 2): For morphisms of \(k\) -varieties \(f : X \to Y\), let \(X_y \subset X\) be the fiber at \(y \in Y\), and for \(x \in X_y\), set \(L_y := \kappa(y) \hookrightarrow \kappa(x) =: k_x\). We defined:
- The map \(f_y^1 : \cup_{x \in X_y} \mathcal{D}(k_x|k) \to \mathcal{D}(L_y|k), v_x \mapsto w_y := v_x|_{L_y}\) induced by \(f\).
- The fiber \(\mathcal{D}_{w_y} := \cup_{x \in X_y} \{v_x \in \mathcal{D}(k_x|k) | w_y = v_x|_{L_y}\}\) of \(f_y^1\) at \(w_y \in \mathcal{D}(L_y|k)\).
- The canonical embedding \(l_y := L_yw_y \hookrightarrow k_xv_x =: k_x\) of residue fields for \(v_x \in \mathcal{D}_{w_y}\).

**Definition 3.1.** Let \(k, \Sigma_k, \) and \(\mathcal{U}\) be as in Remarks/Notations 2.4. Recalling Definition 2.10, for morphisms of \(k\)-varieties \(f : X \to Y\), define:
1) \(w_y \in \mathcal{D}(L_y|k)\) is \(k\)-pseudo-split under \(f\), if for all \(\mathcal{U}\) and all \(k\)-embeddings \(j : l_y \hookrightarrow \kappa_{l_y}\) there is \(v_x \in \mathcal{D}_{w_y}\) such that \(k_x|_{l_y}\) is \(j\)-split, and \(e(v_xw_y) = 1\) if \(w_y\) is non-trivial.
2) We say that \(f\) is \(k\)-pseudo-split if all \(w_y \in \mathcal{D}(L_y|k), y \in Y\), are \(k\)-pseudo-split under \(f\).

Finally, the generalization of hypothesis \((\star)\) we were hinting at is the following hypothesis:

\((\star)_{\Sigma_k}\) \(f : X \to Y\) is a \(\Sigma_k\)-pseudo-split morphism of \(k\)-varieties.

Correspondingly, the natural generalization of \((\text{Srj})\) from Introduction is the property:

\((\text{Srj})_{\Sigma_k}\) \(f^{k_U} : X(k_v) \to Y(k_v)\) is surjective for all \(v \in U_A\) for some \(A \subset k^\times\).

Notice that for number fields \(k\) and \(\Sigma_k = \mathbb{P}(k)\) one has: The hypotheses \((\star)\) and \((\star)_{\Sigma_k}\) are equivalent, and so are properties \((\text{Srj})\) and \((\text{Srj})_{\Sigma_k}\). Further, \((\text{qAKE})_{\Sigma_k}\) holds (by the usual Ax–Kochen–Ershov Principle). Hence Theorem 1.6 follows from the more general:

**Theorem 3.2.** In Notations/Remarks 2.4, let \(k\) endowed with \(\Sigma_k\) satisfy \((\text{qAKE})_{\Sigma_k}\). Then for a morphism \(f : X \to Y\) of \(k\)-varieties the following hold:

1) If \(f\) satisfies hypothesis \((\star)_{\Sigma_k}\), then \(f\) has property \((\text{Srj})_{\Sigma_k}\).
2) Let \(\text{char}(k) = 0\). Then \(f\) satisfies hypothesis \((\star)_{\Sigma_k}\) iff \(f\) has property \((\text{Srj})_{\Sigma_k}\).

The proof of Theorem 3.2 is reduced to proving the Key Lemma 3.3 below as follows: First, by Fact 2.1, the property \((\text{Srj})_{\Sigma_k}\) is equivalent to \(f^{k_U} : X(k_v) \to Y(k_v)\) being surjective for all \(U\). Second, by Fact 2.3 combined with Fact 2.6, and the hypothesis \((\text{qAKE})_{\Sigma_k}\), the surjectivity of \(f^{k_U}\) is equivalent to the surjectivity of \(f^{k_U} : X(k_{i_U}) \to Y(k_{i_U})\). Hence the property \((\text{Srj})_{\Sigma_k}\) is equivalent to the following condition in terms of ultrafilters:

\((\text{Srj})_{U}\) \(f^{k_U} : X(k_{i_U}) \to Y(k_{i_U})\) is surjective for every \(U\).

This reduces the proof of Theorem 3.2 to proving the following:

**Key Lemma 3.3.** Let \(k, \Sigma_k\) be as in Notations/Remarks 2.4, and hypothesis \(\text{(qAKE)}_{\Sigma_k}\) be satisfied. Then for a morphism \(f : X \to Y\) of \(k\)-varieties the following hold:

1) If \(f\) satisfies hypothesis \((\star)_{\Sigma_k}\), then \(f\) has the property \((\text{Srj})_U\).
2) Let \(\text{char}(k) = 0\). Then \(f\) satisfies hypothesis \((\star)_{\Sigma_k}\) iff \(f\) has the property \((\text{Srj})_U\).
3.2. Proof of the Key Lemma 3.3.

We begin by recalling basic facts from valuation theory, which are well known to experts.

**Fact 3.4.** Let $\Omega, w$ be a Henselian field with $\text{char}(\Omega w) = 0$. Then every subfield $l \subset \Omega$ with $w|_l$ trivial is contained in a field of representatives $\kappa' \subset \Omega$ for $\Omega w$.

**Proof.** This is a well known consequence of the Hensel Lemma. □

We next recall basic facts about valuations without (transcendence) defect, see [BOU], Ch. VI, and [Ku], for some/more details on (special cases of) this. Let $\Omega, w$ be a valued field with $w|_{\kappa_0}$ trivial on the prime field $\kappa_0$ of $\Omega$. One says that $w$ has no (transcendence) defect if there exists a transcendence basis of $\Omega|_{\kappa_0}$ of the form $\mathcal{T}_w \cup \mathcal{T}$ satisfying the following: First, $w\mathcal{T}_w$ is a basis of the $\mathbb{Q}$-vector space $w\Omega \otimes \mathbb{Q}$, and second, $\mathcal{T}$ consists of $w$-units such that its image $\mathcal{T}w$ in the residue field $\Omega w$ is a transcendence basis of $\Omega w|_{\kappa_0}$. In particular, if $\kappa'_T \subset \Omega$ is the relative algebraic closure of $\kappa_0(\mathcal{T})$ in $\Omega$, then $\kappa'_T$ is a maximal subfield of $\Omega$ such that $w$ is trivial on $\kappa'_T$, and further, $\Omega w$ is algebraic over $\kappa'_T w'$. Moreover, if $w$ is Henselian, then Hensel Lemma implies that $\Omega w$ is purely inseparable over $\kappa'_T w$.

One of the main properties of valuations $w$ without defect is that for any subfield $N \subset \Omega$, the restriction of $w$ to $N$ is a valuation without defect as well, see [Ku]. In particular, if $l \subset \Omega$ is any subfield such that $w|_l$ is trivial, and $N|_l$ is a function field, then $w|_N$ is a prime divisor of the function field $N|l$.

Hence for the field $k_{\iota} = \kappa'_T(\pi_\iota)^h$ endowed $v_\iota$ from Notations/Remarks 2.4, 4), c), one has:

**Fact 3.5.** Let $l \subset k_{\iota}$ be a subfield with $v_\iota$ trivial on $l$. Let $N|l$ be a function field and $N \hookrightarrow k_{\iota}$ be an $l$-embedding. Then $w := v_\iota|_N$ is either trivial, or a prime divisor of $N|l$.

**Proof.** This is an immediate consequence of the discussion above. □

**Fact 3.6.** Let $N^h$ be the Henselization of a function field $N|l$ w.r.t. a prime divisor $w$. Let $\kappa' \subset \Omega$ be a field of representatives for $N_w$, and $\pi \in N$ have $w(\pi) = 1$. Then $N^h = \kappa'(\pi)^h$.

**Proof.** The Henselian subfield $\tilde{N} := \kappa'(\pi)^h$ of $N^h$ satisfies $\tilde{N}w = N^hw$ and $w\tilde{N} = wN$. Since $w$ has no defect, the fundamental equality holds. Hence $[N^h:\tilde{N}] = e(N^h|\tilde{N})f(N^h|\tilde{N}) = 1$, thus finally implying $N^h = \tilde{N} = \kappa'(\pi)^h$. □

3.2.1. Proof of assertion 1) of the Key Lemma 3.3.

Let $y_{\iota} \in Y(k_{\iota})$ be defined by a point $y \in Y$ and a $k$-embedding $\iota : L_y \hookrightarrow k_{\iota}$. By Fact 3.5 above, $w := v_\iota := v_{\iota}|_{L_y} \in \mathcal{D}(L_y|k)$ is either trivial or a prime divisor of $L_y|k$, and let $j : l_y \hookrightarrow \kappa_{\iota}$ be the corresponding $k$-embedding of the residue fields. Since $f$ is $\Sigma_k$-pseudo-split, there is $x \in X_y$ and $v := v_x \in \mathcal{D}(K_x|k)$ on $K_x = \kappa(x)$ such that $w = v|_{L_y}$, the residue field embedding $k_x|l_y$ is $j$-split, and $e(v|w) = 1$ if $w$ is non-trivial. Hence by definitions, $k_x|l_y$ is separably generated, and $j : l_y \hookrightarrow \kappa_{\iota}$ has a prolongation $\iota : k_x \hookrightarrow \kappa_{\iota}$. Let $\mathcal{T}_0$ be a separable transcendence basis of $k_x$ over $l_y$, and $\mathcal{T} \subset K_x$ be a preimage of $\mathcal{T}_0$ under the canonical residue field projection $\mathcal{O}_v \hookrightarrow K_xv_x$. One has the following:

- Setting $N := L_y$ and $M := K_x$, one has $Nu = l_y$, $k_x = Mv$, and further, $\mathcal{T}_0$ is a separable transcendence basis of $Mv$ over $Nu$, and $\mathcal{T} \subset M$ is a preimage of $\mathcal{T}_0$ under $\mathcal{O}_v \hookrightarrow Mv$.
- Set $N_\mathcal{T} := N(\mathcal{T}) \subset M$. Since $w = v|_N$, it follows by mere definition that $w_\mathcal{T} := v|_{N_\mathcal{T}}$ is the Gauss valuation of $N_\mathcal{T}$ defined by $w$ and $\mathcal{T}$. 


- Setting $\kappa_N := j(Nw) \leftrightarrow \nu(Mv) =: \kappa_M$, it follows that $\nu(T_0)$ is a separable transcendence basis of $\kappa_M$ over $\kappa_N$.

- Setting $\nu_{it} := j_{it}(N) \subset k_{it}$, let $T_{it} \subset k_{it}$ be a preimage of $\nu(T_0)$ under $O_{it} \to \kappa_{it}$, and set $N_{\kappa_{it}} := N_{it}(T_{it})$. Then the restriction $w_{\kappa_{it}}$ of $\nu_{it}$ to $N_{\kappa_{it}}$ is the Gauss valuation of $\nu_{it}$ defined by $w_{it} = \nu_{it}|N_{\kappa_{it}}$ and $T_{it}$. Hence one has a $k$-isomorphism of valued fields

$$j_{\kappa_{it}} : N_T \to N_{\kappa_{it}} \subset k_{it}.$$  

- Finally, let $N_{\kappa_{it}}^h \subset N_{\kappa_{it}}^h \subset k_{it}$ be the Henselizations of $\nu_{it} \subset N_{\kappa_{it}}$ in $k_{it}$. Then since $\kappa_M$ is finite separable over the residue field $N_{\kappa_{it}}w_{\kappa_{it}} = \kappa_N(\nu(T_0))$, one has: There exists a unique algebraic unramified subextension $M_{\kappa_{it}}^0|N_{\kappa_{it}}^h$ of $k_{it}|N_{\kappa_{it}}^h$ with residue field $M_{it}^0|k_{it} = \kappa_M$.

Finally, one has the following case-by-case discussion:

**Case 1.** $v$ is trivial. Then $w$ is trivial, hence $N = Nw \hookrightarrow Mv = M$, and $\bar{y} \in Y(k_{it})$ is defined by the $k$-embedding $j_{it} : \kappa(x) = N \to N_{it} \subset k_{it}$. In particular, in the above notation, the valuations $w_T$ and $w_{\kappa_{it}}$ are trivial, thus $N = N^h \hookrightarrow N_{\kappa_{it}}^h = N_{\kappa_{it}}$, and $M_{\kappa_{it}}^0|N_{\kappa_{it}}^h$ is a finite separable extension of $N_{\kappa_{it}}$ such that the residue map $O_{it} \to \kappa_{it}$ defines an isomorphism $M_{\kappa_{it}}^0 \to \kappa_M$. Hence if $v_0 : \kappa_M \to M_{\kappa_{it}}^0$ is the inverse of the isomorphism $M_{\kappa_{it}}^0 \to \kappa_M$, one has:

$$v_0 : M \xrightarrow{x} \kappa_M \xrightarrow{v_0} M_{\kappa_{it}}^0 \subset k_{it}$$

is an isomorphism prolonging $j_{it} : N \to k_{it}$, thus defining $\bar{x} \in X(k_{it})$ such that $f^{k_{it}}(\bar{x}) = \bar{y}$.

**Case 2.** $v$ is non-trivial and $w$ is trivial, hence $N = Nw$. Then we can view $v$ as a prime divisor of $M|N$, and in the above notation one has: Let $T \subset M$ be a preimage of a separable transcendence basis $T_0 \subset Mv$ of $Mv|N$, and $N_T = N(T)$, then $w_T := w_N|N_T$ is trivial, and the relative algebraic closure $M^0$ of $N(T)$ in $M^h$ is a field of representatives for $Mv$. In particular, if $\pi \in M$ has $v(\pi) = 1$, then $M^h = M^0(\pi)^h$ by Fact 3.6.

Next, let $T_{it} \subset k_{it}$ be a preimage of $\nu(T_0) \subset k_{it}$ under the canonical residue map $O_{it} \to \kappa_{it}$. Then $v_{it}$ is trivial on $N_{\kappa_{it}} = N_{it}(T_{it})$, and $\kappa_M = \nu(Mv)$ has a unique preimage $M_{\kappa_{it}}^0 \subset k_{it}$ which is algebraic over $N_{\kappa_{it}}$. Finally, the $k$-isomorphism $M_0 \to Mv \to M_{\kappa_{it}}^0$ together with $\pi \mapsto \pi_{it}$ give rise to a $k$-embeddings of fields

$$v_{it} : M \xrightarrow{\iota} M^h = M^0(\pi)^h \xrightarrow{\pi_{it}} M_{\kappa_{it}}^0(\pi_{it})^h \subset k_{it},$$

whose restriction to $N = Nw$ is $j_{it}$. Hence the $k_{it}$-rational point $\bar{x} \in X(k_{it})$ defined by $v_{it} : M \xrightarrow{\iota} k_{it}$ satisfies $f^{k_{it}}(\bar{x}) = \bar{y}$.

**Case 3.** $w$ is non-trivial. Let $\pi \in N$ be such that $w(\pi) = 1$, hence $v(\pi) = 1$ by the fact that $e(v|w) = 1$. Then $N_T = N(T) \hookrightarrow M$ gives rise to the embedding of the Henselizations $M^h|N_T^h$. Reasoning as above, the unique unramified subextension $M_0|N_T^h$ of $M^h|N_T^h$ satisfies $M^h = M_0$, and $N_T \to N_{\kappa_{it}}$ together with $\pi \mapsto \pi_{it}$ gives rise to a $k$-embedding $v_{it} : M \to k_{it}$ prolonging $j_{it} : N \to k$, etc. Hence finally, one gets a point $\bar{x} \in X(k_{it})$ such that $f^{k_{it}}(\bar{x}) = \bar{y}$.

3.2.2. **Proof of assertion 2) of the Key Lemma 3.3.**

Since the implication $\Rightarrow$ is actually assertion 1) of the Key lemma, it is left to prove the converse implication, that is, that property $(\mathbb{S}_j)_{\Sigma_k}$ implies the hypothesis $(\ast)_{\Sigma_k}$. In Notations/Remarks 2.4, suppose that $f^{k_{it}} : X(k_{it}) \to Y(k_{it})$ is surjective for a given $U$. Let $y \in Y$, $N := L_y = \kappa(y)$, and $w := w_y \in D(N|k)$, and a $k$-embedding $j : Nw = l_y \hookrightarrow k_{it}$ be
given. We show that there is \( x \in X_y \) such that setting \( M := \kappa(x) \) there is \( v \in D(M|k) \) such that \( w = v|_N, e(v|w) = 1 \), and \( Nw = I_y \hookrightarrow k_x = Mv \) is \( j \)-split.

Indeed, given \( w \), we define a particular \( k_\mathfrak{u} \)-rational point \( \tilde{y} = \tilde{y}_w \in Y(k_\mathfrak{u}) \) as follows: First, if \( w \) is trivial, let \( \tilde{y}_w \) be defined by the \( k \)-embedding \( j : N = \kappa(y) \hookrightarrow k_\mathfrak{u} \subset k_\mathfrak{u} \). Second, if \( w \) is non-trivial, hence a prime divisor of \( N|k \), let \( k_\mathfrak{u} \subset N^h \) be a field of representatives for \( Nw \).

(\text{Note that since char}(k) = 0, such a field of representatives exists.) Thus by Fact 3.6, one has \( N^h = \kappa_w(\pi)^h \). Hence setting \( \kappa_\mathfrak{u}^w = j(Nw) \subset k_\mathfrak{u} \subset k_\mathfrak{u} \), one has that \( N^h \) has a canonical \( k \)-embedding \( j_\mathfrak{u}^h : N^h = \kappa_w(\pi)^h \hookrightarrow \kappa_\mathfrak{u}^w(\pi_\mathfrak{u})^h \subset k_\mathfrak{u} \) via \( j : \kappa_w \to Nw \to \kappa_\mathfrak{u}^w \subset k_\mathfrak{u} \), \( \pi \mapsto \pi_\mathfrak{u} \).

Let \( \tilde{y} \in Y(k_\mathfrak{u}) \) be defined by the \( k \)-embedding \( j_\mathfrak{u} := j_\mathfrak{u}^h|_N : N \to N^h \hookrightarrow k_\mathfrak{u} \). Then by property \((S\mathfrak{r})_\Sigma \), there is some \( \bar{x} \in X(k_\mathfrak{u}) \) such that \( f^{k_\mathfrak{u}}(\bar{x}) = \tilde{y} \), and let \( \bar{x} \) be defined by some \( x \in X \) and a \( k \)-embedding \( \nu_\mathfrak{u} : M = \kappa(x) \hookrightarrow k_\mathfrak{u} \). Then by mere definition one has \( f(x) = y \), and the canonical \( k \)-embedding \( f_{xy} : N = \kappa(y) \hookrightarrow \kappa(x) = M \) satisfies \( \nu_\mathfrak{u} \circ f_{xy} = j_\mathfrak{u} \).

Hence setting \( v := \nu_\mathfrak{u}|_M \), one has \( w = v|_N \), and the following hold: First, one has a canonical \( k \)-embedding \( Nw \to Mv \hookrightarrow \kappa_\mathfrak{u} \). Second, one has canonical embeddings \( wN \to vM \hookrightarrow \nu_\mathfrak{u}k_\mathfrak{u} \); and if \( w \) is non-trivial, then by the definition of \( w \) one has: \( w(\pi) = 1 = \nu_\mathfrak{u}(\pi_\mathfrak{u}) \), hence \( wN \to vM \hookrightarrow \nu_\mathfrak{u}k_\mathfrak{u} \) are isomorphisms, and \( e(v|w) = 1 \). Finally, since \( j : Nw \hookrightarrow \kappa_\mathfrak{u} \) prolongs to a \( k \)-embedding \( Mv \hookrightarrow \kappa_\mathfrak{u} \), it follows that \( Mv|Nw \) is \( j \)-split.

3.3. Final Remarks.

First, it is believed that the hypothesis \((q\mathfrak{AKE})_\Sigma \) always holds, in particular, assertion 1) of Theorem 3.2 should hold unconditionally. Second, the question whether assertion 2) of Theorem 3.2 holds in positive characteristic, is related to subtle questions concerning the relationship between ramification index and purely inseparable non-liftable extensions of the residue field of prime divisors. Hence it is an interesting (and maybe subtle) question whether assertion 2) of Theorem 3.2 holds (un)conditionally in positive characteristic.

4. Proof of Theorem 1.4

By mere definitions, Theorem 1.4 is a consequence of Theorem 1.6 and Theorem 4.1 below. The latter relates the pseudo-splitness of prime divisors in extensions of function fields and pseudo-splitness of morphisms of proper integral v.r.l. varieties over arbitrary fields \( k \). We say that a function field \( N|k \) is \textit{valuation regular-like} (v.r.l.), if \( N|k \) has a co-final system of \textit{proper v.r.l.} models \((Z_\alpha)_\alpha \). In particular, by Hironaka’s Desingularization Theorem, one has:

\textit{If char}(k) = 0, every function field \( N|k \) is \textit{v.r.l.}

**Theorem 4.1.** Let \( f : X \to Y \) be a dominant morphisms of proper v.r.l. \( k \)-varieties, and suppose that the function fields \( K = k(X), L = k(Y) \) are v.r.l. Then \( f : X \to Y \) is pseudo-split in the sense of Definition 1.5 if and only if \( D(L|k) \) is pseudo-split in \( D(K|k) \).

**Proof.** Since \( K|L \) is an extension of v.r.l. function fields over \( k \), there are cofinal systems \( (f_\alpha : X_\alpha \to Y_\alpha)_{\alpha \in I} \) of dominant morphisms of proper v.r.l. \( k \)-varieties defining \( K|L \). In particular, the structure morphisms \( X_{\alpha'} \to X_{\alpha'} \) and \( Y_{\alpha'} \to Y_{\alpha'} \) are proper morphisms for all \( \alpha \leq \alpha' \). Further, w.l.o.g., we can and will replace the given projective system \((f_\alpha)_{\alpha \in I} \) by any subsystem \((f'_{\alpha'})_{\alpha' \in I'} \) indexed by any co-final segment \( I' \subset I \). In particular, w.l.o.g., we can and will suppose that every \( f_\alpha : X_\alpha \to Y_\alpha \) dominates the given \( f : X \to Y \).

We conclude this preparation by summarizing a few well known facts, to be used later.
Fact 4.2. Let \( v \in \text{Val}_k(N) \) have center \( x_\alpha \in X_\alpha \) for \( \alpha \in I \). Setting \( w := v|_L \), one has:

1) The center of \( w \) on \( Y_\alpha \) is \( y_\alpha = f_\alpha(x_\alpha) \), and one has:
   \[
   m_v = \bigcup_\alpha m_{x_\alpha} \subset \bigcup_\alpha O_{x_\alpha} = O_v, \quad m_w = \bigcup_\alpha m_{y_\alpha} \subset \bigcup_\alpha O_{y_\alpha} = O_w,
   \]
   and therefore, \( Lw = \bigcup_\alpha \kappa(y_\alpha) \hookrightarrow \bigcup_\alpha \kappa(x_\alpha) = Kv \) canonically.

2) If \( v \in \mathcal{D}(K|k) \), then \( O_v = O_{x_\alpha} \) and \( O_w = O_{y_\alpha} \) for \( \alpha \in I_v \) in a cofinal segment \( I_v \subset I \).

4.1. The implication “\( \Rightarrow \)”. 

Given \( w \in \mathcal{D}(L|k) \), we show that \( w \) is pseudo-split in \( \mathcal{D}(K|k) \).

**Case 1.** \( w \) is the trivial valuation of \( L|k \). Then the center \( y \in Y \) of \( w \) is the generic point \( y = \eta_Y \) of \( Y \), and \( X_y = X_L \) is the generic fiber of \( f: X \rightarrow Y \). Further, \( Lw = L \). Since \( w \) is pseudo-split under \( f \), for every co-procylic extension \( l'|L \) there exists \( x \in X_L \) and \( v_x \in \mathcal{D}(K_x|k) \) with \( x \mid L \) split above \( l' \). Since \( K \) is regular-like, there exists \( \bar{v} \in \text{Val}_k(K) \) having center \( x \in X_L \subset X \) such that \( K\bar{v} = K_x \). The valuation theoretical composition \( v := v_x \circ \bar{v} \) is trivial on \( L \) under \( L \hookrightarrow K \), hence \( w = v|_L \), and \( k_x = K_xv_x = Kv \). Further, by Fact 4.2, if \( x_\alpha \in X_\alpha \) is the center of \( v \) on \( X_\alpha \), one has \( O_v = \bigcup_\alpha O_{x_\alpha} \), \( m_v = \bigcup_\alpha m_{x_\alpha} \), and \( k_x = Kv = \bigcup_\alpha \kappa(x_\alpha) \). In particular, since \( k_x | k \) is finitely generated, there exists a cofinal segment \( I_x \subset I \) such that \( k_x = Kv = \kappa(x_\alpha) \) for \( \alpha \in I_x \). Since \( K|k \) is regular-like, for every \( x_\alpha \in X_\alpha \), there exists \( v_\alpha \in \mathcal{D}(K|k) \) with center \( x_\alpha \in X_\alpha \) such that \( Kv_\alpha | \kappa(x_\alpha) \) is a regular field extension. In particular, for \( \alpha \in I_x \) one has: \( k_\alpha = \kappa(x_\alpha) \) and \( k_x = \kappa(x_\alpha) \hookrightarrow Kv_\alpha \) is a regular field extension. Hence since \( k_x | l \) is split above \( l' \), and \( Kv_\alpha | k_x \) is a regular extension, by transitivity of splitness, it follows that \( Kv_\alpha | L \) is split above \( l' \). Finally, since \( v_\alpha | L \) is trivial, hence \( w = v_\alpha|_L \), it follows that \( w \) is pseudo-split in \( \mathcal{D}(K|k) \), as claimed.

**Case 2.** \( w \) is non-trivial, hence \( w \in \mathcal{D}(L|k) \) is a prime divisor of \( L|k \). Let \( y_\alpha \in Y_\alpha \) be the center of \( w \) on \( Y_\alpha \). By Fact 4.2, there is a co-final segment \( I_w \subset I \) such that \( O_w = O_{y_\alpha} \), thus \( m_w = m_{y_\alpha} \) and \( Lw = \kappa(y_\alpha) \) for \( \alpha \in I_w \). Letting \( y = \eta_Y \) be the generic point of \( Y \), one has \( L_y = L \), and \( L \in \mathcal{D}(L_y) \), and \( X_y = X_L \) is the generic fiber of \( f: X \rightarrow Y \).

Let \( l'|Lw \) be a co-procylic extension. Then \( w \in \mathcal{D}(L_y|k) \) being split under \( f \) implies that there exists \( x \in X_y = X_L \) and a prime divisor \( v_x \in \mathcal{D}(K_x|k) \) with \( w = v_x|_L \) under \( L \hookrightarrow K \) such that \( e(v_x|w) = 1 \) and \( k_x | Lw \) is split above \( l' \). Let \( \pi \in L \) satisfy \( w(\pi) = 1 \), hence in particular, \( v_x(\pi) = 1 \) under the \( k \)-embedding \( L = L_y \hookrightarrow K_x \). Since \( K|k \) is regular-like, there exists \( \bar{v} \in \text{Val}_k(K) \) with center \( x \in X \) and \( K\bar{v} = \kappa(x) = K_x \). In particular, \( \bar{v}|L \) is trivial on \( L \) under \( L \hookrightarrow K \), and the valuation theoretical composition \( v := v_x \circ \bar{v} \in \text{Val}_k(K) \) satisfies:

a) \( Kv = K_xv_x = k_x \), and \( w = v|_L \) under \( L \hookrightarrow K \), thus \( O_w = O_v \cap L \).

b) Since \( wL = v_xK_x \hookrightarrow vK \), it follows that \( v(\pi) \) is the minimal positive element of \( vK \).

c) In particular, \( m_v = \pi O_v \), hence \( \pi \in m_v \setminus m_v^2 \).

Recalling that \( f_\alpha: X_\alpha \rightarrow Y_\alpha \) are proper morphisms, since \( w = v|_L \) has the center \( y_\alpha \in Y_\alpha \), it follows that \( v \) has a (unique) center \( x_\alpha \in X_\alpha \), and \( f(x_\alpha) = y_\alpha \). In particular, since \( w = v|_L \), by Fact 4.2 one has: First, since \( k_x = Kv \) is finitely generated over \( k \), there is a cofinal segment \( I_x \subset I \) such that \( K\bar{v} = k_x = \kappa(x_\alpha) \) for all \( \alpha \in I_x \). Recalling that \( I_w \subset I \) is a cofinal segment such that \( Lw = \kappa(y_\alpha) \) for all \( \alpha \in I_w \), it follows that \( I' := I_w \cap I_x \) is a cofinal segment in \( I \) such that for all \( \alpha \in I' \) the following hold:

\[
O_w = O_{y_\alpha} = O_{x_\alpha} \cap L, \quad m_w = m_{y_\alpha} = m_{x_\alpha} \cap L, \quad Lw = \kappa(y_\alpha) \hookrightarrow \kappa(x_\alpha) = k_x.
\]
In particular, \( \pi \in m_{x_\alpha} \), and since \( \pi \not\in m_{x_\alpha}^2 \), one has that \( \pi \not\in m_{x_\alpha}^2 \) for all \( \alpha \in I' \).

Since \( K|k \) is regular-like, there exists \( v_\alpha \in D(K|k) \) with center \( x_\alpha \in X_\alpha \) such that
\[ K_{v_\alpha} \kappa(x_\alpha) \] is a regular field extension, and \( v_\alpha(\pi) = 1 \), because \( \pi \in m_{x_\alpha} \setminus m_{x_\alpha}^2 \). Therefore \( w_\alpha := v_\alpha|_L \) lies in \( D(L|k) \), and \( m_{w_\alpha} = m_{v_\alpha} \cap L \). Since \( m_{w_\alpha} = m_{x_\alpha} \cap L \), one has:
\[
m_w = m_{y_\alpha} = m_{x_\alpha} \cap L \subset m_{v_\alpha} \cap L = m_{w_\alpha}, \quad \text{hence} \quad D_w \supset D_{w_\alpha}.
\]
Since \( w, w_\alpha \) are discrete valuations, one must have \( w = w_\alpha \). Recalling that \( w_\alpha := v_\alpha|_L \), we finally get \( w = v_\alpha|_L \), hence \( e(v_\alpha|_L) = 1 \), because \( v_\alpha(\pi) = 1 = w_\alpha(\pi) \). Since \( k_x|Lw \) is \( l' \)-split and \( k_x = \kappa(x_\alpha) \hookrightarrow K_{v_\alpha} \) is a regular field extension for \( \alpha \in I' \), it follows that \( K_{v_\alpha}|Lw \) is split above \( l' \). Conclude that \( w \) is pseudo-split in \( D(K|k) \).

4.2. The implication “\( \Leftarrow \)”.

Setting \( L_y := \kappa(y) \) for \( y \in Y \), we have to show that every \( w_y \in D(L_y|k) \) is pseudo-split under \( f \) in the sense of Definition 1.5. First, if \( y = \eta_y \) is the generic point of \( Y \), hence \( L_y = k(Y) := L \), then the implication follows directly from the fact that \( D(L|k) \) is pseudo-split in \( D(K|k) \). Hence w.l.o.g., \( y \neq \eta_y \).

Case 1. \( w_y \) is the trivial valuation of \( L_y \), i.e., \( L_y = Lw_y = l_y \). First, since \( L|k \) is regular-like, there exists \( w \in D(L|k) \) having center \( y \in Y \) such that \( Lw|l_y \) is a regular field extension. Let \( l'|l_y \) be a co-procyclic extension, and \( l'_w|Lw \) be a co-procyclic extension with \( l' = l_y \cap l'_w \). Since \( D(L|k) \) is pseudo-split in \( D(K|k) \), there is \( v \in D(K|k) \) such that \( e(v|w) = 1 \) and \( K_{v}|Lw \) is split above \( l'_w \). Hence if \( x \in X \) is the center of \( v \) on \( X \), then \( y = f(x) \) is the center of \( w \) on \( Y \), \( x \) lies in the fiber \( x \in X_y \) of \( f \) at \( y \), and there are canonical \( k \)-embeddings
\[
L_y \hookrightarrow K_x = \kappa(x) \hookrightarrow K_v.
\]
In particular, since \( K_v|Lw \) is split above \( l'_w \) and \( l' = l_y \cap l'_w \), and \( Lw|L_y \) is a regular field extension, by the transitivity of splitness, it follows that \( K_v|L_y \) is split above \( l' \). Hence finally, since \( K_x|L_y \) is a subextension of \( K_v|L_y \), it follows that \( K_x|L_y \) is split above \( l' \). Hence letting \( v_x \) be the trivial valuation of \( K_x \), it follows that \( e(v_x|w_y) = 1 \), and \( l_y = L_y \hookrightarrow K_x = k_x \) is split above \( l' \), as claimed.

Case 2. \( w_y \in D(L_y|k) \) is non-trivial. Since \( L|k \) is regular-like, there exits \( w_L \in D(L|k) \) having center \( y \) on \( Y \) and \( Lw_L = L_y \). Next let \( w := w_y \circ w_L \) be the valuation theoretical composition of \( w_y \) and \( w_L \), and notice that \( Lw = L_yw_y = l_y \), and \( wL = w_yL_y \times w_L[L \times L \\text{lexicographically ordered.} \) In particular, if \( \pi \in O_{w_L} \) is any element whose image in \( L_y \) is a uniformizing parameter of \( w_y \), then \( 1_y = w(\pi) \in wL \) is the unique minimal positive element, and \( m_w = \pi O_w \). Then letting \( y_\alpha \in Y_\alpha \) be the center of \( w \) on \( Y_\alpha \), one has: \( O_w = \cup_{\alpha} O_{y_\alpha} \), \( m_w = \cup_{\alpha} m_{y_\alpha} \) and \( m_{y_\alpha} = m_w \cap O_{y_\alpha} \), thus \( l_y = Lw = \cup_{\alpha} \kappa(y_\alpha) \). In particular, since \( \pi \in m_w \) and \( l_y/k \) is finitely generated, there is a cofinal segment \( I_y \subset I \) such that the following hold:

a) \( \pi \not\in m_{y_\alpha} \) for all \( \alpha \in I \), and \( \pi \in m_{y_\alpha} \) for all \( \alpha \in I_y \).

b) \( \kappa(y_\alpha) \subset l_y \) for all \( \alpha \in I \), and \( \kappa(y_\alpha) = l_y \) for all \( \alpha \in I_y \).

Finally, since \( L|k \) is regular-like, taking into account Fact 4.2, there is \( w_\alpha \in D(L|k) \) such that \( O_{w_\alpha} \) dominates the local ring \( O_{y_\alpha} \), and further: \( w_\alpha(a) = 1 \) for all \( a \in m_{y_\alpha} \setminus m_{y_\alpha}^2 \), and \( \kappa(y_\alpha) \hookrightarrow Lw_\alpha := l_\alpha \) is a regular field extension. Therefore, for \( \alpha \in I_y \) the following hold:

- \( l_y = \kappa(y_\alpha) \hookrightarrow l_\alpha \) is a regular field extension.
- \( w_\alpha(\pi) = 1 \), hence \( \pi \) generates \( m_{w_\alpha} \).
Next let \( l'|l_y \) be a co-procyclic extension, and using that \( l_\alpha|l_y \) is a regular field extension, let \( l'_\alpha|l_\alpha \) be any co-procyclic extension such that \( l' = \overline{I_y} \cap l'_\alpha \). Since \( D(L|k) \) is pseudo-split in \( D(K|k) \), there is a prime divisor \( v_\alpha \in D(K|k) \) with \( w_\alpha = v_\alpha|L \) such that \( e(v_\alpha|w_\alpha) = 1 \), and setting \( k_\alpha := K\overline{v_\alpha} \) one has: \( k_\alpha|l_\alpha \) is \( l'_\alpha \)-split. Then taking into account the \textit{transitivity of splintness}, since \( l' = \overline{I_y} \cap (l'_\alpha) = \overline{I_y} \cap l_\alpha \), one finally gets:

- \( w_\alpha(\pi) = 1 = v_\alpha(\pi) \) under \( L \hookrightarrow K \), and \( k_\alpha|l_\alpha \) is split above \( l'|l_\alpha \) for \( \alpha \in I_y \).

Now let \( P_I \) be the pre-filter on \( I \) formed by the cofinite subsets \( I' \subset I_y \), and \( U \) be an ultrafilter on \( I \) containing \( P_I \). Consider the corresponding ultrapowers \( ^*L_{\alpha} \hookrightarrow ^*K_{\alpha} \) of \( L \hookrightarrow K \), endowed with the corresponding ultraproducts of valuations rings

\[
\mathcal{O}_{^*w_\alpha} = \prod_{\alpha} \mathcal{O}_{w_\alpha}/U \hookrightarrow \prod_{\alpha} \mathcal{O}_{v_\alpha}/U =: \mathcal{O}_{\nu_{\alpha}},
\]

having value groups and residue fields as follows:

\[
^*w_\alpha \cdot L_{\alpha} = \mathbb{Z}/U = ^*v_{\alpha} \cdot K_{\alpha}, \quad ^*l_{\alpha} = \prod_{\alpha} l_{\alpha}/U = ^*L_{\alpha} \cdot ^*w_\alpha \hookrightarrow ^*K_{\alpha} \cdot ^*v_{\alpha} = \prod_{\alpha} k_{\alpha}/U = ^*k_{\alpha}.
\]

One has: First, since for every \( \alpha', \alpha'' \in I \) there exists \( \alpha \in I_y \) with \( \mathcal{O}_{y_{\alpha'}}, \mathcal{O}_{y_{\alpha''}} \subset \mathcal{O}_{y_{\alpha}} \subset \mathcal{O}_{w_\alpha} \), it follows that \( \mathcal{O}_{y_{\alpha}} = \bigcup_{\alpha} \mathcal{O}_{y_{\alpha}} \subset \mathcal{O}_{w_\alpha} \). Hence setting \( w' := ^*w_{\alpha}|L, v := ^*v_{\alpha}|K \), one finally has: First, \( \mathcal{O}_w \subset \mathcal{O}_{w'} \subset \mathcal{O}_v \), where the later inclusion is defined via \( L \hookrightarrow K \). Second, since \( w_\alpha(\pi) = 1 = v_\alpha(\pi) \), it follows that \( ^*w_{\alpha}(\pi) = ^*v_{\alpha}(\pi) \) is the minimal positive element in both value groups \( ^*w_{\alpha}|L_{\alpha} = ^*v_{\alpha}|K_{\alpha} \). Therefore, \( \pi \in \mathcal{O}_w \), \( \mathcal{O}_{w'} \) is the element of minimal positive value, thus \( \mathcal{O}_w = \mathcal{O}_{w'} \) by general valuation theory. Further, since \( \pi \in \mathcal{O}_{\nu_{\alpha}} \) is an element of minimal positive value, it follows that \( \pi \in \mathcal{O}_w \) is an element of minimal positive value as well. Finally, recalling that \( w = w_y \circ w_L \), hence by mere definitions, \( \mathcal{O}_{w_L} = \mathcal{O}_w[1/\pi] \), it follows that \( \mathcal{O}_{\nu_{K}} := \mathcal{O}_w[1/\pi] \) is a \( K \)-valuation ring such that \( \mathcal{O}_{\nu_{K}} \cap L = \mathcal{O}_{w_L} \). In particular, since \( X \) is proper, \( v_K \) has a center on \( X \), say \( x \in Y \). One has:

a) Since \( \mathcal{O}_{w_L} \hookrightarrow \mathcal{O}_{\nu_{K}} \) under \( L \hookrightarrow K \), one has \( f(x) = y \), thus \( L_y = \kappa(y) \hookrightarrow \kappa(x) =: K_x \).

b) \( \mathcal{O}_{w_y} = \mathcal{O}_w/m_{w_L} \hookrightarrow \mathcal{O}_v/m_{v_K} =: \mathcal{O}_{v_K} \) are DVRs of \( L_y \hookrightarrow K_x \) with \( w_y(\pi) = 1 = v_x(\pi) \), thus \( m_{w_y} = \pi \mathcal{O}_{w_y} \subset \pi \mathcal{O}_{v_K} = m_{v_K} \), and \( k_x|l_y \) is a \( k \)-subextension of \( ^*l_{\alpha} \).

c) Since \( k_\alpha|l_\alpha \) is split above \( l'|l_\alpha \) for all \( \alpha \in I_y \), it follows that \( ^*k_{\alpha}|^*l_{\alpha} \) is split above \( l'|l_{\alpha} \), thus \( k_x|l_y \) is split above \( l' \) by the transitivity of splintness.

Hence \( e(v_x|w_y) = 1 \), and \( k_x|l_y \) is split above \( l' \), thus completing the proof of Case 2.

This completes the proof of Theorem 4.1.

\[\square\]

References


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