PRO-ℓ ABELIAN-BY-CENTRAL GALOIS THEORY OF PRIME DIVISORS

FLORIAN POP

Abstract. In the present paper I show that one can recover much of the inertia structure of (quasi) divisors of a function field $K|k$ over an algebraically closed base field k from the maximal *pro-ℓ abelian-by-central* Galois theory of K. The results play a central role in the birational anabelian geometry and related questions.

1. Introduction

The present paper is one of the major technical steps toward tackling a program initiated by Bogomolov [Bo] at the beginning of the 1990's, whose final aim is to recover function fields from their pro-ℓ abelian-by-central Galois theory. This program goes beyond Grothendieck's birational anabelian Program as initiated in [G1], [G2]. Let us introduce notations as follows:

- ℓ is a fixed rational prime number.
- Function fields $K|k$ with k algebraically closed of characteristic $\neq \ell$.
- Maximal pro-ℓ *abelian-by-central* extensions $K''|K$ of maximal pro-ℓ *abelian* extensions $K'|K$ of K.
- The canonical projection $pr : \mathrm{Gal}(K''|K) \to \mathrm{Gal}(K'|K)$ between the corresponding Galois groups.

Remark that $pr : \mathrm{Gal}(K''|K) \to \mathrm{Gal}(K'|K)$ can be recovered group theoretically from $\mathrm{Gal}(K''|K)$, as its kernel is exactly the commutator subgroup $[\mathrm{Gal}(K''|K), \mathrm{Gal}(K''|K)]$. Actually, if $G^{(1)} = G_K$, and for $i \geq 1$ we let $G^{(i+1)} := [G^{(i)}, G^{(1)}](G^{(i)})^{\ell^\infty}$ be the closed subgroup of $G^{(i)}$ generated by all the commutators $[x, y]$ with $x \in G^{(i)}$, $y \in G^{(1)}$ and the ℓ^∞-powers of all the $z \in G^{(i)}$, then the $G^{(i)}$, $i \geq 1$, are the descending central ℓ^∞ terms of the absolute Galois group G_K, and $\mathrm{Gal}(K'|K) = G^{(1)}/G^{(2)}$, and $\mathrm{Gal}(K''|K) = G^{(1)}/G^{(3)}$. Further, denoting by $G^{(\infty)}$ the intersection of all the $G^{(i)}$, it follows that $G_K(\ell) := G_K/G^{(\infty)}$ is the maximal pro-ℓ quotient of G_K; see e.g. [NSW], page 220.

The Program initiated by Bogomolov mentioned above has as ultimate goal to recover function fields $K|k$ as above from $\mathrm{Gal}(K''|K)$ in a functorial way. (Recall/remark that Bogomolov denotes $\mathrm{Gal}(K''|K)$ by PGal^c_K.) If completed, this Program would go far beyond Grothendieck's birational anabelian geometry; see [P2] for a historical note on birational anabelian geometry, and [P5], Introduction, for a historical note on Bogomolov's Program

Date: Variant Feb 2009.

1991 *Mathematics Subject Classification.* Primary 12E, 12F, 12G, 12J; Secondary 12E30, 12F10, 12G99.

Key words and phrases. anabelian geometry, function fields, valuations, Zariski prime divisors, Hilbert decomposition theory, pro-ℓ Galois theory.

Supported by NSF grants DMS-0401056 and DMS 0801144.

and an outline of a strategy to tackle this Program; and for an early history beginning even before [G1], [G2], see [NSW], Chapter XII, and also the original sources [Ne], [Uch] and [Kh]. The strategy presented in [P5] to tackle Bogomolov's Program was developed by the author during the last decade, and roughly speaking, it goes as follows: First, using Kummer theory, one recovers the ℓ-adic completion $\widehat{K^\times} \cong \mathrm{Hom}_{\mathrm{cont}}\big(\mathrm{Gal}(K'|K), \mathbb{Z}_\ell\big)$ of the multiplicative group $K^\times$ of K. Second, let $\jmath : K^\times \to \widehat{K^\times}$ be the ℓ-adic completion functor. Since the base field k is algebraically closed, $k^\times$ is ℓ-divisible, and further, $K^\times/k^\times$ is a free abelian group, as it is the group of principal divisors on any "sufficiently rich" model $X \to k$ of $K|k$. Therefore, $K^\times/k^\times \cong \jmath(K^\times)$ inside $\widehat{K^\times}$. Thus interpreting $K^\times/k^\times$ as the projectivization $\mathcal{P}(K,+)$ of the infinite dimensional k-vector space $(K,+)$, and using the *Fundamental Theorem of projective geometries,* see e.g. Artin [Ar], the problem of recovering the field K from $\mathrm{Gal}(K''|K)$ in a functorial way is reduced (roughly speaking) to recovering $\mathcal{P}(K,+) = K^\times/k^\times \cong \jmath(K^\times)$ inside $\widehat{K^\times}$ together with its "collineations", i.e., the projective lines inside $\mathcal{P}(K,+)$. The strategy described in [P5] for achieving these goals has two main parts, a *local theory,* and a *global theory.* The global theory was developed in loc.cit. [P5], and it has as input the *geometric decomposition graphs* of $K|k$ together with their *rational projections,* and as output reconstructs the function field $K|k$ in a functorial way. The local theory —which is "under construction", has as input the Galois information $\mathrm{Gal}(K''|K)$, and aims at recovering the geometric decomposition graphs of $K|k$ together with their rational projections. See loc.cit. [P5] for definitions and details.

The results of the present manuscript represent a first step of the local theory. Supplemented by ideas and methods developed in [P3], one can complete the local theory, if not in general, at least in the case k is an *algebraic closure of a finite field or a global field*; thus finally complete Bogomolov's Program in the case k is an algebraic closure of a finite field or a global field (and a manuscript concerning this is in preparation).

Finally, in order to put the results of this paper in the right perspective, let us recall that the manuscript [P3] presents a strategy to recover $K|k$ in a functorial way from the full pro-ℓ Galois group $G_K(\ell)$ of $K|k$, in the case k is an algebraic closure of a finite field. Working with the full pro-ℓ Galois group $G_K(\ell)$ and under the hypothesis that k is an algebraic closure of a finite field, leads to major simplifications of the local theory. In the case k is not algebraic over a finite field, the situation becomes much more complicate, because the non-trivial valuations of the base field k play into the game, as shown in [P4]: The "usual methods" do not recover the *prime divisors* of $K|k$ from $G_K(\ell)$ —which are needed in the local theory, but the *quasi prime divisors,* which represent a natural generalization of the former ones. The technical issue of recovering/distinguishing the prime divisors among the quasi prime divisors is not resolved yet for an arbitrary algebraically closed base field k.

The present paper shares similarities with [P4], where similar results were obtained, but working with $G_K(\ell)$ and the canonical projection $pr(\ell) : G_K(\ell) \to \mathrm{Gal}(K'|K)$. The point I want to stress is that $G_K(\ell)$ is a much richer object than $\mathrm{Gal}(K''|K)$, and one can recover the latter group from the former one as mentioned above. Thus the results of the present paper are stronger than, and cannot be deduced from, the ones in [P4]. Whereas a key technical tool used in [P4] was a method to recover valuations from $G_K(\ell)$, as developed by Ware [W], Koenigsmann [Ko], Efrat [Ef], and others, here we rely on the theory of "commuting pairs" as developed in Bogomolov–Tschinkel [B–T]. It looks like that part of the results proved here could be obtained under a much weaker hypothesis, namely using just the first two terms of the ℓ central series for G_K in stead of the ℓ^∞ central series, as suggested by results from Mahé–Mináč–Smith [MMS] in the case $\ell = 2$.

In order to present the results proved in the paper, let me first mention briefly facts introduced later in detail:

Let v be a valuation of K, and v' some prolongation of v to K'. Let $T_{v'} \subseteq Z_{v'}$ be the inertia, respectively decomposition, groups of v' in $\mathrm{Gal}(K'|K)$. By Hilbert decomposition theory for valuations, the groups $T_{v'} \subseteq Z_{v'}$ of the several prolongations v' of v to K' are conjugated; thus since $\mathrm{Gal}(K'|K)$ is abelian, the groups $T_{v'} \subseteq Z_{v'}$ depend on v only, and not on its prolongations v' to K'. We will denote these groups by $T_v \subseteq Z_v$, and call them the inertia, respectively decomposition, groups at v. We also remark that the residue field $K'v'$ of v' is actually a maximal pro-ℓ abelian extension $K'v' = (Kv)'$ of the residue field Kv of v, see Fact 2.1, 3).

Recall that a (Zariski) prime divisor of a function field $K|k$ is any valuation v of K which "originates from geometry", i.e., the valuation ring of v equals the local ring $\mathcal{O}_{X,x_v}$ of the generic point x_v of some Weil prime divisor of some normal model $X \to k$ of $K|k$. Thus $vK \cong \mathbb{Z}$ and $Kv|k$ is a function field satisfying $\mathrm{td}(Kv\,|\,k) = \mathrm{td}(K|k) - 1$, where $\mathrm{td}(\cdot|\cdot)$ denotes the transcendence degree. Again, by Hilbert decomposition theory for valuations, see e.g. [BOU], it follows that the following hold:

$$T_v \;\cong\; \mathbb{Z}_\ell \quad \text{and} \quad Z_v \cong T_v \times \mathrm{Gal}(K'v'|Kv) \cong \mathbb{Z}_\ell \times \mathrm{Gal}(K'v'|Kv).$$

For a prime divisor v, we will call Z_v endowed with T_v a divisorial subgroup of $\mathrm{Gal}(K'|K)$.

Ideally, one would like to recover the divisorial subgroups of $\mathrm{Gal}(K'|K)$ from $\mathrm{Gal}(K''|K)$, as these play an essential role in recovering the function field $K|k$ from $\mathrm{Gal}(K'|K)$. This is indeed possible if k is the algebraic closure of a finite field, see below. Unfortunately, there are serious difficulties when one tries to do the same in the case k is not an algebraic closure of a finite field, as the non-trivial valuations of k play themselves into the game. Therefore one is led to considering the following generalization of prime divisors, see e.g. [P4], Appendix: A valuation v of K is called quasi divisorial, or a quasi-divisor of K, if the valuation ring $\mathcal{O}_v$ of v is maximal among the valuation rings of valuations of K satisfying:

i) $vK/vk \cong \mathbb{Z}$ as abstract groups.

ii) $Kv\,|\,kv$ is a function field with $\mathrm{td}(Kv\,|\,kv) = \mathrm{td}(K|k) - 1$.

Remark that a quasi-divisor v of K is a prime divisor if and only if v is trivial on k. In particular, in the case where k is an algebraic closure of a finite field, the quasi-divisors and the prime divisors of $K|k$ coincide (as all valuations of K are trivial on k).

Finally, for a Galois extension $\tilde{K}|K$ and its Galois group $\mathrm{Gal}(\tilde{K}|K)$, we will say that a subgroup Z of $\mathrm{Gal}(\tilde{K}|K)$ endowed with a subgroup T of Z is a quasi-divisorial subgroup of $\mathrm{Gal}(\tilde{K}|K)$, if $T \subseteq Z$ are the inertia, respectively the decomposition, groups above of some quasi-divisor v of K.

It was the main result in [P4] to show that the quasi-divisorial subgroups of $G_K(\ell)$ can be recovered from $G_K(\ell)$. Hence via $pr(\ell) : G_K(\ell) \to \mathrm{Gal}(K'|K)$, one finally recovers the quasi-divisorial subgroups $T_v \subseteq Z_v$ of $\mathrm{Gal}(K'|K)$. We further mention that in order to distinguish the divisorial groups from the quasi divisorial ones —for general algebraically closed base fields k, one used in loc.cit. the following construction: For $t \in K$ a non-constant function, we denoted by K_t the relative algebraic closure of $k(t)$ in K. Thus $K_t|k$ is a function field in one variable, and one has canonical surjective projections $p_t : G_K(\ell) \to G_{K_t}(\ell)$, respectively $p''_t : \mathrm{Gal}(K''|K) \to \mathrm{Gal}(K''_t|K_t)$ and $p'_t : \mathrm{Gal}(K'|K) \to \mathrm{Gal}(K'_t|K_t)$. Finally, it was shown in loc.cit. that a quasi divisorial subgroup Z of $G_K(\ell)$ is divisorial if and only if there exist $t \in K$ such that the image of Z in $G_{K_t}(\ell)$ is an open subgroup.

In the above notations, the main results of this paper can be summarized as follows:

Theorem 1.1. *Let $K|k$ be a function field over the algebraically closed field k, $\mathrm{char}(k) \neq \ell$. Let $pr : \mathrm{Gal}(K''|K) \to \mathrm{Gal}(K'|K)$ be the canonical projection, and for subgroups T, Z, Δ of $\mathrm{Gal}(K'|K)$, let T'', Z'', Δ'' denote their preimages in $\mathrm{Gal}(K''|K)$. Then the following hold:*

(1) *The transcendence degree $d = \mathrm{td}(K|k)$ is the maximal integer d such that there exist closed subgroups $\Delta \cong \mathbb{Z}_\ell^d$ of $\mathrm{Gal}(K'|K)$ with Δ'' Abelian.*

(2) *Suppose that $d := \mathrm{td}(K|k) > 1$. Let $T \subset Z$ be closed subgroups of $\mathrm{Gal}(K'|K)$. Then Z endowed with T is a quasi divisorial subgroup of $\mathrm{Gal}(K'|K)$ if and only Z is maximal in the set of closed subgroups of $\mathrm{Gal}(K'|K)$ which satisfy:*

i) *Z contains a closed subgroup $\Delta \cong \mathbb{Z}_\ell^d$ such that Δ'' is Abelian.*
ii) *$T \cong \mathbb{Z}_\ell$, and T'' is the center of Z''.*

(3) *A quasi divisorial subgroup Z of $\mathrm{Gal}(K'|K)$ is a divisorial subgroup $\iff$ $p_t(Z)$ is open in $\mathrm{Gal}(K'_t|K_t)$ for some non-constant $t \in K$.*

Actually the above Theorem is a special case of the more general assertions Proposition 4.2, and Theorems 4.4, and 5.2, which deal with generalized (quasi) prime r-divisors; and the above Theorem corresponds to the case $r = 1$.

2. Basic facts from valuation theory

A) *Hilbert decomposition in abelian extensions*

Let K be a field of characteristic $\neq \ell$ containing all the ℓ^∞ roots of unity. In particular, the ℓ-adic Tate module $\mathbb{T}_{\ell,K} = \varprojlim_n \mu_n$, $n = \ell^e$, is (non-canonically) isomorphic to $\mathbb{Z}_\ell$ as a G_K module; and let us fix such an isomorphism $\imath_K : \mathbb{T}_{\ell,K} \to \mathbb{Z}_\ell$. By Kummer Theory, for every $n = \ell^e$ one has a canonical non-degenerate pairing

$$K^\times/n \times \mathrm{Gal}(K'|K)/n \to \mu_n \,,$$

which by Pontrjagin duality gives rise to canonical isomorphisms $\mathrm{Gal}(K'|K)/n = \mathrm{Hom}(K^\times, \mu_n)$ and $K^\times/n = \mathrm{Hom}_{\mathrm{cont}}(G_K, \mu_n)$. By taking limits over all $n = \ell^e$, we get finally isomorphisms:

$$\mathrm{Gal}(K'|K) = \mathrm{Hom}(K^\times, \mathbb{T}_{\ell,K}) \xrightarrow{\imath_K} \mathrm{Hom}(K^\times, \mathbb{Z}_\ell).$$

In the above context, let v be a valuation of K, and v' some prolongation of v to K'. Suppose that the residual characteristic $p = \mathrm{char}(Kv) \neq \ell$. We denote by $T_v \subseteq Z_v$ the inertia, respectively decomposition, groups of $v'|v$ in $\mathrm{Gal}(K'|K)$. We remark that the ramification group V_v of $v'|v$ is trivial, as $p = \mathrm{char}(Kv) \neq \ell$ does not divide the order of $\mathrm{Gal}(K'|K)$, thus that of Z_v; and further, $T_v \subseteq Z_v$ depend only on v, as $K'|K$ is Abelian. Finally, we denote by K^T and K^Z the corresponding fixed fields in K'.

Fact 2.1. In the above context one has the following:

1) Let $U^1_v = 1 + \mathfrak{m}_v \leq K^\times$ be the group of principal v-units in K. Then $K^Z|K$ is the Abelian extension of K obtained by adjoining the ℓ^∞ roots of all the elements $x \in U^1_v$, i.e., $K^Z = K[\,\sqrt[\ell^\infty]{U^1_v}\,]$. In particular:

a) Z_v is pure in $\mathrm{Gal}(K'|K)$, i.e., $\mathrm{Gal}(K'|K)/Z_v = \mathrm{Gal}(K^Z|K)$ is torsion free.

b) $K^Z = K' \iff K^\times/n = U^1_v/n$ for all $n = \ell^e > 0$.

2) Let U_v be the group of v-units in K. Then $K^T|K$ is the Abelian extension of K obtained by adjoining the ℓ^∞ roots of all the elements $x \in U_v$, i.e., $K^T = K[\,\sqrt[\ell^\infty]{U_v}\,]$. In particular:

a) T_v is pure in $\mathrm{Gal}(K'|K)$, i.e., $\mathrm{Gal}(K'|K)/T_v = \mathrm{Gal}(K^T|K)$ is torsion free.

b) $K^T = K' \iff K^\times/n = U_v/n$ for all $n = \ell^e > 0 \iff K^Tv' = K'v'$.

c) $K^Z = K^T \iff K^Zv' = K'v' \iff U_v/n = U^1_v/n$ for all $n = \ell^e > 0$.

3) In particular, $v'K'$ is the ℓ-divisible hull of $vK = v'K^T$ in $\mathbb{Q} \otimes vK$, and $K'v' = (Kv)'$. Hence setting $\delta_v = \dim(vK/\ell)$, we have:

a) $T_v = \mathrm{Hom}(vK, \mathbb{T}_\ell) \cong \mathbb{Z}_\ell^{\delta_v}$ (non-canonically), and $G_v := Z_v/T_v = \mathrm{Gal}\big((Kv)'|Kv\big)$.

b) Isomorphisms (latter non-canonical) of pro-ℓ groups:

$$Z_v \cong T_v \times G_v \cong \mathbb{Z}_\ell^{\delta_v} \times \mathrm{Gal}\big((Kv)'|Kv\big)\,.$$

Proof. To 1): Let K^h be some Henselization of K containing K^Z. Then by general decomposition theory, $K^Z = K^h \cap K'$. We first prove that $\sqrt[\ell^\infty]{U^1_v} \subseteq K^Z$. Equivalently, if $a = 1 + x$, $x \in \mathfrak{m}_v$, is some principal v-unit, we have to show that $\sqrt[n]{a} \in K^Z$ for all $n = \ell^e$. Recall that by hypothesis we have: $p = \mathrm{char}(Kv) \neq \ell$, and that K contains the ℓ^∞ roots of unity. Since $a \equiv 1 \pmod{\mathfrak{m}_v}$, it follows that $X^n - a \equiv X^n - 1 \pmod{\mathfrak{m}_v}$, hence $X^n - a$ has n distinct roots $(\mathrm{mod}\,\mathfrak{m}_v)$. Thus by Hensel's Lemma, $X^n - a$ has n distinct roots in K^h, i.e., $\sqrt[n]{a}$ is contained in K^h. Since $\sqrt[n]{a}$ is contained in K' too, we finally deduce that $\sqrt[n]{a}$ is contained in $K^Z = K^h \cap K'$. For the converse, consider some $a \in K^\times$ such that $\sqrt[n]{a} \in K^Z$ for some $n = \ell^e$. We show that $\sqrt[n]{a}$ is contained in $K[\,\sqrt[\ell^\infty]{U^1_v}\,]$. Indeed, since K and K^Z have equal value groups, and $\sqrt[n]{a} \in K^Z$, it follows that $\exists\, b \in K$ such that $v\,\sqrt[n]{a} = v\,b$, hence $v\,a = n \cdot v\,b$. Therefore, $c := a/b^n \in U_v$, and $\sqrt[n]{c} = 1/b\,\sqrt[n]{a}$ is contained in K^Z. Since K and K^Z have equal residue fields, it follows that $\exists\, d \in U_v$ such that $\sqrt[n]{c} \equiv d \bmod \mathfrak{m}_v$, hence $c \equiv d^n \bmod \mathfrak{m}_v$. Thus finally $a_1 := c/d^n$ satisfies: $a_1 \in U^1_v$ is a principal unit, and second, $\sqrt[n]{a_1} = 1/d\,\sqrt[n]{c} = 1/(bd)\,\sqrt[n]{a}$. Thus finally $\sqrt[n]{a} = bd\,\sqrt[n]{a_1}$ is contained in $K[\,\sqrt[\ell^\infty]{U^1_v}\,]$, as claimed.

For the proof of assertion a), remark that by the discussion above we have $K^Z = K[\sqrt[\ell^\infty]{U^1_v}\,]$. On the other hand, $K[\sqrt[\ell^\infty]{U^1_v}\,] = \cup_n K_n$, where $K_n = K[\sqrt[n]{U^1_v}\,]$ and $n = \ell^e$ all the powers of ℓ. Thus setting $\mathrm{Gal}\big(K_n|\,K\big) =: G_n$, we have: First, $\mathrm{Gal}(K^Z|K) = \varprojlim_n G_n$; and second, by Kummer theory, one has a canonical isomorphism of profinite groups $G_n = \mathrm{Hom}(U^1_v/U^1_{v,n}, \mu_n)$, where $U^1_{v,n}$ is the group of all principal v-units which are n^{th} powers in $K^\times$. Let $(x_i)_{i\in I}$ be a family of elements in U^1_v defining an $\mathbb{F}_\ell$-basis of U^1_v/ℓ.

Claim. $U^1_v/U^1_{v,n}$ is a free $\mathbb{Z}/n$-module on the family $(x_i)_i$.

In order to prove the Claim, we first remark that $U^1_{v,n}$ consists of the n^{th}-powers of elements of U^1_v. Indeed, if $u = a^n$ with $u \in U^1_v$ and $a \in K^\times$, then $a \in U_v$ is a v-unit, and moreover, $a^n \equiv u \equiv 1 \pmod{\mathfrak{m}_v}$. Thus $a \equiv \zeta \pmod{\mathfrak{m}_v}$ for some $\zeta \in \mu_n$. But then $u_1 := a/\zeta$ lies in U^1_v

and $u_1^n = a^n = u$, i.e., u is an n^{th}-power inside U_v^1. Finally, with $(x_i)_{i\in I}$ as defined above, the assertion of the Claim is equivalent to the following: Let $\prod_i x_i^{m_i}$ be an n^{th}-power for some exponents m_i. Then all the m_i are divisible by n. Now in order to prove this last assertion (equivalent to the Claim), w.l.o.g. we can suppose that not all the exponents m_i are 0. We set $m_i = mr_i$ with m and r_i natural numbers, and $m = \ell^{e'}$ the largest power of ℓ which divides all the m_i; in particular, at least one of the r_i is not divisible by ℓ. But then we have: First, $x := \prod_i x_i^{r_i}$ is not an ℓ^{th} power in U_v^1, by the hypothesis that $(x_i)_i$ defines an $\mathbb{F}_\ell$-basis of U_v^1/ℓ. Thus by the remarks above, this means that x is not an ℓ^{th} power in $K^\times$. Second, $x^m = \prod_i x_i^{m_i}$ is an n^{th} power in $K^\times$, by hypothesis. Now since K contains the ℓ^∞ roots of unity, it follows by Kummer theory that m must be divisible by n. Equivalently, all the m_i are divisible by n, as claimed.

Coming back to the proof of assertion a), using the Claim we see that $G_n \cong \mu_n^I$ canonically by Kummer theory. Thus taking limits over all n, we get $\mathrm{Gal}(K^Z|K) \cong \mathbb{T}_\ell^I$ canonically. Since $\mathbb{T}_\ell \cong \mathbb{Z}_\ell$, we get $\mathrm{Gal}(K^Z|K) \cong \mathbb{Z}_\ell^I$, thus torsion free.

Finally, the assertion b) is just the translation via Kummer theory of the fact that we have equalities $K' = K[\sqrt[\ell^\infty]{K^\times}\,] = [\sqrt[\ell^\infty]{U_v^1}\,] = K^Z$.

The proof of 2) is similar, and therefore we will omit the details. And finally, 3) is just a translation in Galois terms of the assertions 1) and 2). $\square$

In the above notations, recall that for given valuation v, one can recover $\mathcal{O}_v$ from $\mathfrak{m}_v$, respectively U_v^1, respectively U_v. Indeed, if $\mathfrak{m}_v$ is given, then $K\backslash\mathcal{O}_v = \{\, x \in K^\times \mid x^{-1} \in \mathfrak{m}_v \,\}$, hence one gets $\mathcal{O}_v$; if U_v^1 is given, then $\mathfrak{m}_v = U_v^1 - 1$, and recover $\mathcal{O}_v$ as above; and if U_v is given, then $U_v^1 = \{\, x \in U_v \mid x \notin U_v - 1 \,\}$, etc. Finally, recall that for given valuations v, w of K, with valuation rings $\mathcal{O}_v$, respectively $\mathcal{O}_w$, we say that $w \le v$, or that w is a coarsening of v, if $\mathcal{O}_v \subseteq \mathcal{O}_w$. From the discussion above we deduce that for given valuations v, w, of K, the following assertions are equivalent:

i) w is a coarsening of v.
ii) $\mathfrak{m}_w \subseteq \mathfrak{m}_v$.
iii) $U_w^1 \subseteq U_v^1$.
iv) $U_v \subseteq U_w$.

These facts have the following Galois theoretic translation:

Fact 2.2. In the previous context and notations, the following hold:

1) $Z_v \subseteq Z_w \iff K_w^Z \subseteq K_v^Z \iff U_w^1/n \subseteq U_v^1/n$ for all $n = \ell^e$.
2) $T_w \subseteq T_v \iff K_v^T \subseteq K_w^T \iff U_v/n \subseteq U_w/n$ for all $n = \ell^e$.
3) In particular, if $w \le v$, then $Z_v \subseteq Z_w$ and $T_w \subseteq T_v$.[1]

B) *Pro-ℓ Abelian form of two results of F. K. Schmidt*

In this subsection we give the abelian pro-ℓ form of two results of F. K. Schmidt and generalizations of these like the ones in Pop [P1], The local theory. See also Endler–Engler [E–E].

[1]This is actually true for all Galois extensions $\tilde{K}|K$, and not just for $K'|K$. But then one has to start with valuations $\tilde{v}$ and coarsenings $\tilde{w}$ of those on $\tilde{K}$, etc.

Let v be a fixed valuation of K, and $v'|v$ a fixed prolongation of v to K'. Let further $\Lambda|K$ be a fixed sub-extension of $K'|K$ containing K^Z. Let $\mathcal{V}'_{\Lambda,v'}$ be the set of all coarsenings w' of v' such that $\Lambda w' = (Kw)'$, where w is the restriction of w' to K. We have the following:

Fact/Definition 2.3. In the above notations, let $\mathcal{V}_{\Lambda,v}$ be the restriction of $\mathcal{V}'_{\Lambda,v'}$ to K.

1) $\mathcal{V}_{\Lambda,v}$ *depends on* v *and* Λ *only,* and not on the specific prolongation v' of v to K'.

In fact $\mathcal{V}_{\Lambda,v}$ consists of all the coarsenings w of v such that $\Lambda w' = (Kw)'$ for *some* prolongation w' of w to K' (and equivalently, for *every* prolongation w' of w to K').

2) More precisely, $w \in \mathcal{V}_{\Lambda,v} \iff K^T_w \subseteq \Lambda \iff \mathrm{Gal}(K'|\Lambda) \subseteq T_w$.

In particular, $v \in \mathcal{V}_{v,\Lambda} \iff K^T_v \subseteq \Lambda \iff \mathrm{Gal}(K'|\Lambda) \subseteq T_v$.

3) We set $\mathcal{V}^0_{\Lambda,v} = \mathcal{V}_{\Lambda,v} \cup \{v\}$. By general valuation theory, $\mathcal{V}^0_{\Lambda,v}$ has an infimum whose valuation ring is the union of all the valuation rings $\mathcal{O}_w$ with $w \in \mathcal{V}^0_{\Lambda,v}$. We denote this valuation by

$$v_\Lambda := \inf \mathcal{V}^0_{\Lambda,v}$$

and call it the abelian pro-ℓ Λ-core of v.

Proof. To 1): If $\tilde{v}$ is another prolongation of v to K', then there exists some $\sigma \in \mathrm{Gal}(K'|K)$ such that $\tilde{v} = v' \circ \sigma^{-1} := \sigma(v')$, and so σ defines a bijection $\mathcal{V}_{v',\Lambda} \to \mathcal{V}_{\tilde{v},\sigma(\Lambda)}$ by $w \mapsto w \circ \sigma^{-1}$. Note that since $\Lambda|K$ is abelian, thus in particular Galois, one has $\Lambda = \sigma(\Lambda)$. Thus for $w' \in \mathcal{V}_{v',\Lambda}$, and $\tilde{w} = \sigma(w') := w' \circ \sigma$ one has: σ gives rise to an Kw-isomorphism of the residue fields $(Kw)' = \Lambda w' \to \sigma(\Lambda)\sigma(w') = \Lambda\tilde{w}$.

The proof of the remaining assertions is clear.

To 2): Let w' be a coarsening of v'. Then $U^1_w \subseteq U^1_v$, thus by Fact 2.1 it follows that $K^Z_w \subseteq K^Z_v$. (This is actually true for any Galois extension $\tilde{K}|K$ and any $\tilde{w}$ coarsening of $\tilde{v}$, correspondingly.) Further, by general decomposition theory, $K^T_w|K^Z_w$ is the unique minimal one among all the sub-extensions of $K'|K^Z_w$ having residue field equal to $(Kw)'$. (Again, this is true for any Galois extension $\tilde{K}|K$, etc.) Now since by hypothesis $K^Z_w \subseteq K^Z_v \subseteq \Lambda$, we have: $K^T_w \subseteq \Lambda$, provided $\Lambda w' = (Kw)'$. Therefore, $w \in \mathcal{V}^0_{v,\Lambda} \iff K^T_w \subseteq \Lambda$, by the discussion above. ◻

Proposition 2.4. *In the above context and notations, suppose that $\Lambda \neq K'$ is a proper sub-extension of $K'|K$ containing K^Z, thus in particular, $K^Z \neq K'$. Then the abelian pro-ℓ Λ-core v_Λ of v is non-trivial and lies in $\mathcal{V}^0_{v,\Lambda}$. Consequently:*

(1) *If v_1 is a valuation of K satisfying $v_1 < v_\Lambda$, then $\Lambda v'_1 \neq (Kv_1)'$. Further, if $\Lambda v' = (Kv)'$, then $\Lambda v'_\Lambda = (Kv)'$, and v_Λ is the minimal coarsening of v with this property.*

(2) *$(Kv_{K^Z})' = Kv_{K^Z} \iff (Kv)' = Kv \iff U_v/n = U^1_v/n$ for all $n = \ell^e$, where v_{K^Z} is the abelian pro-ℓ core of v for $\Lambda = K^Z$.*

(3) *If v has rank one, or if $Kv \neq (Kv)'$, then v equals its abelian pro-ℓ K^Z-core v_{K^Z}.*

Proof. If $\mathcal{V}_{v,\Lambda}$ is empty, i.e., $\Lambda v' \neq (Kv)'$, then $\mathcal{V}^0_{v,\Lambda} = \{v\}$, hence $v_\Lambda = v$, and there is nothing to show. Now suppose that $\mathcal{V}_{v,\Lambda}$ is non-empty. Then $\Lambda v' = (Kv)'$, hence $v \in \mathcal{V}_{v,\Lambda}$, and we will show that actually $v_\Lambda \in \mathcal{V}_{v,\Lambda}$. Equivalently, by Fact/Definition 2.3, 2), above, we have to show that $K^T_{v_\Lambda} \subseteq \Lambda$. Thus by the description of K^T given in Fact 2.1, 2), we have to show that for every given v_Λ-unit x, and every $n = \ell^e$, one has: $\sqrt[n]{x} \in \Lambda$. On the other hand, $v_\Lambda = \inf w$, $w \in \mathcal{V}_{v,\Lambda}$. Hence $\exists\ w \in \mathcal{V}_{v,\Lambda}$ such that x is a w-unit. But then since $w \in \mathcal{V}_{v,\Lambda}$, by Fact/Definition 2.3, 2), it follows that $K^T_w \subseteq \Lambda$. As $\sqrt[n]{x} \in K^T_w$, we finally get $\sqrt[n]{x} \in \Lambda$, as claimed.

The assertions (1), (2), (3) are immediate consequences of the main assertion of the Proposition proved above, and we omit their proof. $\square$

Proposition 2.5. *In the above context, the following hold:*

(1) *Let v_1, v_2 be valuations of K such that $K^Z_{v_1}$, $K^Z_{v_2}$ are contained in some $\Lambda \neq K'$. Then the abelian pro-ℓ Λ-cores of v_1 and v_2 are comparable.*

(2) *Let v_1, v_2 be valuations of K which equal their abelian pro-ℓ K^Z-cores, respectively. If $K^Z_{v_1} = K^Z_{v_2}$, then v_1 and v_2 are comparable. (Obviously, if $K^Z_w \subset K^Z_v$ strictly, then $w < v$.)*

(3) *Let v be a valuation of K, and $\Lambda_1 \subseteq \Lambda_2$ subfields of K' with $K^Z_v \subseteq \Lambda_1$, and $\mathrm{Gal}(\Lambda_2|\Lambda_1)$ generated by torsion elements. Then the abelian pro-ℓ Λ_i-cores of v are equal $v_{\Lambda_1} = v_{\Lambda_2}$.*

Proof. To (1): We first make the following observation: Suppose that v_1 and v_2 are independent valuations of K. Then $K^\times = U^1_{v_1} \cdot U^1_{v_2}$. (Indeed, this follows immediately from the Approximation Theorem for independent valuations). In particular, if v_1 and v_2 are independent, then K' equals the compositum $K^Z_{v_1} K^Z_{v_2}$ inside K'. Now since by hypothesis we have $K^Z_{v_1}, K^Z_{v_2} \subseteq \Lambda \neq K'$, it follows that v_1 and v_2 are not independent. Let v be the maximal common coarsening of v_1 and v_2. (By general valuation theory, the valuation ideal $\mathfrak{m}_v$ is the maximal common ideal of $\mathcal{O}_{v_1}$ and $\mathcal{O}_{v_2}$.) Denote $w_i = v_i/v$ on the residue field $L := Kv$. Then we have:

- If both w_1 and w_2 are non-trivial, then they are independent.
- $L^Z_{w_i} = (K^Z_{v_i})v' \subseteq \Lambda v' \subseteq K'v' = (Kv)'$ for $i = 1, 2$.

Therefore, by the discussion above, we either have $\Lambda v' = (Kv)'$, or otherwise at least one of the w_i is the trivial valuation.

First, consider the case where one of the w_i is trivial. Equivalently, $v_i = \min(v_1, v_2)$, hence v_1 and v_2 are comparable. Thus any two coarsenings of v_1 and v_2 are comparable, hence their abelian pro-ℓ Λ-cores too.

Second, consider the case where $\Lambda v' = (Kv)'$. Then by the definition of $\mathcal{V}'_{v_i,\Lambda}$, it follows that $v \in \mathcal{V}'_{v_i,\Lambda}$, as v is by definition a coarsening of v_i, $i = 1, 2$. Hence finally the abelian pro-ℓ Λ-cores of v_1 and v_2 are both some coarsenings of v, thus comparable.

To (2): We apply assertion 1) above with $K^Z_{v_1} =: \Lambda := K^Z_{v_2}$.

To (3): By hypothesis, for every coarsening w of v we have: $\mathrm{Gal}(\Lambda_2 w'|\Lambda_1 w')$ is generated by torsion elements. Hence by Kummer Theory, $\Lambda_2 w' = (Kw)' \iff \Lambda_1 w' = (Kw)'$. Thus $\mathcal{V}_{\Lambda_1,v} = \mathcal{V}_{\Lambda_2,v}$. Hence finally the abelian pro-ℓ Λ_i-cores, $i = 1, 2$, of v are equal. $\square$

3. Hilbert decomposition in pro-ℓ Abelian-by-Central Extensions

In this section we work/keep the notations from Introduction and the previous sections concerning field extensions $K|k$ and the canonical projection $pr : \mathrm{Gal}(K''|K) \to \mathrm{Gal}(K'|K)$.

Fact/Definition 3.1. In the above notations we have the following:

1) For a family $\Sigma = (\sigma_i)_i$ of elements of $\mathrm{Gal}(K'|K)$, let Δ_Σ be the closed subgroup generated by Σ. Then the following are equivalent:

i) $\exists$ preimages $\sigma'_i \in \mathrm{Gal}(K''|K)$ (all i) which commute with each other.

ii) The preimage Δ''_Σ in $\mathrm{Gal}(K''|K)$ of Δ_Σ is Abelian.

We say that a family of elements $\Sigma = (\sigma_i)_i$ of $\mathrm{Gal}(K'|K)$ is commuting liftable, for short c.l., if Σ satisfies the above equivalent conditions i), ii).

2) For a family $(\Delta_i)_i$ of subgroups of $\mathrm{Gal}(K'|K)$ the following are equivalent:

i) All families $(\sigma_i)_i$ with $\sigma_i \in \Delta_i$ are c.l.

ii) If Δ''_i is the preimage of Δ_i in $\mathrm{Gal}(K''|K)$, then $[\Delta''_i, \Delta''_j] = (1)$ for all $i \neq j$.

We say that a family of subgroups $(\Delta_i)_i$ of $\mathrm{Gal}(K'|K)$, is commuting liftable, for short c.l., if it satisfies the equivalent conditions i), ii) above.

3) We will say that a subgroup Δ of $\mathrm{Gal}(K'|K)$ is commuting liftable, for short c.l., if its preimage Δ'' in $\mathrm{Gal}(K''|K)$ is commutative.

4) We finally remark the following: For subgroups $T \subseteq Z$ of $\mathrm{Gal}(K'|K)$, let $T'' \subseteq Z''$ be their preimages in $\mathrm{Gal}(K''|K)$. Then the following are equivalent:

i) Both T and (T, Z) are c.l.

ii) T'' is contained in the center of Z''.

5) In particular, given a closed subgroup Z of $\mathrm{Gal}(K'|K)$, there exists a unique maximal (closed) subgroup T of Z such that T and (T, Z) are c.l. Indeed, denoting by Z'' the preimage of Z in $\mathrm{Gal}(K''|K)$, and denoting by T'' its center, the group T is the image of T'' in $\mathrm{Gal}(K'|K)$ under the canonical projection $pr : \mathrm{Gal}(K''|K) \to \mathrm{Gal}(K'|K)$.

We next recall the following fundamental fact announced by Bogomolov [Bo], see Bogomolov–Tschinkel [B–T], Proposition 6.4.1 and Corollary 6.4.2, for a proof.

Key Fact 3.2. (Bogomolov–Tschinkel). In the notations from above, let $\sigma, \tau \in \mathrm{Gal}(K'|K)$ be c.l. elements of $\mathrm{Gal}(K'|K)$ such that the closed subgroup $<\sigma, \tau>$ generated by σ, τ is not pro-cyclic. Then there exists a valuation v of K with the following properties:

i) $<\sigma, \tau> \subseteq Z_v$.

ii) $<\sigma, \tau> \cap T_v$ is non-trivial, and $\mathrm{char}(Kv) \neq \ell$.

A) *Inertia elements*

Recall that in the notations from above, we say that an element $\sigma \in \mathrm{Gal}(K'|K)$ is an ℓ-*inertia element,* for short *inertia element,* if there exists a valuation v of K such that $\sigma \in T_v$ and $\mathrm{char}(Kv) \neq \ell$. Clearly, the set of all the inertia elements at v is exactly T_v.

Fact 3.3. Let $\sigma \neq 1$ be an inertia element of $\mathrm{Gal}(K'|K)$. Then there exists a valuation v_σ of K, which we call the *canonical valuation* for σ such that the following hold:

i) $\sigma \in T_{v_\sigma}$, i.e., σ is inertia element at v_σ.

ii) If σ is inertia element at some valuation v, then $v_\sigma \leq v$.

Proof. Construction of v_σ: Let Λ be the fixed field of σ in K'. For every valuation v such that σ is inertia element at v, i.e., $\sigma \in T_v$, let v_Λ be the abelian pro-ℓ Λ-core of v. We claim that $v_\sigma := v_\Lambda$ satisfies the conditions i), ii). Indeed: First, since $\sigma \in T_v$, and $K^T v' = (Kv)'$, we have $K^T_v \subseteq \Lambda$, hence $\Lambda v' = (Kv)'$. Thus by Proposition 2.4, 1), it follows that $\Lambda v'_\Lambda = (Kv_\Lambda)'$. But then one must have $K^T_{v_\Lambda} \subseteq \Lambda$, and therefore, $\mathrm{Gal}(K'|\Lambda) \subseteq T_{v_\Lambda}$; hence $\sigma \in T_{v_\Lambda}$, thus i) holds. Second, in order to prove ii), let v_1 be another valuation of K such that $\sigma \in T_{v_1}$. Let $v_{1,\Lambda}$ to be the abelian pro-ℓ Λ-core of v_1. In particular, by the discussion above, we have

$\Lambda v'_{1,\Lambda} = (Kv_{1,\Lambda})'$. We claim that actually $v_\Lambda = v_{1,\Lambda}$. Indeed, both v_Λ and $v_{1,\Lambda}$ equal their abelian pro-ℓ Λ-cores; hence they are comparable by Proposition 2.5, 1). By contradiction, suppose that $v_{1,\Lambda} \neq v_\Lambda$, say $v_{1,\Lambda} < v_\Lambda$. Since v_Λ equals its abelian pro-ℓ Λ-core, and $v_{1,\Lambda} < v_\Lambda$, it follows by Proposition 2.4, 1), that $\Lambda v'_{1,\Lambda} \neq (Kv_{1,\Lambda})'$, contradiction! Thus $v_\Lambda = v_{1,\Lambda}$. Since $v_{1,\Lambda} \leq v_1$, we finally get $v_\Lambda \leq v_1$. This completes the proof of ii). $\square$

Proposition 3.4. *In the context and the notations form above, the following hold:*

(1) *Let $\Sigma = (\sigma_i)_i$ be a c.l. family of inertia elements. Then the canonical valuations v_{σ_i} are pairwise comparable. Moreover, denoting by $v_\Sigma = \sup_i v_{\sigma_i}$ their supremum, and by Λ the fixed field of Σ in K', one has:*

a) *$\sigma_i \in T_{v_\Sigma}$ for all i.*

b) *v_Σ equals its abelian pro-ℓ Λ-core.*

(2) *Let $Z \subseteq \mathrm{Gal}(K'|K)$ be some subgroup, and $\Sigma_Z = (\sigma_i)_i$ be the family of all inertia elements σ_i in Z such that (σ_i, Z) is c.l. for each i. Then the valuation $v := v_{\Sigma_Z}$ as constructed above with respect to Σ_Z satisfies:*

a) *$Z \subseteq Z_v$.*

b) *$\Sigma_Z = Z \cap T_v$.*

Proof. To (1): For each σ_i let T_i be the closed subgroup of $\mathrm{Gal}(K'|K)$ generated by σ_i, and Λ_i the fixed field of σ_i in K'. Thus $T_i = \mathrm{Gal}(K'|\Lambda_i)$, and $\sigma_i \neq 1$ implies $T_i \cong \mathbb{Z}_\ell$. W.l.o.g. we can suppose that $\sigma_i, \sigma_j \neq 1$. We have the following possibilities:

Case 1) $T_i \cap T_j \neq \{1\}$: Then since $T_i \cong \mathbb{Z}_\ell \cong T_j$, it follows that $T := T_i \cap T_j$ is open in both T_i and T_j, and $T \cong \mathbb{Z}_\ell$. Let $\Lambda = \Lambda_i\Lambda_j$ be the fixed field of T in K'. Note that $\Lambda|\Lambda_i$ is finite, as T is open in T_i. Hence by Proposition 2.5, 3), the abelian pro-ℓ Λ-core $v_{\Lambda,i}$ of v_{σ_i} equals v_{σ_i}. And the same is true correspondingly for v_{σ_j}. On the other hand, reasoning as at the end of the proof of ii) from Fact above, it follows that $v_{\Lambda,i} = v_{\Lambda,j}$. Hence finally $v_{\sigma_j} = v_{\sigma_i}$, as claimed.

Case 2) $T_i \cap T_j = \{1\}$: Let T_{ij} be the closed subgroup T_{ij} generated by σ_i, σ_j in $\mathrm{Gal}(K'|K)$, and let Λ_{ij} be the fixed field of T_{ij} in K'. Since $T_i \cap T_j = \{1\}$, we have $T_{ij} \cong \mathbb{Z}_\ell \times \mathbb{Z}_\ell$, thus T_{ij} is not pro-cyclic. Hence by the Key Fact 3.2 above, there exists a valuation v such that $T_{ij} \subseteq Z_v$, $T := T_v \cap T_{ij}$ is non-trivial, and $\mathrm{char}(Kv) \neq \ell$. Moreover, by replacing v by its abelian pro-ℓ Λ_{ij}-core, we can suppose that actually v equals its Λ_{ij}-core. Finally let us remark that we have $T_{ij}/T = \mathrm{Gal}\big((Kv)'|\Lambda_{ij}v'\big)$, and by Kummer theory, $\mathrm{Gal}\big((Kv)'|\Lambda_{ij}v'\big)$ is of the form $\mathbb{Z}_\ell^r$ for some r. Now since T is non-trivial, and $T_{ij} \cong \mathbb{Z}_\ell \times \mathbb{Z}_\ell$, we finally get: T_{ij}/T is either trivial, or $T_{ij}/T \cong \mathbb{Z}_\ell$ else.

Now let v_i be the abelian pro-ℓ Λ_i-core of v. The we have the following case discussion:

- Suppose $v_i > v_{\sigma_i}$: Reasoning as in the proof of ii) of Fact above, by Proposition 2.4, 1), we get: Since v_i is the abelian pro-ℓ Λ_i-core of v, and $v_{\sigma_i} < v_i$, we have $\Lambda_i v'_{\sigma_i} \neq (Kv_{\sigma_i})'$, contradiction! The same holds correspondingly for v_{σ_j} and the corresponding v_j.

Hence we must have $v_i \leq v_{\sigma_i}$, and $v_j \leq v_{\sigma_j}$.

- Suppose $v_i < v_{\sigma_i}$: Recall that v_{σ_i} equal its abelian pro-ℓ Λ_i-core, and that $\Lambda_i v'_{\sigma_i} = (Kv_{\sigma_i})'$ by the definition/construction of v_{σ_i}. Since $v_i < v_{\sigma_i}$, it follows by Proposition 2.4, 1), that $\Lambda_i v'_i \neq (Kv_i)'$. But then by Proposition 2.4, 3), we get $v_i = v$. Hence finally we have the following situation: $v = v_i < v_{\sigma_i}$, and $\Lambda_i v' \neq (Kv)'$. Equivalently, $\Lambda_i \not\subseteq K_v^T$, and so, $T_i \not\subseteq T_v$.

On the other hand, $\mathrm{Gal}(K'|K)/T_v$ is torsion free by Fact 2.1, and $T_i \cong \mathbb{Z}_\ell$. Since $T_i \not\subseteq T_v$, we finally deduce that $T_i \cap T_v$ is trivial, hence $T_i \cap T$ is trivial. Hence by the remarks above we have: $T_{ij} = T_i T$. On the other hand, $v < v_{\sigma_i}$ implies that $T_v \subseteq T_{v_{\sigma_i}}$, hence $T \subseteq T_{v_{\sigma_i}}$. Since $T_i \subseteq T_{v_{\sigma_i}}$ by the definition of v_{σ_i}, we finally get: T_{ij} is contained in $T_{v_{\sigma_i}}$. Therefore, σ_i, σ_j are both inertia elements at v_{σ_i}. But then reasoning as at the end of the proof of ii) from Fact above, we deduce that the abelian pro-ℓ Λ_j-core of v_{σ_i}, say w_i, equals v_{σ_j}. Thus finally we have $v_{\sigma_j} = w_i \leq v_{\sigma_i}$, hence v_{σ_j} and v_{σ_i} are comparable, as claimed.

- By symmetry, we come to the same conclusion in the case $v_j < v_{\sigma_j}$, etc.

- Thus we are left to analyze the case when $v_{\sigma_i} = v_i$ and $v_{\sigma_j} = v_j$. Now since both v_i and v_j are coarsenings of v, it follows that they are comparable. Equivalently, $v_{\sigma_i} = v_i$ and $v_{\sigma_j} = v_j$ are comparable, as claimed.

To (2): Let Let $\sigma \in \Sigma_Z$ be a non-trivial element, and let v_σ be the canonical valuation attached to σ as defined above at Fact 3.3.

Claim. $Z \subseteq Z_{v_\sigma}$.

Case 1) Z is pro-cyclic: Then $Z \cong \mathbb{Z}_\ell$, and if T_σ is the closed subgroup of Z generated by σ, then T_σ is open in Z. On the other hand, since $\mathrm{Gal}(K'|K)/T_{v_\sigma}$ is torsion free by Fact 2.1, it follows that $Z \subseteq T_{v_\sigma}$. Hence Z consists of inertia elements at v_σ only, and in particular, $Z \subseteq Z_{v_\sigma}$.

Case 2): Z is not pro-cyclic: Let $\tau \in Z$ be any element such that the closed subgroup $Z_{\sigma,\tau}$ generated by σ, τ is not pro-cyclic. We claim that $\tau \in Z_{v_\sigma}$. Indeed, by the Key Fact 3.2 above, it follows that there exists a valuation v having the following properties: $Z_{\sigma,\tau} \subseteq Z_v$, $T := Z_{\sigma,\tau} \cap T_v$ is non-trivial, etc. Let ρ be a generator of T. Then since (σ, Z) is c.l., and $\rho \in Z$, it follows that (σ, ρ) is a c.l. pair of inertia elements of $\mathrm{Gal}(K'|K)$. Hence by assertion (1) above, it follows that the canonical valuations v_σ and v_ρ are comparable. We remark that $v_\rho \leq v$, as the former valuation is a core of the latter one. We have the following case by case discussion:

- Suppose that $v_\sigma \leq v_\rho$: Then $v_\sigma \leq v_\rho \leq v$, hence $Z_v \subseteq Z_{v_\rho} \subseteq Z_{v_\sigma}$. Since $\tau \in Z_{\sigma,\tau} \subseteq Z_v$, we finally get $\tau \in Z_{v_\sigma}$, as claimed.

- Suppose that $v_\sigma > v_\rho$: Then in the notations from above, $\Lambda_\sigma v'_\rho \neq (Kv_\rho)'$, hence T_σ is mapped isomorphically into the residual Galois group $\mathrm{Gal}(K'v'_\rho|\Lambda_\sigma v'_\rho)$. In particular, since $\rho \in T_{v_\rho}$, it follows that $T_\sigma \cap T_\rho$ is trivial, hence σ, ρ generate $Z_{\sigma,\tau}$. On the other hand, $v_\sigma > v_\rho$ implies $T_{v_\rho} \subseteq T_{v_\sigma} \subseteq Z_{v_\sigma}$. Since $\rho \in T_{v_\rho}$, we finally get $\rho \in Z_{v_\sigma}$. Hence finally $\tau \in Z_{v_\sigma}$.

Combining Case 1) and Case 2), we conclude the proof of the Claim.

Now recall that by (1), the valuations v_σ are comparable, and that v denotes their supremum. By general decomposition theory for valuation we then get: Since $Z \subseteq Z_{v_\sigma}$ for all σ, one has $Z \subseteq Z_v$ too. Finally, in order to show that $\Sigma_Z = Z \cap T_v$, we remark the following: First, $\Sigma_Z \subseteq T_v$ by the assertion (1) above. For the converse, recall first that for every valuation v of K, the following hold: The decomposition groups $Z_v(\ell)$ at v in the maximal pro-ℓ quotient $G_K(\ell)$ of G_K have the following structure, see e.g. Pop [P4]: $Z_v(\ell) = T_v(\ell) \cdot G$, where G is a complement of $T_v(\ell)$, thus isomorphic to $G_{Kv}(\ell)$; and moreover, $T_v(\ell)$ is abelian, and G and $T_v(\ell)$ commute element-wise with each other. Thus each $\sigma \in T_v \subseteq Z_v \subseteq \mathrm{Gal}(K'|K)$ has a preimage in $Z_v(\ell)$ which lies in the center of $Z_v(\ell)$. But then its projection into $\mathrm{Gal}(K''|K)$ will commute with the preimage of Z_v in $\mathrm{Gal}(K''|K)$. Hence coming back to the proof of assertion (2) we finally have: If $\sigma \in Z \cap T_v$, then (σ, Z) is c.l. Hence $\sigma \in \Sigma_Z$ by the definition of Σ_Z. $\square$

B) *Inertia elements and the c.l. property*

As a consequence of the above Proposition, one has the following:

Proposition 3.5. *In the context and the notations form above, the following hold:*

(1) *Let Δ be a c.l. subgroup of* $\mathrm{Gal}(K'|K)$. *Then Δ contains a subgroup Σ consisting of inertia elements such that Δ/Σ is pro-cyclic (maybe trivial).*

In particular, there exists a valuation $v := v_\Sigma$ such that $\Delta \subseteq Z_v$, and $\Delta \cap T_v = \Sigma$, and v equals its abelian pro-ℓ Λ-core, where Λ is the fixed field of Σ in K'.

(2) *Let $Z \subseteq \mathrm{Gal}(K'|K)$ be a closed subgroup, and Σ_Z a maximal subgroup of Z such that:*

$$(*) \qquad \Sigma_Z \ \ \textit{and} \ \ (\Sigma_Z, Z) \ \ \textit{are c.l.}$$

Suppose that $\Sigma_Z \neq Z$. Then Σ_Z is the unique maximal subgroup of Z satisfying $(*)$, and it consists of all the inertia elements σ in Z such that (σ, Z) is c.l.

Moreover, there exists a unique valuation v such that $Z \subseteq Z_v$, $\Sigma_Z = Z \cap T_v$, and v equals its abelian pro-ℓ Λ-core, where Λ is the fixed field of Σ_Z in K'.

Proof. To (1): We consider the following two cases:

Case a) All $\sigma \in \Delta$ are inertia elements: Then $\Sigma = \Delta$, and the conclusion follows by applying Proposition 3.4, (1).

Case b) $\exists\, \sigma_0 \in \Delta$ which not an inertia element:

Then for each $\sigma'_i \in \Delta$ such that the closed subgroup Z_{σ_0,σ'_i} generated by σ_0, σ'_i is not pro-cyclic, the following holds: Since (σ'_i, σ_0) is by hypothesis c.l., it follows that there exists a valuation v_i of K such that $Z_{\sigma'_i,\sigma_0} \subseteq Z_{v_i}$, and $T_i := Z_{\sigma'_i,\sigma_0} \cap T_{v_i}$ is not trivial, etc. Moreover, since σ_0 is by assumption not an inertia element, it follows that denoting by σ_i a generator of T, we have: σ_0, σ_i generate topologically $Z_{\sigma'_i,\sigma_0}$, and σ_i is an inertia element in Δ. In particular, if v_{σ_i} is the canonical valuation attached to the inertia element σ_i, then $v_{\sigma_i} \leq v_i$. Hence we have $\sigma_0 \in Z_{v_i} \subseteq Z_{v_{\sigma_i}}$. We next apply Proposition 3.4 above, and get the valuation $v := v_\Sigma = \sup_i v_{\sigma_i}$. Since $\sigma_0 \in Z_{v_{\sigma_i}}$ for all i, by general valuation theory one has $\sigma_0 \in Z_{v_\Sigma}$.

To (2): We first claim that Σ_Z consists of inertia elements. By contradiction, suppose that this is not the case, and let $\sigma_0 \in \Sigma_Z$ be a non-inertia element. Reasoning as in Case b) above and in the notations from there we have: For every $\sigma'_i \in Z$ such that the closed subgroup $Z_{\sigma'_i,\sigma_0}$ generated by σ_0, σ'_i is not pro-cyclic, there exists an inertia element $\sigma_i \in Z_{\sigma'_i,\sigma_0}$, such that $\sigma'_i \in Z_{v_{\sigma_i}}$, and the closed subgroup generated by σ_i, σ_0 equals $Z_{\sigma'_i,\sigma_0}$. Then if v_{σ_i} is the canonical valuation for σ_i, we have: $Z_{\sigma'_i,\sigma_0} \subseteq Z_{v_{\sigma_i}}$, and $Z_{\sigma'_i,\sigma_0} \cap T_{v_{\sigma_i}}$ is generated by σ_i. In particular, if Λ_0 is the fixed field of σ_0 in K', then for every σ_i one has: $\Lambda_0 v'_{\sigma_i} \neq (K v_{\sigma_i})'$, hence v_{σ_i} equals its abelian pro-ℓ Λ_0-core. Now taking into account that all σ_i lie in Δ, and that Δ is c.l., it follows by Proposition 3.4, 1), that $(v_{\sigma_i})_i$ is a family of pairwise comparable valuations. Let $v = \sup_i v_{\sigma_i}$ be the supremum of all these valuations. Since $v_{\sigma_i} \leq v$ for all σ_i, it follows that $T_{v_{\sigma_i}} \subseteq T_v$ for all σ_i, and $\sigma_0 \notin T_v$. And moreover, the family of all the σ_i together with σ_0 generates Z. Hence setting $T := Z \cap T_v$ we get: T together with σ_0 generates topologically Z. But then (σ_0, T) is c.l., hence Z is c.l., contradiction!

Hence Σ_Z consists of inertia elements only, and by hypothesis, Σ_Z is c.l. Let $v := v_{\Sigma_Z}$ be the valuation constructed in Proposition 3.4, 1), for the c.l. family consisting of all the elements from Σ_Z.

Claim. $Z \subseteq Z_v$.

Indeed, let $\sigma_0 \in Z$ be a fixed element $\neq 1$, and $\sigma'_i \in \Sigma_Z$. Then (σ'_i, σ_0) is c.l. by hypothesis of the Proposition. Reasoning as in Case b) above, we obtain σ_i and v_{σ_i} as there. On the other hand, $\sigma'_i \in \Sigma_Z$ is itself an inertia element, as Σ_Z consists of inertia elements only by the discussion above. Since (σ'_i, Z) is c.l. by hypothesis, it follows that (σ'_i, σ_i) is c.l. Since they are inertia elements, it follows by Proposition 3.4, 1), that v_{σ_i} and $v_{\sigma'_i}$ are comparable. And further note that $\sigma_0 \in Z_{v_{\sigma_i}}$. We have the following case by case discussion:

- $v_{\sigma_i} \geq v_{\sigma'_i}$ for all i. Then setting $v_0 := v_{\sigma_i}$ for a fixed i, we have $v_0 \geq \sup_i v_{\sigma'_i} = v_{\Sigma_Z} =: v$ by the definition of v above. Hence $Z_{v_0} \subseteq Z_v$. Since $\sigma_0 \in Z_{v_0}$, we finally $\sigma_0 \in Z_v$, as claimed.

- $v_{\sigma_i} < v_{\sigma'_i}$ for some σ'_i. Then $\sigma'_i \notin T_{v_{\sigma_i}}$, hence (σ_i, σ'_i) and (σ_0, σ'_i) generate the same closed subgroup. Or equivalently, σ_0 is contained in the subgroup generated by (σ_i, σ'_i). On the other hand, $v_{\sigma_i} < v_{\sigma'_i} \leq v$ implies $T_{v_{\sigma_i}} \subseteq T_{v_{\sigma'_i}} \subseteq T_v$, hence $\sigma_i, \sigma'_i \in T_v$. But then $\sigma_0 \in T_v$, and in particular, $\sigma_0 \in Z_v$, as claimed.

Finally we prove that $\Sigma_Z = Z \cap T_v$: First, by the definition of v we have $\Sigma_Z \subseteq T_v$, hence $\Sigma_Z \subseteq Z \cap T_v$. For the converse, apply Proposition 3.4, 2). $\square$

4. Quasi r-divisorial subgroups

In this section we will prove a more general form of the first main result announced in the Introduction. We begin by quickly recalling some basic facts about defectless valuations, see e.g. Pop [P4], Appendix, for more details. We further keep the notations from the previous sections including the notation for the canonical projection $pr : \mathrm{Gal}(K''|K) \to \mathrm{Gal}(K'|K)$.

A) *Generalized quasi divisorial valuations*

Recall that $K|k$ is a function field over the algebraically closed field k with $\mathrm{char}(k) \neq \ell$. For every valuation v on K (and/or on any algebraic extension of K, like for instance K' or K'') one has the following: Since k is algebraically closed, vk is a totally ordered $\mathbb{Q}$-vector space (which is trivial, if the restriction of v to k is trivial). We will denote by r_v the rational rank of the torsion free group vK/vk, and by abuse of language call it the *rational rank of* v. Next remark that the residue field kv is algebraically closed too, and $Kv\,|kv$ is some field extension (not necessarily a function field!). We will denote $\mathrm{td}_v = \mathrm{td}(Kv\,|kv)$ and call it the residual transcendence degree. By general valuation theory, see e.g. [BOU], Ch.6, §10, 3, one has the following:

$$r_v + \mathrm{td}_v \leq \mathrm{td}(K|k).$$

We will say that v has no (transcendence) defect, or that v is defectless, if the above inequality is an equality, i.e., $r_v + \mathrm{td}_v = \mathrm{td}(K|k)$.

Remark/Definition 4.1. Using Fact 5.4 from Pop [P4], it follows that for a valuation v of K and $r \leq \mathrm{td}(K|k)$ the following are equivalent:

i) v is minimal among the valuations w of K satisfying $r_w = r$ and $\mathrm{td}_w = \mathrm{td}(K|k) - r$.

ii) v has no relative defect and satisfies: First, $r_v = r$, and second, $r_{v'} < r$ for any proper coarsening v' of v.

A valuation of K with the equivalent properties i), ii), above is called quasi r-divisorial, or a quasi r-divisor of $K|k$; or simply a generalized quasi divisor, if the rank r is not relevant for the context.

We remark the following:

1) Exactly as in loc.cit. Appendix, Fact 5.5, 2), b), it follows that if v is quasi r-divisorial, then $Kv|kv$ is a function field with $\mathrm{td}(Kv\,|\,kv) = \mathrm{td}(K|k) - r$, and $vK/vk \cong \mathbb{Z}^r$.

2) Every Zariski prime divisor of $K|k$ is a quasi 1-divisor. And conversely, a quasi 1-divisor v of K is a Zariski prime divisor if and only if v is trivial on k.

3) From the additivity of the rational rank $r_{(\cdot)}$, see loc.cit. Fact 5.4, 1), ones gets: If v is quasi r-divisorial on $K|k$, and v_0 is quasi r_0-divisorial on $Kv\,|\,kv$, then the compositum $v_0 \circ v$ is a quasi $(r + r_0)$-divisor of K.

Proposition 4.2. *In the above context, suppose that $d = \mathrm{td}(K|k) > 0$. Let v be a quasi r-divisor of $K|k$, and v'' a prolongation of v to K''. Let $T_{v''} \subseteq Z_{v''}$ be the inertia, respectively decomposition, groups of v'' in $\mathrm{Gal}(K''|K)$, and $G_{v''} = Z_{v''}/T_{v''}$ the Galois group of the corresponding Galois residue field extension $K''v''\,|\,Kv$. Then the following hold:*

(1) *$T_{v''} \cong \mathbb{Z}_\ell^r$, and the canonical exact sequence $1 \to T_{v''} \hookrightarrow Z_{v''} \to G_{v''} \to 1$ is split, i.e., $T_{v''}$ has complements in $Z_{v''}$. And finally, $Z_{v''} \cong T_{v''} \times G_{v''}$ as profinite groups.*

(2) *Under the canonical projection $pr : \mathrm{Gal}(K''|K) \to \mathrm{Gal}(K'|K)$, the above exact sequence maps to $1 \to T_v \hookrightarrow Z_v \to \mathrm{Gal}(K'v'\,|\,Kv) \to 1$, and $T_{v''}$ maps isomorphically onto T_v, and $G_{v''}$ onto $\mathrm{Gal}(K'v'\,|\,Kv)$. In particular, T_v and (T_v, Z_v) are c.l.*

(3) *Moreover, the following hold:*

a) *Z_v contains c.l. subgroups $\cong \mathbb{Z}_\ell^d$, and Z_v is maximal among the closed subgroups Z of $\mathrm{Gal}(K'|K)$ which contain subgroups $T \cong \mathbb{Z}_\ell^r$ such that both T and (T, Z) are c.l.*

b) *$T := T_v$ is the unique maximal subgroup of Z_v such that T and (T, Z_v) are both c.l. Equivalently, if $T''_v \subseteq Z''_v$ are the preimages of $T_v \subseteq Z_v$ in $\mathrm{Gal}(K''|K)$, then T''_v is the center of Z''_v.*

Proof. Assertions (1) and (2) follow immediately from the behavior of the decomposition/inertia groups in towers of algebraic extensions, and Fact 2.1, 2), 3), of [P4].

To 3a): First let us prove that Z_v contains c.l. subgroups $\Delta \cong \mathbb{Z}_\ell^d$: Since v is a quasi r-divisor, we have $\mathrm{td}(Kv\,|\,kv) = \mathrm{td}_v = \mathrm{td}(K|k) - r$. Now if $\mathrm{td}_v = 0$, then we are done by assertion (1) above. If $\mathrm{td}_v > 0$, then we consider any quasi td_v-divisor v_0 on the residue field Kv. Then denoting by $v_1 := v_0 \circ v$ the refinement of v by v_0, we have: First, $v < v_1$, hence $T_v \subseteq T_{v_1}$. Second, $d = \mathrm{td}(K|k) = r + \mathrm{td}_v = r_{v_1}$, hence v_1 is a quasi d-divisor of K. Thus T_v is contained in the group $T_{v_1} \cong \mathbb{Z}_\ell^d$, and by assertion (2) above, it follows that T_{v_1} is c.l.

Second, let $T \subseteq Z$ be a closed subgroup as at 3a) such that $Z_v \subseteq Z$. We claim that $Z = Z_v$. Let Σ be a maximal c.l. subgroup of Z such that $T \subseteq \Sigma$, and (Σ, Z) is c.l. too. Applying Proposition 3.5, let $w := v_\Sigma$ be the resulting valuation from loc.cit. Hence we have $Z \subseteq Z_w$, and $\Sigma = Z \cap T_w$.

Claim. $w \geq v$.

Indeed, suppose by contradiction that $w < v$. Then by the fact that v is a quasi r-divisor, it follows that $r_w < r = r_v$. But then $\dim(wK/\ell) = r_w < r = \dim(vK/\ell)$, hence $T_w \cong \mathbb{Z}_\ell^{r_w}$. Since $T \cong \mathbb{Z}_\ell^r$ and $T \subseteq \Sigma \cong \mathbb{Z}_\ell^{r_w}$, we get a contradiction!

Hence we must have $w \geq v$. Therefore $Z_w \subseteq Z_v$. Since $Z \subseteq Z_w$, we finally have $Z \subseteq Z_v$, as claimed.

To 3b): By assertion (2) above, it follows that T_v and (T_v, Z_v) are c.l. We show that T_v is the unique maximal (closed) subgroup of Z_v with this property. Indeed, let T be a closed subgroup of Z_v as at 3a). Then denoting by Σ the closed subgroup of Z_v generated by T_v and T, since T_v and T, and (T_v, Z_v) and (T, Z_v) are c.l., it follows that Σ and (Σ, Z_v) are c.l. too. Thus w.l.o.g. we can suppose that $T_v \subseteq T$, and that T is maximal with the properties from 3b). Now if $r = d$, then $Z_v = T_v$, thus there is nothing to prove. Thus suppose that $r < d$. Let v_Σ be the unique valuation of K given by Proposition 3.5.

Claim. $v_\Sigma \leq v$.

Indeed, suppose by contradiction that $v_\Sigma > v$. Since v is a quasi r-divisor of K, it follows that $Kv|kv$ is a function field with $\mathrm{td}(Kv|kv) = \mathrm{td}_v = d - r > 0$. Since the valuation $v_0 := v_\Sigma / v$ on Kv is non-trivial, it follows that $Z_{v_0} \subseteq K'v'$ is a proper subgroup of $\mathrm{Gal}(K'|K)$. Taking into account that Z_{v_Σ} is the preimage of $Z_{v_0} \subseteq K'v'$ under the canonical projection $Z_v \to \mathrm{Gal}(Kv'|Kv)$, it follows that Z_{v_Σ} is strictly contained in Z_v, contradiction!

Thus by the Claim above, $v_\Sigma \leq v$. But then $T_{v_\Sigma} \subseteq T_v$, hence $\Sigma = Z_v \cap T_{v_\Sigma} \subseteq T_v$. Thus finally $T \subseteq T_v$, as claimed. $\square$

B) *Characterizing quasi r-divisorial subgroups*

We keep the notations from the previous subsection.

Definition 4.3. We say that a closed subgroup Z of $\mathrm{Gal}(K'|K)$ is an quasi r-divisorial subgroup, if there exists a quasi r-divisor v of $K|k$ such that $Z = Z_v$.

Below we give a characterization of the quasi r-divisorial subgroups of $\mathrm{Gal}(K'|K)$, thus of the quasi r-divisors of K, in terms of the group theoretical information encoded in the Galois group $\mathrm{Gal}(K''|K)$ alone, provided $r < \mathrm{td}(K|k)$.

Theorem 4.4. *Let $K|k$ be a function field over the algebraically closed field k, $\mathrm{char}(k) \neq \ell$. Let $pr : \mathrm{Gal}(K''|K) \to \mathrm{Gal}(K'|K)$ be the canonical projection, and for subgroups T, Z, Δ of $\mathrm{Gal}(K'|K)$, let T'', Z'', Δ'' denote their preimages in $\mathrm{Gal}(K''|K)$. Then the following hold:*

(1) *The transcendence degree $d = \mathrm{td}(K|k)$ is the maximal integer d such that there exists a closed subgroup $\Delta \cong \mathbb{Z}_\ell^d$ of $\mathrm{Gal}(K'|K)$ with Δ'' abelian.*

(2) *Suppose that $d := \mathrm{td}(K|k) > r > 0$. Let $T \subseteq Z$ be closed subgroups of $\mathrm{Gal}(K'|K)$. Then Z endowed with T is a quasi r-divisorial subgroup of $\mathrm{Gal}(K'|K)$ if and only Z is maximal in the set of closed subgroups of $\mathrm{Gal}(K'|K)$ which satisfy:*

i) *Z contains a closed subgroup $\Delta \cong \mathbb{Z}_\ell^d$ such that Δ'' is Abelian.*
ii) *$T \cong \mathbb{Z}_\ell^r$, and T'' is the center of Z''.*

Proof. First recall, see e.g. Fact/Definition 3.1, especially the points 3, 4, 5, that Δ'' being abelian is equivalent to Δ being c.l., and T'' being the center of Z'' is equivalent to T being the maximal subgroup of Z such that (T, Z) is c.l. We will use the c.l. language from now on.

To (1): Let Δ be any c.l. closed non-procyclic subgroup of $\mathrm{Gal}(K'|K)$. Then by Proposition 3.5, (1), it follows that there exists a valuation v of K such that $\Delta \subseteq Z_v$, and setting $T_\Delta := \Delta \cap T_v$, it follows that Δ / T_Δ is pro-cyclic (maybe trivial). Hence we have the cases:

Case 1) $T_\Delta = \Delta$. Then $T_\Delta \cong \mathbb{Z}_\ell^\delta$, and by Fact 2.1, 3), it follows that $\delta_v \geq \delta$. Since $\mathrm{td}(K|k) \geq r_v \geq \delta_v$, we finally get $\mathrm{td}(K|k) \geq \delta$.

Case 2) Δ/T_Δ is non-trivial. Then $\Delta/T_\Delta \cong \mathbb{Z}_\ell$ and $T_\Delta \cong \mathbb{Z}_\ell^{\delta-1}$. Then the image of Δ in $\mathrm{Gal}(Kv'|Kv)$ is non-trivial, hence $\mathrm{Gal}(K'|K)$ is non-trivial. Since kv is algebraically closed, we must have $Kv \neq kv$. Equivalently, $\mathrm{td}_v > 0$. Proceeding as above, we also have $r_v \geq (\delta - 1)$, hence finally: $\mathrm{td}(K|k) \geq r_v + \mathrm{td}_v > (\delta - 1)$. Thus $\mathrm{td}(K|k) \geq \delta$.

We now show the converse inequality: Using Pop [P4], Fact 5.6, one constructs valuations v of K such that $r_v = \mathrm{td}(K|k) =: d$. If v is such a valuation, then $\dim(vK/\ell) = d$. Hence by Fact 2.1, 3), $T_v \cong \mathbb{Z}_\ell^d$; and T_v is a c.l. closed subgroup of $\mathrm{Gal}(K'|K)$.

To 2): By Proposition 4.2, it follows that T_v, Z_v have the properties asked for T, Z at 2).

For the converse, we first remark that $T \neq Z$, as by hypothesis we have: $T \cong \mathbb{Z}_\ell^r$, and Z contains closed subgroups $\Delta \cong \mathbb{Z}_\ell^d$ with $d > r$. And remark that the fact that T'' is the center of Z'' is equivalent to the fact that T is the unique maximal subgroup of Z such that T and (T, Z) are c.l.

Step 1) Consider a maximal c.l. subgroup $\Delta \cong \mathbb{Z}_\ell^d$ of Z. Since T is by hypothesis a c.l. subgroup of Z such that (T, Z) is c.l. too, it follows that the closed subgroup T_1 of Z generated by T and Δ is a c.l. closed subgroup of Z. Hence by the maximality of Δ it follows that $T_1 \subseteq \Delta$, hence $T \subseteq \Delta$.

Step 2) Applying Proposition 3.5, let v_0 be the valuation of K deduced from the data (T, Z). Hence $Z \subseteq Z_{v_0}$, and $T = Z \cap T_{v_0}$.

Let Λ be the fixed field of T in K'. Then we obviously have $K_{v_0}^T \subseteq \Lambda$. Let v be the abelian pro-ℓ Λ-core of v_0. We will eventually show that v is a quasi r-divisor of K, and that $Z = Z_v$, $T = T_v$, thus concluding the proof.

First remark that $T \subseteq T_{v_0}$ implies $K_{v_0}^T \subseteq \Lambda$, and therefore $\Lambda v_0' = (Kv_0)'$. Hence by Proposition 2.4, 1), the same is true for the abelian pro-ℓ Λ-core v of v_0, i.e., $\Lambda v' = (Kv)'$.

Step 3) By Proposition 3.5 applied to Δ, it follows that there exists a valuation v_Δ of K such that $\Delta \subseteq Z_{v_\Delta}$. Moreover, by the discussion in the proof of assertion (1) above, it follows that v_Δ is defectless, and the following hold: Either $r_{v_\Delta} = d$, and moreover, in this case $\Delta = T_{v_\Delta}$, thus Δ consists of inertia elements only. Or $r_{v_\Delta} = d - 1$, and in this case we have $T_{v_\Delta} \cong \mathbb{Z}_\ell^{d-1}$; and since Δ is a c.l. subgroup of $\mathrm{Gal}(K'|K)$, and $T \subseteq \Delta$ consists of inertia elements only, it follows that $T \subseteq T_{v_\Delta}$; and finally, Δ contains non inertia elements of $\mathrm{Gal}(K'|K)$.

Let w be the abelian pro-ℓ Λ-core of v_Δ. Since $T \subseteq T_{v_\Delta}$, we have $K_{v_\Delta}^T \subseteq \Lambda$. Hence $\Lambda v_\Delta' = (Kv_\Delta)'$. But then by Proposition 2.4, 1), the same is true for w, i.e., $\Lambda w' = (Kw)'$.

Claim. $v = w$

Indeed, by contradiction, suppose that that $v \neq w$. First suppose that $v > w$. Since by the conclusion of Step 2) we have: $\Lambda v' = (Kv)'$ and v equals its abelian pro-ℓ Λ-core, it follows by Proposition 2.4, 1) that $\Lambda w' \neq (Kw)'$. But this contradicts the conclusion of Step 3). Second, suppose that $v < w$. Then reasoning as above, we contradict the fact that $\Lambda v' = (Kv)'$. The Claim is proved.

Now since w is a coarsening of v_Δ, and the latter valuation is defectless, the same is true for w, thus for $v = w$. We next claim that v is a quasi r-divisor. Indeed, first remark that we have the following:

- $T \subseteq T_w = T_v$; and since $T \cong \mathbb{Z}_\ell^r$, it follows that $T_v \cong \mathbb{Z}_\ell^\delta$ for some $r \le \delta \le d$.

- Since $v \le v_0$, it follows that $Z_{v_0} \subseteq Z_v$ Hence $Z \subseteq Z_v$, as $Z \subseteq Z_{v_0}$ by the definition of v_0.

- Moreover, T_v is a c.l. subgroup of Z_v such that (T_v, Z_v) is c.l. too. In particular, T is a c.l. subgroup of Z_v, and (T, Z_v) is c.l. too.

But then from the maximality of Z and T, it follows that, first, $Z = Z_v$, and second, $T = T_v$, as claimed. □

5. Characterization of r-divisorial subgroups

Definition 5.1. We say that a quasi r-divisorial valuation of $K|k$ is an r-divisorial valuation or an r-divisor of $K|k$, if v is trivial on K.

Using the idea from Pop [P4], we now show that using the information encoded in "sufficiently many" 1-dimensional projections, one can characterize the r-divisorial subgroups among all the quasi r-divisorial subgroups of $\mathrm{Gal}(K'|K)$. See loc.cit., especially Fact 4.5 for more details.

Theorem 5.2. *Let $K|k$ be a function field as usual with $\mathrm{td}(K|k) > 1$. Then for a given a quasi r-divisorial subgroup $Z \subseteq \mathrm{Gal}(K'|K)$, the following assertions are equivalent:*

i) *Z is an r-divisorial subgroup of $\mathrm{Gal}(K'|K)$.*

ii) *$\exists\ t \in K\backslash k$ such that $p_t(Z) \subseteq \mathrm{Gal}(K'_t|K_t)$ is an open subgroup.*

Proof. The proof is word-by-word identical with the one of Proposition 5.6 from loc.cit., and therefore we will omit the proof here. □

References:

[Ar] Artin, E., Geometric algebra, Interscience Publishers, Inc., New York 1957.

[Bo] Bogomolov, F. A., *On two conjectures in birational algebraic geometry,* in: Algebraic Geometry and Analytic Geometry, ICM-90 Satellite Conference Proceedings, eds A. Fujiki et all, Springer Verlag Tokyo 1991.

[B–T] Bogomolov, F. A. and Tschinkel, Y., *Commuting elements in Galois groups of function fields,* pp. 75-120; in: Motives, Polylogarithms and Hodge theory", eds F.A. Bogomolov, L. Katzarkov, International Press, 2002.

[BOU] Bourbaki, Algèbre commutative, Hermann Paris 1964.

[Ef] Ido Efrat, Valuations, Orderings and Milnor K-Theory, AMS Mathematical Surveys and Monographs, Vol. **124**, 2006.

[E–E] Endler, O. and Engler, A. J., *Fields with Henselian Valuation Rings,* Math. Z. **152** (1977), 191–193.

[E–P] Engler, A. J. and Prestel, A., Valued Fields, Springer Monographs in Mathematics Series, Springer-Verlag Berlin and Heidelberg GmbH & Co. KG, 2005.

[GGA] Geometric Galois Actions I, LMS LNS Vol **242**, eds L. Schneps – P. Lochak, Cambridge Univ. Press 1998.

[G1] Grothendieck, A., *Letter to Faltings, June 1983*, See [GGA].

[G2] Grothendieck, A., *Esquisse d'un programme, 1984.* See [GGA].

[Kh] Koch, H., *Die Galoissche Theorie der p-Erweiterungen,* Math. Monogr. **10**, Berlin 1970.

[Ko] Koenigsmann, J., *Solvable absolute Galois groups are metabelian,* Inventiones Math. **144** (2001), 1–22.

[MMS] Mahé, L., Mináč and Smith, T. L., *Additive structure of multiplicative subgroups of fields and Galois theory,* Doc. Math. **9** (2004), 301-355.

[Mu] Mumford, D., The red book of varieties and schemes, LNM 1358, 2nd edition, Springer Verlag 1999.

[Ne] Neukirch, J., *Über eine algebraische Kennzeichnung der Henselkörper,* J. reine angew. Math. **231** (1968), 75–81.

[NSW] Neukirch, J., Schmidt, A. and Wingberg, K., *Cohomology of Number Fields,* 2nd edition, Grundlehren der Mathematischen Wissenschaften **323**, Springer-Verlag Berlin 2008.

[Pa] Parshin, A. N., *Finiteness Theorems and Hyperbolic Manifolds,* in: The Grothendieck Festschrift III, eds P. Cartier et all, PM Series Vol 88, Birkhäuser Boston Basel Berlin 1990.

[P1] Pop, F., *On Grothendieck's conjecture of birational anabelian geometry,* Ann. of Math. **138** (1994), 145–182.

[P2] Pop, F., *Glimpses of Grothendieck's anabelian geometry,* in: Geometric Galois Actions I, LMS LNS Vol **242**, p. 133–126; eds L. Schneps – P. Lochak, Cambridge Univ. Press 1998.

[P3] Pop, F., *Pro-ℓ birational anabelian geometry over algebraically closed fields* I, Manuscript, Bonn 2003. See: `http://arxiv.org/pdf/math.AG/0307076`.

[P4] Pop, F., *Pro-ℓ Galois theory of Zariski prime divisors,* in: Luminy Proceedings Conference, SMF No **13**; eds Débès et al, Hérmann Paris 2006.

[P5] Pop, F., *Recovering fields from their decomposition graphs,* Manuscript 2007.
See: `http://www.math.upenn.edu/~pop/Research/Papers.html`.

[Ro] Roquette, P., *Zur Theorie der Konstantenreduktion algebraischer Mannigfaltigkeiten,* J. reine angew. Math. **200** (1958), 1–44.

[Sz] Szamuely, T., *Groupes de Galois de corps de type fini (d'après Pop),* Astérisque **294** (2004), 403–431.

[Se] Serre, J.-P., Cohomologie Galoisienne, LNM 5, Springer 1965.

[Uch] Uchida, K., *Isomorphisms of Galois groups of solvably closed Galois extensions,* Tôhoku Math. J. **31** (1979), 359–362.

[Z–S] Zariski, O. and Samuel, P., Commutative Algebra, Vol II, Springer-Verlag, New York, 1975.

Department of Mathematics
University of Pennsylvania
DRL, 209 S 33rd Street
Philadelphia, PA 19104. USA
E-mail address: `pop@math.upenn.edu`
URL: `http://math.penn.edu/~pop`