Lectures on
Anabelian Phenomena
in Geometry and Arithmetic

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PART I: Introduction and motivation

The term "anabelian" was invented by GROTHENDIECK, and a possible translation of it might be "beyond Abelian." The corresponding mathematical notion of "anabelian Geometry" is vague as well, and roughly means that under certain "anabelian hypotheses" one has:

$* * *$ Arithmetic and Geometry are encoded in Galois Theory $* * *$

It is our aim to try to explain the above assertion by presenting/explaining some results in this direction. For GROTHENDIECK's writings concerning this the reader should have a look at [G1], [G2].

A) First examples:

a) *Absolute Galois group and real closed fields*

Let K be an arbitrary field, K^{a} an algebraic closure, K^{s} the separable closure of K inside K^{a}, and finally $G_K = \mathrm{Aut}(K^{\mathrm{s}}|K) = \mathrm{Aut}(K^{\mathrm{a}}|K)$ the absolute Galois group of K. It is a celebrated well known Theorem by ARTIN–SCHREIER from the 1920's which asserts the following: *If G_K is a finite non-trivial group, then $G_K \cong G_{\mathbb{R}}$ and K is real closed.* In particular, $\mathrm{char}(K) = 0$, and $K^{\mathrm{a}} = K[\sqrt{-1}]$. Thus the non-triviality + finiteness of G_K imposes very strong restrictions on K. Nevertheless, the kind of restrictions imposed on K are not on the *isomorphism type of* K as a field, as there is a big variety of isomorphy types of real closed fields (and their classification up to isomorphism seems to be out of reach). The kind of restriction imposed on K is rather one concerning the algebraic behavior of K, namely that the algebraic geometry over K looks like the one over $\mathbb{R}$.

b) *Fundamental group and topology of complex curves*

Let X be a smooth complete curve over an algebraically closed field of characteristic zero. Then using basic results about the structure of algebraic fundamental groups, it follows that the geometric fundamental group $\pi_1(X)$ of X is isomorphic – as a profinite group – to the profinite completion $\widehat{\Gamma}_g$ of the fundamental group Γ_g of the compact orientable topological surface of genus g. Hence $\pi_1(X)$ is the profinite group on $2g$ generators σ_i, τ_i $(1 \leq i \leq g)$ subject to the unique relation $\prod_i [\sigma_i, \tau_i] = 1$. In particular, the genus g of the curve X is encoded in $\pi_1(X)$. But as above, the isomorphy type of the curve X, i.e., of the object under discussion, is not "seen" by its geometric fundamental group $\pi_1(X)$ (which in some sense corresponds to the absolute Galois group of the field K). Precisely, the restriction imposed by $\pi_1(X)$ on X is of topological nature (one on the complex points $X(\mathbb{C})$ of the curve).

B) Galois characterization of global fields

More than forty years after the result of Artin–Schreier, it was Neukirch who realized (in the late 1960's) that there must be a *p-adic* variant of the Artin–Schreier Theorem; and that such a result would have highly interesting consequences for the arithmetic of number fields (and more general, global fields). The situation is as follows: In the notations from a) above, suppose that K is a field of algebraic numbers, i.e., $K \subset \mathbb{Q}^{\mathrm{a}} \subset \mathbb{C}$. Then the Artin–Schreier Theorem asserts that if G_K is finite and non-trivial, K is isomorphic to the field of real algebraic numbers $\mathbb{R}^{\mathrm{abs}} = \mathbb{R} \cap \mathbb{Q}^{\mathrm{a}}$. This means that the only finite non-trivial subgroups of $G_{\mathbb{Q}}$ are the ones generated by the $G_{\mathbb{Q}}$-conjugates of the complex conjugation; in particular, all such subgroups have order 2, and their fixed fields are the conjugates of the field of real algebraic numbers. Now the idea of Neukirch was to understand the fields of algebraic numbers $K \subset \mathbb{Q}^{\mathrm{a}}$ having absolute Galois group G_K isomorphic (as profinite group) to the absolute Galois group $G_{\mathbb{Q}_p}$ of the p-adics $\mathbb{Q}_p$. Note that $G_{\mathbb{Q}_p}$ is much more complicated than $G_{\mathbb{R}}$. It is nevertheless a topologically finitely generated field, and its structure is relatively known, by work of Jakovlev, Poitou, Jannsen–Wingberg, etc., see e.g. [J–W]. Finally, Neukirch proved the following surprising result, which in the case of subfields $K \subset \mathbb{Q}$ is the perfect p-adic analog of the Theorem of Artin–Schreier:

Theorem (See e.g. Neukirch [N1]).
For fields of algebraic numbers $K, K' \subset \mathbb{Q}^{\mathrm{a}}$ the following hold:

(1) *Suppose that $G_K \cong G_{\mathbb{Q}_p}$. Then K is the decomposition field of some prolongation of $|\ |_p$ to $\mathbb{Q}^{\mathrm{a}}$. Or equivalently, K is $G_{\mathbb{Q}}$-conjugated to the field of algebraic p-adic numbers $\mathbb{Q}_p^{\mathrm{abs}}$.*

(2) *Suppose that $G_{K'}$ is isomorphic to an open subgroup of $G_{\mathbb{Q}_p}$. Then there exists a unique $K \subset \mathbb{Q}^{\mathrm{a}}$ as at* (1) *above such that K' is a finite extension of K.*

The Theorem above has the surprising consequence that an *isomorphism of Galois groups* of number fields gives rise functorially to an *arithmetical equivalence* of the number fields under discussion. The precise statement is as follows: For number fields K, let $\mathcal{P}(K)$ denote the set of their places. Let $\Phi : G_K \to G_L$ be an isomorphism of Galois groups of number fields. Then a consequence of the above Theorem reads: Φ *maps the decomposition groups of the places of* K *isomorphically onto the decomposition places of* L*. This bijection respects the arithmetical invariants* $e(\mathfrak{p}|p)$, $f(\mathfrak{p}|p)$ *of the places* $\mathfrak{p}|p$*, thus defines an arithmetical equivalence:*

$$\varphi : \mathcal{P}(K) \to \mathcal{P}(L).$$

Finally, applying basic facts concerning arithmetical equivalence of number fields, one gets: In the above context, suppose that $K|\mathbb{Q}$ is a Galois extension. Then $K \cong L$ as fields. Naturally, this isomorphism is a $\mathbb{Q}$-isomorphism. Since $K|\mathbb{Q}$ is a normal extension, it follows that $K = L$ when viewed as sub-extensions of fixed algebraic closure $\mathbb{Q}^{\mathrm{a}}$. In particular, $G_K = G_L$ as subgroups of $G_{\mathbb{Q}}$. Thus the open normal subgroups of $G_{\mathbb{Q}}$ are *equivariant,* i.e., they are invariant under automorphisms of $G_{\mathbb{Q}}$. This lead NEUKIRCH to the following questions:

1) Does $G_{\mathbb{Q}}$ have inner automorphisms only?

2) Is every isomorphism $\Phi : G_K \to G_L$ as above defined by the conjugation by some element inside $G_{\mathbb{Q}}$?

Finally, the first peak in this development was reached at the beginning of the 1970's, with a *positive answer* to Question 1) by IKEDA [Ik] (and partial results by KOMATSU), and the break through by UCHIDA [U1], [U2], [U3] (and unpublished notes by IWASAWA) showing that the answer to Question 2) is positive. Even more, the following holds:

Theorem. *Let* K *and* L *be global fields. Then the following hold:*

(1) *If* $G_K \cong G_L$ *as profinite groups,* $L \cong K$ *as fields.*

(2) *More precisely, for every profinite group isomorphism* $\Phi : G_K \to G_L$*, there exists a unique field isomorphism* $\phi : L^{\mathrm{s}} \to K^{\mathrm{s}}$ *defining* Φ*, i.e., such that*

$$\Phi(g) = \phi^{-1} \circ g \circ \phi \quad \textit{for all} \quad g \in G_K.$$

In particular, $\phi(L) = K$*. And therefore we have a bijection:*

$$\mathrm{Isom}_{\mathrm{fields}}(L, K) \cong \mathrm{Out}_{\mathrm{prof.gr.}}(G_K, G_L)$$

This is indeed a very remarkable fact: The Galois theory of the global fields encodes the isomorphism type of such fields in a functorial way! Often this result is called the *Galois characterization of global fields.*

We recall briefly the idea of the proof, as it is very instructive for the future developments. First, recall that by results of TATE and SHAFAREVICH, we know that the virtual ℓ-cohomological dimension $\mathrm{vcd}_\ell(K) := \mathrm{vcd}(G_K)$ of a global field K is as follows, see e.g. SERRE [S1], Ch.II:

i) If K is a number field, then $\mathrm{vcd}_\ell(K) = 2$ for all ℓ.

ii) If $\mathrm{char}(K) = p > 0$, then $\mathrm{vcd}_p(K) = 1$, and $\mathrm{vcd}_\ell(K) = 2$ for $\ell \neq p$.

In particular, if $G_K \cong G_L$ then K and L have the same characteristic.

Case 1. $K, L \subset \mathbb{Q}^{\mathrm{a}}$ are number fields. Then the isomorphism $\Phi : G_K \to G_L$ defines an arithmetical equivalence of K and L. Therefore, K and L have the same normal hull M_0 over $\mathbb{Q}$ inside $\mathbb{Q}^{\mathrm{a}}$; and moreover, for every finite normal sub-extension $M|\mathbb{Q}$ of $\mathbb{Q}^{\mathrm{a}}$ which contains K and L one has: Φ maps G_M isomorphically onto itself, thus defines an isomorphism

$$\overline{\Phi}_M : \mathrm{Gal}(M|K) \to \mathrm{Gal}(M|L)$$

In order to conclude, one shows for a properly chosen *Abelian* extension $M_1|M$, every isomorphism $\overline{\Phi}_M$ which can be extended to an isomorphism $\overline{\Phi}_{M_1}$, can also be extended to an automorphism of $\mathrm{Gal}(M|\mathbb{Q})$. Finally, one deduces from this that Φ can be extended to an automorphism of $G_{\mathbb{Q}}$, etc.

Note that the fact that the arithmetical equivalence of normal number fields implies their isomorphism relies on the Chebotarev Density Theorem, thus *analytical methods.* Until now we do not have a purely *algebraic proof* of that fact.

Case 2. K, L are global fields over $\mathbb{F}_p$. First recall that the space of all the non-trivial places $\mathcal{P}(K)$ of K is in a canonical bijection with the closed points of the unique complete smooth model $X \to \mathbb{F}_p$ of K. In particular, given an isomorphism $\Phi : G_K \to G_L$, the "arithmetical equivalence" of K and L, is just a bijection $X^0 \to Y^0$ from the closed points of X to the closed points of the complete smooth model $Y \to \mathbb{F}_p$ of L. And the problem is now to show that this abstract bijection comes from geometry. The way to do it is by using the class field theory of global function fields as follows: First, one recovers the Frobenius elements at each place $\mathfrak{p}$ of K; and then the multiplicative group $K^\times$ by using Artin's reciprocity map; and finally the addition on $K = K^\times \cup \{0\}$. Since the recipe for recovering these objects is invariant under profinite group isomorphisms, it follows that $\Phi : G_K \to G_L$ defines a group isomorphism $\phi_K : K^\times \to L^\times$. Finally, one shows that ϕ_K respects the addition, by reducing it to the case $\phi_K(x+1) = \phi_K(x)+1$. Moreover, by performing this construction for all finite sub-extensions $K_1|K$ of $K^{\mathrm{s}}|K$, and the corresponding sub-extensions $L_1|L$ of $L^{\mathrm{s}}|L$ –which are finite as well, and using the functoriality of the class field theory, one finally gets a field isomorphism $\phi : K^{\mathrm{s}} \to L^{\mathrm{s}}$ which defines Φ, i.e., $\Phi(g) = \phi \circ g \circ \phi^{-1}$ for all $g \in G_K$.

PART II: Grothendieck's Anabelian Geometry

The natural context in which the above results appear as first prominent examples is *Grothendieck's anabelian geometry,* see [G1], [G2]. We will formulate Grothendieck's anabelian conjectures in a more general context later, after having presented the basic facts about étale fundamental groups. But it is easy and appropriate to formulate here the so called *birational anabelian Conjectures,* which involve only the usual absolute Galois group.

A) Warm-up: Birational anabelian Conjectures

The so called birational anabelian Conjectures place the Results by Neukirch, Ikeda, Uchida, et al —at least conjecturally— into a bigger picture. And in their most naive form, these conjectures assert that there should be a "Galois characterization" of the finitely generated infinite fields similar to that of the global fields, i.e., if K and L are such fields and $G_K \cong G_L$, then K and L have finite purely inseparable extensions $K'|K$ and $L'|L$ such that $K' \cong L'$ as fields. (Note that the canonical projections $G_{K'} \to G_K$, $G_{L'} \to G_L$ are isomorphisms, hence every isomorphisms $G_K \cong G_L$ gives rise canonically to an isomorphism $G_{K'} \cong G_{L'}$. This is simply the translation of the fact that Galois Theory "does not see" pure inseparable extensions.) To make a more precise conjecture, recall that for an arbitrary field K we denote by $K^{\mathrm{i}} \subseteq K^{\mathrm{a}}$ its maximal purely inseparable extension in some algebraic closure K^{a} of K. Thus if $\mathrm{char}(K) = 0$, then $K^{\mathrm{i}} = K$. Further, we say that two field homomorphisms $\phi, \psi : L \to K$ differ by an absolute Frobenius twist, if $\psi = \phi \circ \mathrm{Frob}^n$ on L^{i} for some power Frob^n of the absolute Frobenius Frob. Finally, we identify G_K with $G_{K^{\mathrm{i}}}$ via the canonical projection $G_{K^{\mathrm{i}}} \to G_K$, which is an isomorphism.

Birational anabelian Conjectures.

(1) *There exists a group theoretic recipe by which one can recover K^{i} from G_K for every finitely generated infinite field K. In particular, if for such fields K and L one has $G_K \cong G_L$, then $K^{\mathrm{i}} \cong L^{\mathrm{i}}$.*

(2) *Moreover, given such fields K and L, one has the following:*

• Isom-form: *Every isomorphism $\Phi : G_K \to G_L$ is defined by a field isomorphism $\phi : L^{\mathrm{a}} \to K^{\mathrm{a}}$ via $\Phi(g) = \phi^{-1} \circ g \circ \phi$ for $g \in G_K$, and ϕ is unique up to Frobenius twists. In particular, one has $\phi(L^{\mathrm{i}}) = K^{\mathrm{i}}$.*

• Hom-form: *Every open homomorphism $\Phi : G_K \to G_L$ is defined by a field embedding $\phi : L^{\mathrm{a}} \hookrightarrow K^{\mathrm{a}}$, and ϕ is unique up to Frobenius twists. In particular, one has $\phi(L^{\mathrm{i}}) \subseteq K^{\mathrm{i}}$.*

As in the case of global fields, the Isom-form of the Birational anabelian Conjecture is also called the *Galois characterization of the finitely generated infinite fields.* The main known facts are summarized below:

Theorem.

(1) (See Pop [P2], [P3]) *There is a group theoretical recipe by which one can recover in a functorial way finitely generated infinite fields K from their absolute Galois groups G_K.*

Moreover, this recipe works in such a way that it implies the Isom-form *of the birational anabelian Conjecture, i.e., every isomorphism $\Phi : G_K \to G_L$ is defined by an isomorphism $\phi : L^{\mathrm{a}} \to K^{\mathrm{a}}$, and ϕ is unique up to Frobenius twists.*

(2) (See Mochizuki [Mzk3], Theorem B) *The* relative Hom-form *of the birational anabelian Conjecture is true in characteristic zero, which means the following: Given function fields K and L over $\mathbb{Q}$, every open $G_{\mathbb{Q}}$-homomorphism $\Phi : G_K \to G_L$ is defined by a unique field embedding $\phi : L^{\mathrm{a}} \to K^{\mathrm{a}}$, which in particular, maps L into K.*

We give here the sketch of the proof of the Isom-form. Mochizuki's Hom-form relies on his proof of the anabelian conjectures for curves over sub-p-adic fields, and we will say more about that later on.

The main steps of the proof are the following:

The first part of the proof consists in developing a higher dimensional *Local Theory* which, roughly speaking, is a direct generalization of Neukirch's result above concerning the description of the places of global fields. Nevertheless, there are some difficulties with this generalization, because in higher dimensions the finitely generated fields do not have unique normal (or smooth) complete models. Recall that a model $X \to \mathbb{Z}$ for such a field K is by definition a separated, integral scheme of finite type over $\mathbb{Z}$ whose function field is K. We will consider only quasi-projective normal models, maybe satisfying some extra conditions, like regular, etc. In particular, if X is a model of K, then the Kronecker dimension $\dim(K)$ of K equals $\dim(X)$ as a scheme. One has:

• K is a global field if and only if every normal model X of K is an open of either $X_K := \operatorname{Spec} \mathcal{O}_K$ if K is a number field, or of the unique complete smooth model $X_K \to \mathbb{F}_p$ of K, if K is a global function field with $\mathrm{char}(K) = p$. Further, there exists a natural bijection between the *prime Weil divisors* of X_K and the non-archimedean places of K. The basic result by Neukirch [N1] can be interpreted as follows: First let us say that a closed subgroup $Z \subset G_K$ is a *divisorial like subgroup,* if it is isomorphic to a decomposition group $Z_{\mathfrak{q}}$ over some prime $\mathfrak{q}$ of some global field L. Note that the structure of such groups as profinite groups is known, see e.g., Jannsen–Wingberg [J–W]. Then the decomposition groups over the places of K are the *maximal divisorial like subgroups* of G_K.

This gives then the group theoretic recipe for describing the prime Weil divisors of X_K in a functorial way.

• In general, i.e., if K is not necessarily a global field, there is a huge variety of normal complete models $X \to \mathbb{Z}$ of K. In particular, we cannot hope to obtain much information about a single specific model X of K, as in general there is no privileged model for K as in the global field case. (Well, maybe with the exception of arithmetical surfaces, where one could choose the minimal model, but this doesn't help much...) A way to avoid this is to consider –in a first approximation– the space of (Zariski) *prime divisors* $\mathcal{D}^1_K$ of K. This is, by definition, the set of all the discrete valuations v of K defined by the Weil prime divisors of all possible normal models $X \to \mathbb{Z}$ of K.

A prime divisor v of K is called geometrical if the residue field Kv of v has $\mathrm{char}(Kv) = \mathrm{char}(K)$, or equivalently, if v is trivial on the prime field of K, and arithmetical otherwise. Clearly, arithmetical prime divisors exist only if $\mathrm{char}(K) = 0$. If so, and v is defined by a Weil prime divisor X_1 of a normal model $X \to \mathbb{Z}$, then v is geometrical if and only if v is a "horizontal" divisor of $X \to \mathbb{Z}$.

For every prime divisor $v \in \mathcal{D}^1_K$ of K, let Z_v be the decomposition group of some prolongation v^{s} of v to K^{s}. We will call the totality of all the closed subgroups of the form Z_v the *divisorial subgroups* of G_K or of K. Finally, as above, a closed subgroup $Z \subset G_K$ is called *divisorial like subgroup,* if it is isomorphic to a divisorial subgroup of a finitely generated field L with $\dim(L) = \dim(K)$. The main results of the Local Theory are as follows, see [P1]:

a) For a prime divisor v, the numerical data $\mathrm{char}(K)$, $\mathrm{char}(Kv)$, and $\dim(K)$ are group theoretically encoded in Z_v; in particular, whether v is geometric or not. Further, the inertia group $T_v \subset Z_v$ of $v^{\mathrm{s}}|v$, and the canonical projection $\pi_v : Z_v \to G_{Kv}$ are also encoded group theoretically in Z_v. In particular, the residual absolute Galois group G_{Kv} at all the prime divisors v of K is group theoretically encoded in G_K.

b) Every divisorial like subgroup $Z \subset G_K$ is contained in a unique divisorial subgroup Z_v of G_K. Thus the divisorial like subgroups of G_K are exactly the *maximal divisorial like subgroups* of G_K. And the space $\mathcal{D}^1_K$ is in bijection with the conjugacy classes of divisorial subgroups of G_K.

The results from the local theory above suggest that one should try to prove the birational anabelian Conjecture by *induction on* $\dim(K)$. This is the idea for developing a *Global Theory* along the following lines:

For every field Ω and an abelian group A related to Ω, e.g., $A = \mathbb{Z}$ or $A = \mu_{\overline{\Omega}}$ the roots of unity in $\overline{\Omega}$, we consider the `prime to` $\mathrm{char}(\Omega)$ `adic completion` of A denoted $\widehat{A}_\Omega := \varprojlim_m A/mA$, $(m, \mathrm{char}(\Omega)) = 1$, or $\widehat{A}$ if Ω is clear from the context.

First, the Isom-form of the birational anabelian Conjecture for global fields, i.e., $\dim(K) = 1$, is known; and we think of it as the first induction step. Now suppose that $\dim(K) = d > 1$. By the induction hypothesis, suppose that the Isom-form of the birational anabelian Conjecture is true in dimension $< d$. Then one recovers the field K^{i} up to Frobenius twists from G_K along the following steps (from which it it will be clear what we mean by a "group theoretic recipe").

<u>Step 1)</u> Recover the cyclotomic character $\chi_K : G_K \to \widehat{\mathbb{Z}}^\times$ of G_K.

The recipe is as follows: Since $\dim(Kv) = \dim(K) - 1 < d$, the cyclotomic character χ_{Kv} is "known" for each prime divisor $v \in \mathcal{D}^1_K$. Thus

$$\chi_v : Z_v \xrightarrow{\pi_v} G_{Kv} \xrightarrow{\chi_{Kv}} \widehat{\mathbb{Z}}^\times$$

is known for all $v \in \mathcal{D}^1_K$. On the other hand, using the higher dimensional Chebotarev Density Theorem, see e.g., Serre [S3], it follows that $\ker(\chi_K)$ is the closed subgroup of G_K generated by all the $\ker(\chi_v)$, $v \in \mathcal{D}^1_K$. Thus χ_K is the unique character $\chi : G_K \to \widehat{\mathbb{Z}}^\times$ which coincides with χ_v on each Z_v.

Next let $\imath : \mathbb{T}_K \to \widehat{\mathbb{Z}}(1)$ be a fixed identification of $\mathbb{Z}(1)$ with the Tate G_K-module $\mathbb{T}_K = \varprojlim_m \mu_m$. The Kummer Theory gives to:

$$K^\times \xrightarrow{\jmath_K} \widehat{K^\times} \xrightarrow{\hat{\delta}} \mathrm{H}^1(K, \widehat{\mathbb{Z}}(1)),$$

where $\widehat{K^\times}$ is the prime to $\mathrm{char}(K)$ adic completion of $K^\times$ and $\jmath_K$ is the completion homomorphism; and "functorial" means that performing this construction for finite extensions $M|K$ inside $K^{\mathrm{a}}|K$, we get corresponding commutative "inclusion-restriction" diagrams (which we omit to write down here). An essential point to make here is that the completion morphism $\jmath_K : K^\times \to \widehat{K^\times}$ is injective. This follows by induction on $\dim(K)$ by using the following fact: Let $K|k$ be the function field of a geometrically irreducible complete curve $X \to k$. Then $K^\times/k^\times$ is the group of principal divisors of X, thus a free Abelian group. And for K a number field, one knows that $K^\times/\mu_K$ is a free Abelian group.

<u>Step 2)</u> Recover the geometric small sets of prime divisors.

We will say that a subset $\mathcal{D} \subset \mathcal{D}^1_K$ of prime divisors is *geometric,* if there exists a quasi-projective normal model $X \to k$ of K such that $\mathcal{D} = \mathcal{D}_X$ is the set of prime divisors of K defined by the Weil prime divisors of X. Here, k is the field of constants of K. It is a quite technical point to show —by induction on the (absolute) transcendence degree $d = \mathrm{td}(K)$, that the geometric sets of prime divisors can be recovered from G_K, see Pop [P3]. Next let $\mathcal{D} = \mathcal{D}_X$ be a geometric set of prime divisors. One has a canonical exact sequence

$$1 \to U_{\mathcal{D}} \to K^\times \to \mathrm{Div}(X) \to \mathfrak{Cl}(X) \to 0,$$

where $U_{\mathcal{D}}$ are the units in the ring of global sections on X, and the other notations are standard. Since the base field k is either finite or a number field, the Weil divisor class group $\mathfrak{Cl}(X)$ is finitely generated. Thus if X is "sufficiently small", then $\mathfrak{Cl}(X) = 0$. A geometric set of prime divisors $\mathcal{D} = \mathcal{D}_X$ will be called a *small geometric set* of prime divisors, if the adic completion $\widehat{\mathfrak{Cl}}(X)$ is trivial. One shows that the small geometric sets of prime divisors can be recovered form G_K, see loc.cit. In this process, one shows that the adic completion of the above exact sequence can be recovered form G_K too:

$$1 \to \widehat{U}_{\mathcal{D}} \to \widehat{K^\times} \to \widehat{\mathrm{Div}}(X) = \widehat{\oplus}_v \widehat{\mathbb{Z}} \to \widehat{\mathfrak{Cl}}(X) \to 0,$$

<u>Step 3)</u> Recover the multiplicative group $K^\times$ inside $\widehat{K^\times}$.

Let $\mathcal{D} = \mathcal{D}_X$ be a small geometric set of prime divisors of K. The resulting exact sequence defined above becomes $1 \to \widehat{U}_{\mathcal{D}} \to \widehat{K^\times} \to \widehat{\oplus}_v \widehat{\mathbb{Z}} \to 0$, as $\widehat{\mathfrak{Cl}}(X) = 0$. Next let $v \in \mathcal{D}$ be arbitrary. Then the group of global units $U_{\mathcal{D}}$ is contained in the group of v-units $\mathcal{O}_v^\times$. Thus the $(\mathrm{mod}\,\mathfrak{m}_v)$ reduction $p_v : \mathcal{O}_v^\times \to Kv^\times$ is defined on

$U_{\mathcal{D}}$. Using some arguments involving Hilbertian fields, one shows that there exist "many" $v \in \mathcal{D}$ such that $U_{\mathcal{D}}$ as well as $\widehat{U}_{\mathcal{D}}$ are actually mapped isomorphically into $Kv^{\times}$, respectively $\widehat{Kv}^{\times}$; and moreover, that inside $\widehat{Kv}^{\times}$ one has

$$(*) \qquad p_v(U_{\mathcal{D}}) = \hat{p}_v(\widehat{U}_{\mathcal{D}}) \cap Kv^{\times}.$$

On the Galois theoretic side, the reduction map p_v is defined by the restriction coming from the inclusion $Z_v \hookrightarrow G_K$. And moreover, since $U_{\mathcal{D}}$ is contained in the v-units, it follows that under the restriction map $\widehat{K} \to \mathrm{H}^1(Z_v, \widehat{\mathbb{Z}}(1))$, the image of $\widehat{U}_{\mathcal{D}}$ is contained in the image of $\inf(\pi_v) : \widehat{Kv}^{\times} \to \mathrm{H}^1(Z_v, \widehat{\mathbb{Z}}(1))$ defined by the canonical projection $\pi_v : Z_v \to G_{Kv}$.

Finally, the recipe to recover $K^{\times}$ inside $\widehat{K^{\times}}$ is as follows. First, for each small geometric set of prime divisors $\mathcal{D}$ and v as above, $U_{\mathcal{D}}$ is exactly the preimage of $\hat{p}_v(\widehat{U}_{\mathcal{D}}) \cap Kv^{\times}$, by assertion $(*)$ above. Since $K^{\times} = \cup_{\mathcal{D}} U_{\mathcal{D}}$, when $\mathcal{D}$ runs over smaller and smaller (small) geometric sets of prime divisors, we finally recover $K^{\times}$ inside $\widehat{K^{\times}}$.

Step 4) Define the addition in $K = K^{\times} \cup \{0\}$.

This is easily done using the induction hypothesis: Let namely $x, y \in K^{\times}$ be given. Then $x + y = 0$ iff $x/y = -1$, and this fact is encoded in $K^{\times}$. Now suppose that $x + y \neq 0$. Then $x + y = z$ in K iff for all v such that x, y, z are all v-units one has: $p_v(x) + p_v(y) = p_v(z)$. On the other hand, this last fact is encoded in the field structure of G_{Kv}, which we already know.

Finally, in order to conclude the proof of the Isom-form of the birational anabelian Conjecture, we proceed as follows: Let $\Phi : G_K \to G_L$ be an isomorphism of absolute Galois group $\Phi : G_K \to G_L$. Then the recipe of recovering the fields K and L are "identified" via Φ, and shows that the p-divisible hulls of K and L inside $\widehat{K^{\times}} \cong \widehat{L^{\times}}$ must be the same, where $p = \mathrm{char}(K)$. This finally leads to an isomorphism $\phi : L^{\mathrm{a}} \to K^{\mathrm{a}}$ which defines Φ. Its uniqueness up to Frobenius twists follows from the fact that given two such field isomorphisms ϕ', ϕ'', then setting $\phi := \phi'^{-1} \circ \phi''$ we obtain an automorphism of K^{a} which commutes with G_K. And one checks that any such automorphism is a Frobenius twist.

B) Anabelian Conjectures for Curves

a) *Étale fundamental groups*

Let X be a connected scheme endowed with a geometric base point $\overline{x}$. Recall that the étale fundamental group $\pi_1(X, \overline{x})$ of $(X, \overline{x})$ is the automorphism group of the fiber functor on the category of all the étale connected covers of X. The étale fundamental group is functorial in the following sense: Let connected schemes with geometric base points $(X, \overline{x})$ and $(Y, \overline{y})$, and a morphism $\phi : X \to Y$ be given such that $\overline{y} = \phi \circ \overline{x}$. Then ϕ gives rise to a morphism between the fiber functors $\mathcal{F}_{\overline{x}}$ and $\mathcal{F}_{\overline{y}}$, which induces a continuous morphism of profinite groups

$\pi_1(\phi) : \pi_1(X, \overline{x}) \to \pi_1(Y, \overline{y})$ in the canonical way. In particular, setting $Y = X$ and $\overline{y}$ some geometric point of X, a "path" from between $\overline{x}$ and $\overline{y}$, gives rise to an inner automorphism of $\pi_1(X, \overline{x})$. In other words, $\pi_1(X, \overline{x})$ is *determined by X up to inner automorphisms.* (This means that the situation is completely parallel to the one in the case of the topological fundamental group.) A basic property of the étale fundamental group that it is invariant under *universal homeomorphisms,* hence under purely inseparable covers and Frobenius twists.

Next let $\mathcal{G}$ be the category of all profinite groups and outer continuous homomorphisms as morphisms. The objects of $\mathcal{G}$ are the profinite groups, and for given objects G and H, a $\mathcal{G}$-morphism from G to H is a set of the form $\mathrm{Inn}(H) \circ f$, where $f : G \to H$ is a morphism of profinite groups, and $\mathrm{Inn}(H)$ is the set of all the inner automorphisms of H. Clearly, if $f, g : G \to H$ differ by an inner automorphism of G, then $\mathrm{Inn}(H) \circ f = \mathrm{Inn}(H) \circ g$, thus they define the same $\mathcal{G}$-homomorphism from G to H. Further, $\mathrm{Inn}(H) \circ f$ is a $\mathcal{G}$-isomorphism if and only if $f : G \to H$ is an isomorphism of profinite groups.

Therefore, viewing the étale fundamental group π_1 as having values in $\mathcal{G}$ rather than in the category of profinite groups, the relevance of the geometric points $\overline{x}$ vanishes. Therefore, we will simply write $\pi_1(X)$ for the fundamental group of a connected scheme X.

In the same way, if S is a connected base scheme, and X is a connected S-scheme, then the structure morphism $\varphi_X : X \to S$ gives rise to an *augmentation morphism* $p_X : \pi_1(X) \to \pi_1(S)$. Thus the category $\mathfrak{Sch}_S$ of all the S-schemes is mapped by π_1 into the category $\mathcal{G}_S$ of all the $\pi_1(S)$-groups, i.e. the profinite groups G with an "augmentation" morphism $pr_G : G \to \pi_1(S)$.

Now let us consider the more specific situation when the base scheme S is a field, and the k-schemes X are geometrically connected. Denote $\overline{X} = X \times_k k^{\mathrm{s}}$ the base to the separable closure of k (in some fixed "universal field"), and remark that by the facts above one has an exact sequence of profinite groups of the form

$$1 \to \pi_1(\overline{X}) \to \pi_1(X) \to G_k \to 1.$$

In particular, we have a representation $\rho_X : G_k \to \mathrm{Out}(\pi_1(\overline{X})) = \mathrm{Aut}_{\mathcal{G}}(\pi_1(\overline{X}))$ which encodes most of the information carried by the exact sequence above. The group $\pi_1(\overline{X})$ is called the *algebraic* (or *geometric*) fundamental group of X. In general, little is known about $\pi_1(\overline{X})$, and in particular, even less about $\pi_1(X)$. Nevertheless, if X is a k-variety, and $k \subset \mathbb{C}$, then the base change to $\mathbb{C}$ gives a realization of $\pi_1(\overline{X})$ as the *profinite completion* of the topological fundamental group of $X^{\mathrm{an}} = X(\mathbb{C})$.

In terms of function fields, if $X \to k$ is geometrically integral and normal, one has the following: Let $k(X) \hookrightarrow k(\overline{X})$ be the function fields of $\overline{X} \to X$. Then the algebraic fundamental group $\pi_1(\overline{X})$ is (canonically) isomorphic to the Galois group of a maximal unramified Galois field extension $\mathcal{K}_{\overline{X}} \mid k(\overline{X})$.

Finally, we recall that $\pi_1(X)$ is a *birational invariant* in the case X is complete and regular. In other words, if X and X' are birationally equivalent complete regular k-varieties, then $\pi_1(X) \cong \pi_1(X')$ and $\pi_1(\overline{X}) \cong \pi_1(\overline{X'})$ canonically.

b) *Étale fundamental groups of curves*

Specializing even more, we turn our attention to curves, and give a short review of the basic known facts in this case. In this discussion we will suppose that X is a *smooth connected* curve, having a *smooth completion* say X_0 over k. We denote $S = X_0 \backslash X$, and $\overline{S} = \overline{X}_0 \backslash \overline{X}$. We will say that X is a (g, r) curve, if X_0 has (geometric) genus g, and $|\overline{S}| = r$. We will say that X is a *hyperbolic curve,* if its Euler–Poincaré characteristic $2 - 2g - r$ is negative. And we will say that a curve X as above is *virtually hyperbolic,* if it has an étale connected cover $X' \to X$ such that X' is a *hyperbolic curve* in the sense above. (Note that every étale cover $f : X' \to X$ as above is smooth and has a smooth completion which is a (g', r')-curve with $g \leq g'$ and $r' \leq r \deg(f)$ over some finite $k'|k$.)

In the above notations, let $X \to k$ be a (g, r) curve. Then a short list of the known facts about the algebraic fundamental group $\pi_1(\overline{X})$ is as follows. First, let $\Gamma_{g,r}$ be the fundamental group of the orientable compact topological surface of genus g with r punctures. Thus

$$\Gamma_{g,r} = < a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_r \mid \prod_i [a_i, b_i] \prod c_j = 1 >$$

is the discrete group on $2g + r$ generators $a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_r$ with the given unique relation. (The generators a_i, b_i, c_j have a precise interpretation as loops around the handles, respectively around the missing points.) In particular, if $r > 0$, then $\Gamma_{g,r}$ is the discrete free group on $2g + r - 1$ generators. It is well known that $\Gamma_{g,r}$ is *residually finite,* i.e., $\Gamma_{g,r}$ injects into its profinite completion:

$$\Gamma_{g,r} \hookrightarrow \widehat{\Gamma}_{g,r} \, .$$

Finally, given a fixed prime number p, respectively arbitrary prime numbers ℓ, we will denote by $\widehat{\Gamma}_{g,r} \to \widehat{\Gamma}'_{g,r}$ the maximal prime-p quotient of $\widehat{\Gamma}_{g,r}$, and by $\widehat{\Gamma}_{g,r} \to \widehat{\Gamma}^{\ell}_{g,r}$ the maximal pro-ℓ quotient of $\Gamma_{g,r}$.

<u>Case 1.</u> $\mathrm{char}(k) = 0$.

Using the remark above, in the case $k \hookrightarrow \mathbb{C}$, it follows that $\pi_1(\overline{X}) \cong \widehat{\Gamma}_{g,r}$ via the base change $X \times_k \mathbb{C} \to \overline{X}$. If $\kappa(\overline{X})$ is the function field of $\overline{X}$, then $\pi_1(\overline{X})$ is the Galois group of a maximal Galois unramified field extension $\mathcal{K}_{k(\overline{X})}|k(\overline{X})$. Moreover, the loops $c_j \in \Gamma_{g,r}$ around the missing points $x_i \in \overline{X}_0 \backslash \overline{X}$ are canonical generators of inertia groups T_{x_i} over these points in $\pi_1(\overline{X})$. In particular we have:

a) X is a complete curve of genus g if and only if $\pi_1(\overline{X})$ has $2g$ generators a_i, b_i with the single relation $\prod_i [a_i, b_i] = 1$, provided X is not $\mathbb{A}^1_k$.

b) X is of type (g, r) with $r > 0$ if and only if $\pi_1(\overline{X})$ is a profinite free group on $2g + r - 1$ generators, provided X is not k-isomorphic to $\mathbb{A}^1_k$ or $\mathbb{P}^1_k$.

Clearly, the dichotomy between the above subcases a) and b) can be as well deduced from the pro-ℓ maximal quotient $\pi_1^\ell(\overline{X})$ of $\pi_1(\overline{X})$, by simply replacing "profinite" by "pro-ℓ".

Further, the following conditions on X are equivalent:

(i) X is hyperbolic

(ii) X is virtually hyperbolic.

(iii) $\pi_1(\overline{X})$ is non-Abelian, or equivalently, (iii)$^\ell$ $\pi_1^\ell(\overline{X})$ is non-Abelian.

Case 2. $\mathrm{char}(k) > 0$.

First recall that the *tame fundamental group* $\pi_1^{\mathrm{t}}(X)$ of X is the maximal quotient of $\pi_1(X)$ which classifies étale connected covers $X' \to X$ whose ramification above the missing points $x_i \in \overline{X}_0 \backslash \overline{X}$ is tame. We will denote by $\pi_1^{\mathrm{t}}(\overline{X})$ the tame quotient of $\pi_1(\overline{X})$, and call it the *tame algebraic fundamental group* of X. Now the main technical tools used in understanding $\pi_1(\overline{X})$ and its tame quotient $\pi_1^{\mathrm{t}}(\overline{X})$ are the following two facts:

Shafarevich's Theorem.
In the context above, set $\mathrm{char}(k) = p > 0$, *and denote by* $\pi_1^p(\overline{X})$ *the maximal pro-p quotient of* $\pi_1(\overline{X})$. *Further let* $r_{X_0} = \dim_{\mathbb{F}_p} \mathrm{Jac}_{\overline{X}_0}[p]$ *denote the Hasse–Witt invariant of the complete curve* $\overline{X}_0$. *Then one has:*

(1) *If* $X = X_0$, *then* $\pi_1^p(\overline{X})$ *is a pro-p free group on* $r_{X_0} \leq g$ *generators.*

(2) *If* X *is affine, then* $\pi_1^p(X)$ *is a pro-p free group on* $|k^{\mathrm{a}}|$ *generators.*

Let k be an arbitrary base field, and v a complete discrete valuation with valuation ring $R = R_v$ of k and residue field $kv = \kappa$. Let $X \to k$ be a smooth curve which has a smooth completion $X_0 \to k$. We will say that $X \to k$ *has good reduction* at v, if the following hold: $X_0 \to k$ has a smooth model $X_{0,R} \to R$ over R, and there exists an étale divisor $S_R \to R$ of $X_{0,R}$ such that the generic fiber of the complement $X_{0,R} \backslash S_R =: X_R \to R$ is X.

Now let $X \to k$ be a hyperbolic curve having good reduction at v. In the notations from above, let $X_s \to \kappa$ be the special fiber of $X_R \to R$. Then the canonical diagram of schemes

$$\begin{array}{ccccc} X & \hookrightarrow & X_R & \hookleftarrow & X_s \\ \downarrow & & \downarrow & & \downarrow \\ k & \hookrightarrow & R & \hookleftarrow & \kappa \end{array}$$

gives rise to a diagram of fundamental groups as follows:

$$\begin{array}{ccccc} \pi_1^{\mathrm{t}}(X) & \to & \pi_1^{\mathrm{t}}(X_R) & \leftarrow & \pi_1^{\mathrm{t}}(X_s) \\ \downarrow & & \downarrow & & \downarrow \\ G_k & \to & G_k^{\mathrm{t}} & \leftarrow & G_\kappa \end{array}$$

where $\pi_1^{\mathrm{t}}(X_R)$ is the "tame fundamental group" of X_R, i.e., the maximal quotient of $\pi_1(X)$ classifying connected covers of X_R which have ramification only along S_R and the generic point of the special fiber, and this ramification is tame, and G_k^{t} is the Galois group of the maximal tamely ramified extension of k. The fundamental result concerning the fundamental groups in the diagram above is the following:

Grothendieck's Specialization Theorem.

In the context above, let X be a smooth curve of type (g, r). Further let R^{t} be the extension of R to k^{t}, and $\overline{X}_R := X_R \times_R R^{\mathrm{t}}$. Then one has:

(1) $\pi_1^{\mathrm{t}}(\overline{X}) \to \pi_1^{\mathrm{t}}(\overline{X}_R)$ *is surjective, and* $\pi_1^{\mathrm{t}}(\overline{X}_R) \leftarrow \pi_1^{\mathrm{t}}(\overline{X}_s)$ *is an isomorphism. The resulting surjective homomorphism*

$$\mathrm{sp}_v : \pi_1^{\mathrm{t}}(\overline{X}) \to \pi_1^{\mathrm{t}}(\overline{X}_s)$$

is called the specialization homomorphism *of tame fundamental groups at v. In particular, $\pi_1^{\mathrm{t}}(\overline{X})$ is a quotient of $\widehat{\Gamma}_{g,r}$ in such a way that the generators c_j are mapped to inertia elements at the missing points $x_i \in \overline{X}_0 \backslash \overline{X}$.*

(2) *Further, let* $\mathrm{char}(k) = p$, *and denote by $\pi_1'(\overline{X})$ the maximal prime to p quotient of $\pi_1(\overline{X})$ (which then equals the maximal prime to p quotient of $\pi_1^{\mathrm{t}}(\overline{X})$ too). Then $\pi_1'(\overline{X}) \cong \widehat{\Gamma}'_{g,r}$, and* sp_v *maps $\pi_1'(\overline{X})$ isomorphically onto $\pi_1'(\overline{X}_s)$. In particular, $\pi_1'(\overline{X})$ depends on (g, r) only.*

Combining Shafarevich's Theorem and Grothendieck's Specialization Theorem above, we immediately see that the following facts and invariants of $X \to k$ are encoded in $\pi_1(\overline{X})$:

a) If $\ell \neq p$, then $\pi_1^{\ell}(\overline{X}) \cong \widehat{\Gamma}^{\ell}_{g,r}$, and $\pi_1^{p}(\overline{X}) \not\cong \widehat{\Gamma}^{p}_{g,r}$. Therefore, $p = \mathrm{char}(k)$ can be recovered from $\pi_1(\overline{X})$, provided X is not $\mathbb{P}^1_k$.

a)$^{\mathrm{t}}$ The same is true correspondingly concerning the tame fundamental group $\pi_1^{\mathrm{t}}(\overline{X})$, provided X is not isomorphic to $\mathbb{A}^1_k$ or $\mathbb{P}^1_k$.

b) $X \to k$ is complete if and only if $\pi_1^{p}(\overline{X})$ is finitely generated.

b)$^{\mathrm{t}}$ Correspondingly, $X \to k$ is complete if and only if $\pi_1^{\ell}(\overline{X})$ not pro-ℓ-free, provided X is not isomorphic to $\mathbb{A}^1_k$.

c) In particular, if X is complete, then $\pi_1^{\ell}(\overline{X})$ has $2g$ generators, thus g can be recovered from $\pi_1^{\ell}(\overline{X})$.

Finally, concerning the virtual hyperbolicity of X we have the following:

d) In positive characteristic, every affine curve X is virtually hyperbolic.

Remark.

Clearly, the applicability of Grothendieck's Specialization Theorem is limited by the fact that one would need *a priori* criterions for the good reduction of the given curve X at the (completions of k with respect to the) discrete valuations v of k. At least in the case of hyperbolic curves $X \to k$ such criteria do exist. The setting is as follows: Let $X \to k$ be a hyperbolic curve, and let v be a discrete valuation of k. Let $T_v \subseteq Z_v$ be the inertia, respectively the decomposition, groups of some prolongation of v to k^{s}. Recall the canonical projections $\pi_1(X) \to G_k$ and $\pi_1^{\mathrm{t}}(X) \to G_k$ and the resulting Galois representations $\rho_X : G_k \to \mathrm{Out}(\pi_1(\overline{X}))$ and $\rho_X^{\mathrm{t}} : G_k \to \mathrm{Out}(\pi_1^{\mathrm{t}}(\overline{X}))$. Then one can characterize the fact that X has (potentially) good reduction at v as follows, see Oda [O] in the case of complete hyperbolic curves, and by Tamagawa [T1] in the case of arbitrary hyperbolic curves:

In the above notations, $X \to k$ has good reduction at v if and only if the representation ρ_X^{t} is trivial on T_v.

The concrete picture of how to apply the above remark in studying fundamental groups of hyperbolic curves $X \to k$ over either finitely generated infinite base field k or finitely generated fields over some fixed base field k_0 is as follows: Let $X \to k$ be a smooth curve of type (g, r). Further let S be a smooth model of k over $\mathbb{Z}$, if k is a finitely generated field, respectively over the base k_0 otherwise. For every closed point $s \in S$, there exists a discrete valuation v_s whose valuation ring R_s dominates the local ring $\mathcal{O}_{S,s}$, and having residue field $\kappa_{v_s} = \kappa(s)$. Let us choose such a valuation v_s. Then in the previous notations, X has good reduction at s if and only if ρ_X^{t} is trivial on the inertia group T_s over the point s. Note that by the uniqueness of the smooth model $X_{R_s} \to R_s$ —in the case it does exist, the existence of such a good reduction *does not depend on the concrete valuation v_s* used. One should also remark here, that in the context above, $X \to k$ has good reduction on a Zariski open subset of S. This follows e.g., from the Jacobian Criterion for smoothness.

Finally, we now come to announcing Grothendieck's anabelian Conjectures for Curves and the Section Conjectures.

Let $\mathcal{P}$ be a property defined for some category of schemes X. We will say that the property $\mathcal{P}$ is an *anabelian property,* if it is encoded in $\pi_1(X)$ in a group theoretical way, or in other words, if $\mathcal{P}$ can be recovered by a group theoretic recipe from $\pi_1(X)$. In particular, if X has the property $\mathcal{P}$, and $\pi_1(X) \cong \pi_1(Y)$, then Y has the property $\mathcal{P}$.

Examples:

a) In the category of all the fields K, the property *"K is real closed"* is anabelian. This is the Theorem of Artin–Schreier from above.

b) In the category of all the smooth k-curves X which are not isomorphic to $\mathbb{A}^1_k$, the property: *"X is complete and has genus g"* is anabelian. This follows from the structure theorems for the fundamental group of complete curves as discussed above.

We will say that a scheme X is *anabelian* if the isomorphy type of X up to some natural transformations, which are not encoded in Galois Theory, can be recovered *group theoretically* from $\pi_1(X)$ in a functorial way; or equivalently, if there exists a *group theoretic recipe* to recover the isomorphy type of X, up to the natural transformations in discussion, from $\pi_1(X)$. Typical examples of such "natural transformations" which are not seen by Galois Theory are the radicial covers and the birational equivalence of complete regular schemes. Concretely, for k-varieties $X \to k$ with $\mathrm{char}(k) = p > 0$, there are two typical radicial covers: First the maximal purely inseparable cover $X^{\mathrm{i}} \to k^{\mathrm{i}}$. And second, the Frobenius twists $X(n) \to k$ of $X \to k$ and/or $X^{\mathrm{i}}(n) \to k^{\mathrm{i}}$ of $X^{\mathrm{i}} \to k^{\mathrm{i}}$ obtained by acting by Frob^n "on the coefficients": $X(n) := X \times_{\mathrm{Frob}^n} k \to k$. In the same way, if $X \to k$ and $Y \to k$ are complete regular k varieties which are birationally equivalent, then $\pi_1(X)$ and $\pi_1(Y)$ are canonically isomorphic, but X and Y might be very different.

A good set of examples of anabelian schemes are the *finitely generated infinite fields,* as we have seen in the previous section. Given such a field K, one has $\pi_1(\mathrm{Spec}\,K) = G_K$, and by the birational anabelian Conjectures we know that K can be recovered from $\pi_1(K)$ in a functorial way, up to pure inseparable extensions and Frobenius twists.

Anabelian Conjecture for Curves (absolute form)

(1) *Let $X \to k$ be a virtually hyperbolic curve over a finitely generated base field k. Then X is anabelian in the sense that the isomorphism type of X can be recovered from $\pi_1(X)$ up to purely inseparable covers and Frobenius twists.*

(2) *Moreover, given such curves $X \to k$ and $Y \to l$, one has the following:*

• Isom-form: *Every isomorphism $\Phi : \pi_1(X) \to \pi_1(Y)$ is defined by an isomorphism $\phi : X^{\mathrm{i}} \to Y^{\mathrm{i}}$, and ϕ is unique up to Frobenius twists.*

• Hom-form: *Every open homomorphism $\Phi : \pi_1(X) \to \pi_1(Y)$ is defined by a dominant morphism $\phi : X^{\mathrm{i}} \to Y^{\mathrm{i}}$, and ϕ is unique up to Frobenius twists.*

One could as well consider a relative form of the above conjecture as follows:

Anabelian Conjecture for Curves (relative form)

(1) *Let $X \to k$ be a virtually hyperbolic curve over a finitely generated base field k. Then $X \to k$ is anabelian in the sense that $X \to k$ can be recovered from $\pi_1(X) \to G_k$ up to purely inseparable covers and Frobenius twists.*

(2) *Moreover, given such curves $X \to k$ and $Y \to k$, one has the following:*

• Isom-form: *Every G_k-isomorphism $\Phi : \pi_1(X) \to \pi_1(Y)$ is defined by a unique k^{i}-isomorphism $\phi : X^{\mathrm{i}}(n) \to Y^{\mathrm{i}}$ for some n-twist.*

• Hom-form: *Every open G_k-homomorphism $\Phi : \pi_1(X) \to \pi_1(Y)$ is defined by a unique dominant k^{i}-morphism $\phi : X^{\mathrm{i}}(n) \to Y^{\mathrm{i}}$ of some n-twist.*

Concerning **Higher dimensional anabelian Conjectures,** there are only vague ideas. There are some obvious necessary conditions which higher dimensional varieties X have to satisfy in order to be anabelian (like being of general type, being $K(\pi_1)$, etc.). Also, easy counter-examples show that one cannot expect a naive Hom-form of the conjectures. See GROTHENDIECK [G2], and IHARA–NAKAMURA [I–N], MOCHIZUKI [Mzk3], [Mzk4] for more about this.

Remark (Standard reduction technique).

Before going into the details concerning the known facts about the anabelian Conjectures for curves, let us set the technical frame for a fact used several times below. Let $X \to k$ be a smooth curve over the field k. Suppose that k is either a finitely generated infinite field, or a function field over some base field k_0. Let $S \to \mathbb{Z}$, respectively $S \to k_0$ be a smooth model of k.

Next let $X \to k$ be a hyperbolic curve, say with smooth completion X_0. Let $\pi_1(\overline{X}) \to G_k$, respectively $\pi_1^{\mathrm{t}}(\overline{X}) \to G_k$ be the corresponding canonical projections. Then choosing for each closed point $s \in S$ a discrete valuation v_s which dominates the local ring of s, we have the Oda–Tamagawa Criterion (mentioned above) for deciding whether $X \to k$ has good reduction at s. We also know, that $X \to k$ has good reduction on a Zariski open subset of S. In particular, the Oda–Tamagawa Criterion is a group theoretic criterion for describing the Zariski open subset of S on which $X \to k$ has good reduction. Moreover, if s is a point of good reduction of $X \to k$, then Grothendieck's Specialization Theorem for π_1^{t} gives a commutative diagram of the following form:

$$\begin{array}{ccc} \pi_1^{\mathrm{t}}(X_{k_v}) & \xrightarrow{\mathrm{sp}_s} & \pi_1^{\mathrm{t}}(X_s) \\ \downarrow & & \downarrow \\ Z_v & \xrightarrow{pr_v} & G_{\kappa(s)} \end{array}$$

where $k_v \subset k^{\mathrm{s}}$ is the decomposition field over v defining Z_v inside G_k. Therefore, given a point $s \in S$, one can recover the fact that X has good reduction at s, and if this the case, also the canonical projection $\pi_1^{\mathrm{t}}(X_s) \to G_{\kappa(s)}$ from the following data: $\pi_1^{\mathrm{t}}(X) \to G_k$ endowed with a decomposition group Z_v above a discrete valuation v whose valuation ring R_v dominates the local ring $\mathcal{O}_{S,s}$.

We conclude that the set of points $s \in S$ of good reduction of $X \to k$ as well as the canonical projections $\pi_1^{\mathrm{t}}(X_s) \to G_{\kappa(s)}$ at such points *can be recovered form* $\pi_1^{\mathrm{t}}(X) \to G_k$ if we endow G_k with decomposition groups over some discrete valuations v_s dominating $\mathcal{O}_{S,s}$ (all closed points $s \in S$)

I) TAMAGAWA*'s Results concerning affine hyperbolic curves*

In this subsection we will sketch a proof of the following result by AKIO TAMAGAWA concerning affine hyperbolic curves.

Theorem. (See Tamagawa [T1])

(1) *There exists a group theoretic recipe by which one can recover an affine smooth connected curve X defined over a finite field from $\pi_1(X)$. Moreover, if X is hyperbolic, then this recipe recovers X from $\pi_1^{\mathrm{t}}(X)$.*

Further, the absolute and the relative Isom-form *of the anabelian conjecture for curves holds for affine curves over finite fields; and its tame form holds for affine hyperbolic curves over finite fields.*

(2) *There exists a group theoretic recipe by which one can recover affine hyperbolic curves $X \to k$ defined over finitely generated fields k of characteristic zero from $\pi_1(X)$.*

Further, the absolute and the relative Isom-form *of the anabelian conjecture for Curves holds for affine hyperbolic curves over finitely generated fields of characteristic zero.*

The strategy of the proof is as follows:

First consider the case when the base field k is finitely generated and has $\mathrm{char}(k) = 0$. We claim that the canonical exact sequence

$$(*) \qquad 1 \to \pi_1(\overline{X}) \to \pi_1(X) \to G_k \to 1,$$

is encoded in $\pi_1(X)$. Indeed, recall that the algebraic fundamental group $\pi_1(\overline{X})$ is a finitely generated normal subgroup of $\pi_1(X)$. Therefore, since G_k has no proper finitely generated normal subgroups, see e.g. [F–J], Ch.16, Proposition 16.11.6, it follows that $\pi_1(\overline{X})$ is the unique maximal finitely generated normal subgroup of $\pi_1(X)$. Thus the exact sequence above can be recovered from $\pi_1(X)$. Further, by either using the characterization of the geometric inertia elements in $\pi_1(X)$ given by Nakamura [Na1], or by using specialization techniques, one finally recovers the projection $\pi_1(X) \to \pi_1(X_0)$, where X_0 is the completion of X.

After having recovered the exact sequence $(*)$ above, one reduces the case of hyperbolic affine curves over finitely generated fields of characteristic zero to the π_1^{t}-case of affine hyperbolic curves over finite fields. This is done by using the "standard reduction technique" mentioned above.

Tamagawa also shows that the absolute form of the Isom-conjecture and the relative one are roughly speaking equivalent (using the birational anabelian Conjecture described at the beginning of Part II).

We now turn our attention to the case of affine curves over finite fields, respectively the π_1^{t}-case of hyperbolic curves over finite fields. Tamagawa's approach is a tremendous refinement of Uchida's strategy to tackle the birational case, i.e., to prove the birational anabelian Conjecture for global function fields. (Naturally, since $\pi_1(X)$ seems to encode much less information than the absolute Galois group $G_{k(X)}$, the things might/should be much more intricate in the case of curves.) A rough approximation of Tamagawa's proof is as follows. Let $X \to k$

be an affine smooth geometrically connected curve, where k is a finite field with $\mathrm{char}(k) = p$. As usual let X_0 be the smooth completion of X. Thus we have surjective canonical projections

$$\pi_1(X) \to \pi_1(X_0) \to G_k \to 1$$

and correspondingly for the tame fundamental groups.

The first part of the proof consists in developing a *Local Theory,* which as in the birational case, will give a description of the closed points $x \in X_0$ in terms of the conjugacy classes of the decomposition groups $Z_x \subset \pi_1(X)$ above each closed point $x \in X$.

The steps for doing this are as follows:

Step 1) Recovering the several arithmetical invariants.

Here TAMAGAWA shows that the canonical projections $\pi_1(X) \to G_k$, thus $\pi_1(\overline{X})$, as well as $\pi_1(\overline{X}) \to \pi_1^{\mathrm{t}}(\overline{X})$ are encoded in $\pi_1(X)$. Further, by a combinatorial argument he recovers the Frobenius element $\varphi_{\mathrm{k}} \in G_k$. In particular, one gets the *cyclotomic character* of $\pi_1(X)$, and so one knows the ℓ-adic cohomology of $\pi_1(X)$, as well as the Galois action of G_k on the ℓ-adic Galois cohomology groups of $\pi_1(\overline{X})$.

The next essential remark is that after replacing X by some "sufficiently general" finite étale cover $Y \to X$, the completion $Y_0 \to k$ of Y is itself hyperbolic. In particular, the ℓ-adic Galois cohomology groups $\mathrm{H}^i(\pi_1(\overline{Y}_0), \mathbb{Z}_\ell(r))$ of $\pi_1(\overline{Y}_0)$ are the same as the ℓ-adic étale cohomology $\mathrm{H}^i(\overline{Y}_0, \mathbb{Z}_\ell(r))$ of $\overline{Y}_0$. Thus by the remarks above, one can recover the ℓ-adic cohomology of $\overline{Y}_0$ for every étale cover $Y \to X$ having a hyperbolic completion Y_0.

This is a fundamental observation in TAMAGAWA's approach, as it can be used in order to tackle the following problem:

Which sections of the canonical projection $pr_X : \pi_1(X) \to G_k$ are defined by points $x \in X_0(k)$ in the way as described in the Section Conjecture?

We remark that since $G_k \cong \widehat{\mathbb{Z}}$ is profinite free on one generator, one cannot expect that all such sections are defined by points as asked by the Section Conjecture. (Indeed, there are uncountable many such conjugacy classes of sections, thus too "many" in oder to be defined by points, even if X_0 has no k-rational points.)

Here is TAMAGAWA's answer: Let $s : G_k \to \pi_1(X)$ be a given section. For every open neighborhood $U \subset \pi_1(X)$ of $s(G_k)$, we denote by $X_U \to X$ the finite étale cover of X classified by U. First, since U projects onto G_k, the curve $X_U \to k$ is geometrically connected. Further, we have in tautological way: $\pi_1(X_U) = U$, and $\overline{U} := U \cap \pi_1(\overline{X}) = \pi_1(\overline{X}_U)$. Let $X_{U,0}$ be the smooth completion of X_U. Then by Step 1), the canonical projection $\pi_1(X_U) \to \pi_1(X_{U,0})$ can be recovered from $U = \pi_1(X_U)$, thus from $\pi_1(X)$ endowed with the section $s : G \to \pi_1(X)$.

Now we remark that for U sufficiently small, the complete curve $X_{U,0}$ is hyperbolic, both in the case X is affine, or if X was hyperbolic and we were working with $\pi_1^{\mathrm{t}}(X)$. We set $U_0 := \pi_1(X_{U,0})$ and view it as quotient of $\pi_1(X_U)$, and $\overline{U}_0 = \pi_1(\overline{X}_{U,0})$. Since $X_{U,0}$ is complete and hyperbolic, the ℓ-adic cohomology group $\mathrm{H}^i_{\mathrm{et}}(\overline{X}_{U_0}, \mathbb{Z}_\ell(1))$ equals the Galois cohomology group $\mathrm{H}^i(\pi_1(\overline{X}_{U_0}), \mathbb{Z}_\ell(1))$, thus the cohomology group $\mathrm{H}^i(\overline{U}_0, \mathbb{Z}_\ell(1))$ for all i.

Finally, since the Frobenius element $\varphi_k \in G_k$ is known, by applying the Lefschetz Trace Formula, we can recover the *number of k-rational points* of X_{U_0}:

$$|X_{U_0}(k)| = \textstyle\sum_{i=0}^2 (-1)^i \mathrm{Tr}(\varphi_k) |\mathrm{H}^i(\overline{U}_0, \mathbb{Z}_\ell(1))$$

In this way we obtain the technical input for the following:

Proposition. *Let $s : G_k \to \pi_1(X)$ be a section of $pr_X : \pi_1(X) \to G_k$. Then s is defined by a point of $X_0(k)$ if and only if for every open sufficiently small neighborhood U of $s(G_k)$ as above, one has $X_{U_0}(k) \neq \emptyset$.*

Step 2) Recover the decomposition groups Z_x over closed points.

This is done using the Proposition above. Actually, using the Artin's Reciprocity law, one shows that in the case of a *complete hyperbolic curve,* like the X_{U_0} above, the set $X_{U_0}(k)$ is in bijection with the conjugacy classes of sections s defining points. And this gives a recipe to recover the points $X_0(k)$ which come from points in $X_{U_0}(k)$ for some U as above; thus finally for recovering all the points in $X_0(k)$. By replacing k by finite extension $l|k$, one recovers in a functorial way $X(l)$ too, etc. Thus finally one recovers the closed points x of X_0 as being in bijection with the conjugacy classes of decomposition groups $Z_x \subset \pi_1(X)$. Correspondingly the same is done for π_1^{t}-case.

The second part of the proof is to develop a *Global Theory,* as done by Uchida in the birational case. Naturally, $\pi_1(X)$ endowed with all the decomposition groups over the closed points of X_0 carries much less information than $G_{k(X)}$ endowed with all the decomposition groups Z_v over the places v of $k(X)$.

Step 3) Recover the multiplicative group $k(X)^\times$ together with the valuations $v_x : k(X) \to \mathbb{Z}$.

Since the Frobenius elements $\varphi_x \in Z_x$ are known for closed points $x \in X_0$, by applying global class field theory as in the birational case, one gets the multiplicative group $k(X)^\times$ together with the valuations $v_x : k(X)^\times \to \mathbb{Z}$.

Step 4) Recovering the addition on $k(X)^\times \cup \{0\}$.

This is much more difficult than that in the birational case. And here is where the hypothesis that X is affine is used. Namely, if $x \in X_0 \backslash X$ is any point "at infinity", then from a decomposition group Z_x over x, one finally can recover the evaluation map

$$\mathfrak{p}_x : k(X) \to k^{\mathrm{a}} \cup \infty.$$

One proceeds by applying the following:

Proposition. *Let $X_0 \to \kappa$ be a complete smooth curve over an algebraically closed field κ. Suppose that the multiplicative group $k(X_0)^\times$ together with the valuations $v_x : k(X_0)^\times \to \mathbb{Z}$ at closed points $x \in X_0$, and the evaluation of the functions at at least three k-points x_0, x_1, x_∞ of X are known. Then from these data the structure field of $k(X_0)$ can be recovered.*

In order to conclude the proof of TAMAGAWA's theorem above over finite fields, we remark that all closed points of X_0 were recovered from $\pi_1(X)$ via the decomposition groups above them. In particular, such a point lies in X if and only if the inertia group above such a point is trivial. This gives the recipe to identify X inside X_0. Thus we finally have a group theoretic recipe for recovering the curve $X \to k$ from its fundamental group $\pi_1(X)$.

Finally, the functoriality of the recipe for recovering X shows that any isomorphism of fundamental groups $\Phi : \pi_1(X) \to \pi_1(Y)$ is defined by an isomorphism of function fields $\phi : k(X_0) \to k(Y_0)$ which induces an isomorphism of schemes $X \to Y$. The uniqueness of ϕ up to Frobenius twists follows the same pattern as in the birational case, but using the fact that the center of $\pi_1(X)$ is trivial in the cases under discussion.

II) *Mochizuki's results over sub-p-adic fields*

In this subsection we discuss briefly some of MOCHIZUKI's results concerning hyperbolic curves in characteristic zero and applications of these results to other anabelian questions.

The first such result was announced by MOCHIZUKI shortly after TAMAGAWA's theorem discussed above. The result deals with hyperbolic curves over finitely generated fields of characteristic zero, and more or less extends the corresponding result by TAMAGAWA to *complete hyperbolic curves.* The proof relies heavily on TAMAGAWA's theorem, but MOCHIZUKI's strategy for the proof goes beyond TAMAGAWA's approach.

Theorem (See MOCHIZUKI [Mzk1]).
The hyperbolic curves over finitely generated fields of characteristic zero are anabelian. Further, both the relative and the absolute Isom-form of the anabelian Conjecture for hyperbolic curves over such fields hold.

We indicate briefly the idea of the proof. Let $X \to k$ be a hyperbolic curve over a finitely generated field k of characteristic zero. Proceeding as TAMAGAWA did in the case of affine hyperbolic curves, we can recover the exact sequence

$$1 \to \pi_1(\overline{X}) \to \pi_1(X) \to G_k \to 1,$$

and also the projection $\pi_1(X) \to \pi_1(X_0)$, where X_0 is the completion of X. In this way one reduces the question to the case of *complete hyperbolic curves.*

Thus let $X \to k$ be a complete hyperbolic curve over some finitely generated field k of characteristic zero. The idea of MOCHIZUKI is to reduce the problem in this case to the π_1^{t}-case of affine hyperbolic curves over finite fields and then use TAMAGAWA's theorem for affine hyperbolic curves over finite fields. In order to do that, MOCHIZUKI uses log-schemes and log-fundamental groups. In essence one does the following: In the context above, recall the setting explained in the "standard reduction technique". In the notations from there, let v_s be a discrete valuation of k with valuation ring R_s dominating some closed point $s \in S$ such that the residue field equals $\kappa(s)$, thus finite. Let $p = \mathrm{char}(\kappa(s))$, thus $\kappa(s)$ is finite over $\mathbb{F}_p$. In the case $X \to k$ has good reduction at s —and this is the case on a Zariski open subset of S, in particular if p is big enough, the special fiber $X_s \to \kappa(s)$ of $X_{R_s} \to R_s$ at s is a complete hyperbolic curve. Thus one <u>cannot</u> apply and use TAMAGAWA's theorem in order to recover $X_s \to \kappa(s)$ from $\pi_1^{\mathrm{t}}(X_s)$ (even if the projection $\pi_1^{\mathrm{t}}(X_s) \to G_{\kappa_v}$ is known). Nevertheless, the fact that X has good reduction at v is encoded in the canonical exact sequence $\pi_1(X) \to G_k$ endowed with a decomposition group $Z_{v_s} \subset G_k$ above v_s by Oda's Criterion for good reduction of hyperbolic complete curves.

In the above notations, suppose that $X_{R_s} \to R_s$ is smooth. Let us consider a finite Galois étale cover $Y^{(p)} \to X$ such that its geometric part $\overline{Y}^{(p)} \to \overline{X}$ is the maximal p-elementary Abelian cover of $\overline{X}$. After enlarging k, we eventually can suppose that $Y^{(p)} \to k$ is geometrically connected, and that $\mathrm{Aut}_X(Y^{(p)})$ is defined over k. Under this hypothesis one has:

$$\deg(Y^{(p)} \to X) = p^{2g_X} \quad (g_X \text{ is the genus of } X).$$

On the other hand, considering the maximal geometric p-elementary Abelian cover $Z^{(p)} \to X_s$, and recalling that $r_{X_s} \le g_{X_s} = g_X$ denotes the Hasse–Witt invariant of X_s, we see that

$$\deg(Z^{(p)} \to X_s) = p^{r_{X_v}} \le p^{g_X}.$$

We conclude that $Y^{(p)} \to k$ does not have potentially good reduction. Moreover, for p getting larger, the special fiber $Y_s^{(p)} \to \kappa_s$ of the stable model of $Y^{(p)} \to k$ (which is defined over some finite extension $l|k$ and corresponding extensions R_w of R_{v_s}, etc.) has "many" double points. We set $Y_s^{(p)} = \cup_i Y_i$, where Y_i are the irreducible components of Y_s. For each Y_i, let U_i be the smooth part of Y_s inside Y_i. Since Y_s is connected, it follows that each $U_i \to \kappa_w$ is an affine hyperbolic curve over the finite field κ_w.

Now it is part of the theory of log-fundamental groups, that the tame fundamental group $\pi_1^{\mathrm{t}}(U_i)$ can be recovered from $\pi_1(Y^{(p)}) \to G_l$. Thus applying TAMAGAWA's theorem for the π_1^{t}-case of affine hyperbolic curves over finite fields, we can recover $U_i \to \kappa_w$ in a functorial way from $\pi_1^{\mathrm{t}}(U_i)$. And finally, one can recover $Y_s^{(p)} \to \kappa_w$, and $X_s \to \kappa_v$ as well.

One concludes by using the standard reduction/globalization techniques.

The result above by Mochizuki is the precursor of his much stronger result concerning hyperbolic curves over sub-p-adic fields as explained below. First let us introduce Mochizuki's notations. A *sub-p-adic field k* is any field which can be embedded into some function field over $\mathbb{Q}_p$. Let k be a sub-p-adic field, and let $X \to k$ be a geometrically connected scheme over k. Consider the exact sequence of fundamental groups

$$1 \to \pi_1(\overline{X}) \to \pi_1(X) \to G_k \to 1.$$

We denote $\Delta_X := \pi^p(\overline{X})$ the maximal pro-p quotient of $\pi_1(\overline{X})$, and remark that the kernel N of the map $\pi_1(\overline{X}) \to \Delta(X)$ is a characteristic subgroup of $\pi_1(\overline{X})$, i.e., it is invariant under all automorphisms of $\pi_1(\overline{X})$. In particular, N is invariant under the conjugation in $\pi_1(X)$. Thus N is a normal subgroup in $\pi_1(X)$ too. We will set $\Pi_X = \pi_1(X)/N$. Therefore, the above exact sequence gives rise to a canonical exact sequence of fundamental groups:

$$1 \to \Delta_X = \pi_1^p(\overline{X}) \to \Pi_X = \pi_1(X)/N \to G_k \to 1.$$

With these notations, the main result by Mochizuki can be stated as follows:

Theorem (See Mochizuki [Mzk3]).

Let $Y \to k$ be a geometrically integral hyperbolic curve over a sub-p-adic field. Then Y can be recovered from the canonical projection $\Pi_Y \to G_k$.

Moreover, this recipe is functorial in such a way that it implies the following Hom-form *of the relative anabelian Conjecture for curves: Let $X \to k$ be a geometrically integral smooth variety. Then every open G_k-homomorphism $\Phi : \Pi_X \to \Pi_Y$ is defined in a functorial way by a unique dominant k-morphism $\phi : X \to Y$.*

The main tools used by Mochizuki are the p-adic Tate–Hodge Theory and Faltings' Theory of almost étale morphisms. The proof is very technical and difficult to follow for non-experts (maybe even for experts!). I will nevertheless try to summarize here the main points in the proof (which are though more intricate and complex, than I might suggest here...). I should also mention that in this case we do not have a recipe to recover $X \to k$ from $\pi_X \to G_k$, which is as explicit as in the previous cases. The main difficulty in this respect lies in not having an as *explicit local theory* as in the previous case. In particular and unfortunately, until now we do not have a way of describing $X(k)$, i.e., we do not have any kind of an answer to the Section Conjecture so far.

Coming back to the proof of the Theorem above, the first observation is that via more or less standard specialization techniques, the problem is reduced to the following case: $k|\mathbb{Q}_p$ is a finite extension, and $X \to k$ is a smooth hyperbolic curve, and $Y \to k$ is a complete hyperbolic curve. And finally replacing X by the étale cover classified by the image of Φ, one can suppose that Φ is surjective.

One should remark that a further reduction step to the case where X is complete is not at all trivial, and it is one of the facts which complicates things a lot. Naturally, by using the canonical projection $\Pi_{k(X)} \to \Pi_X$, one might reformulate the problem above correspondingly, and ask whether every surjective G_k-morphism $\Phi : \Pi_{k(X)} \to \Pi_Y$ is defined by a unique morphism of function fields $k(Y) \hookrightarrow k(X)$ over k. We will nevertheless take this last reduction step for granted, and only outline the proof of the following:

Let $X \to k$ and $Y \to k$ be complete hyperbolic curves, where $k|\mathbb{Q}_p$ is finite. Then every surjective G_k-homomorphism $\Phi : \Pi_X \to \Pi_Y$ is defined by a unique k-morphism $\phi : X \to Y$ in a functorial way.

The *Local Theory* in this case is as follows: As mentioned above, one has no clue at all how to recover $X(k)$ from the given data $\Pi_X \to G_k$, and the situation is not better if we replace Π_X by the full étale fundamental group $\pi_1(X)$. Nevertheless, Mochizuki develops another kind of a "Local Theory", which fortunately does the right job for the problem. The kind of points one can recover are as follows: Let R be the valuation ring of k, and let $X_R \to R$ be a semi-stable model of $X \to k$ (such models exist after enlarging the base field k). Let $(X_i)_i$ be the irreducible components of the special fiber $X_s \to \kappa$ of $X_R \to R$. If $\eta_i \in X_R$ is the generic point of X_i, then the local ring $\mathcal{O}_{X_R,\eta_i}$ is a discrete valuation ring of $k(X)$ dominating R, and the residue field $\kappa(\eta_i)$ is the function field of $X_i \to \kappa$. Let us call such points η_i *arithmetical points* of X (arising for the several models X_R).

Next let (L, v) be a discrete complete valued field over k such that the valuation of L prolongs the p-adic valuation of k, and the residue field $Lv \,|\, kv$ is a function field in one variable. Remark that the completion of $k(X)$ with respect to the valuation $v := v_{\eta_i}$ defined by an arithmetical point is actually such a discrete complete valued field over k. We denote for short $\mathcal{H}_L^\Omega = \mathrm{H}^1(G_{Lk^\mathrm{a}}, \widehat{\mathcal{O}}_{L^\mathrm{a}}(1))/(\text{torsion})$, where $\widehat{\mathcal{O}}_{L^\mathrm{a}}$ is the completion of the valuation ring of L^a (similar to the completion $\mathbb{C}_p$ of the algebraic closure of $\mathbb{Q}_p^\mathrm{a}$).

We will say that a G_k-homomorphism $\Phi_X : G_L \to \Pi_X$ is *non-degenerate,* if the induced map on the p-adic cohomology

$$\mathrm{H}^1(\Delta_X, \mathbb{Z}_p(1)) \xrightarrow{\mathrm{infl}_{\Phi_X}} \mathrm{H}^1(G_L, \widehat{\mathcal{O}}_{L^\mathrm{a}}(1)) \xrightarrow{\mathrm{can}} \mathcal{H}_L^\Omega$$

is non-trivial. Now the main technical points of the proof are as follows:

1) In the context above, let $\Phi : \Pi_X \to \Pi_Y$ be an open G_k-homomorphism. Then there exists a non-degenerate G_k-homomorphism $\Phi_X : G_L \to \Pi_X$ such that the composition $\Phi_Y := \Phi \circ \Phi_X$ is a non-degenerate homomorphism. Thinking of the Local Theory from the birational case, this assertion here corresponds more or less to the characterization of arithmetical prime divisors.

2) Every non-degenerate G_k-morphism $\Phi_X : G_L \to \Pi_X$ as above is of geometrical nature: Given such a Φ_X, there exists a unique *L-rational point $\psi_{\Phi_X} : L \to X$ defining Φ_X in a functorial way.*

3) In particular, for Φ and Φ_X as at 1) above, there exist L-rational points $\psi_{\Phi_X} : L \to X$ and $\psi_{\Phi_Y} : L \to Y$ defining the non-degenerate morphisms Φ_X and $\Phi_Y = \Phi \circ \Phi_X$ in a functorial way.

Finally, MOCHIZUKI's *Global Theory* is a very nice application of the p-adic Hodge–Tate Theory and of Faltings' Theory of almost étale morphisms. The idea is as follows:

First, by the p-adic Hodge–Tate Theory, the sheaf of global differentials on X, say $D_X := \mathrm{H}^0(X, \Omega_X)$, can be recovered from the action of G_k on the $\mathbb{C}_p$-twists with the p-adic cohomology $\mathrm{H}^i_{\mathrm{et}}(\overline{X}, \mathbb{Z}_p(j))$ of $\overline{X}$. On the other hand, since X is a complete hyperbolic curve over a field of characteristic zero, thus $\neq p$, the p-adic cohomology $\mathrm{H}^i_{\mathrm{et}}(\overline{X}, \mathbb{Z}_p(j))$ is the same as the Galois cohomology of Δ_X, thus known.

Let us denote $D^i_X = \mathrm{H}^0(X, \Omega_X^{\otimes i})$, and let $\mathcal{R}^i_X := \ker(D_X^{\otimes i} \to D^i_X)$ be the space of i^{th} homogeneous relations in D^i_X. If X is not a hyperelliptic curve (what we can suppose after replacing X by a properly chosen étale cover whose geometric part is p-elementary Abelian), then the system of all the D^i_X completely defines X. Equivalently, the system of all the data $\mathcal{R}^i_X \subset D_X^{\otimes i}$ completely defines X.

Second, let $\Phi : \Pi_X \to \Pi_Y$ be a surjective G_k-homomorphism. Then Φ induces in a functorial way a morphism of k vector spaces $\imath_\Phi : D_Y \to D_X$; thus also morphisms of k vector space $\imath_\Phi^{\otimes i} : D_Y^{\otimes i} \to D_X^{\otimes i}$ for each $i \geq 1$. And by the general non-sense concerning the canonical embedding, if each $\imath_\Phi^{\otimes i}$ "respects the relations", i.e., it maps $\mathcal{R}^i_Y$ into $\mathcal{R}^i_X$, then $\imath_\Phi$ is defined by some dominant k-morphism $\phi : X \to Y$ in the canonical way.

Finally, in order to check that $\imath_\Phi$ does indeed respect the relations, one uses the Local Theory and Faltings' Theory of almost étale morphisms: Choose a non-degenerate morphism $\Phi_X : G_L \to \Pi_X$ as at 3) above. Let Ω_L denote the *p-adically continuous* k-differentials of L, and Ω^i_L be its powers. Since ψ_{Φ_X} is a non-degenerate point of X, the differential $d_X := d(\jmath_{\Phi_X}) : D_X \to \Omega_L$ of $\jmath_{\Phi_X} : L \to X$ and its powers $d^i_X : D^i_X \hookrightarrow \Omega^i_L$ are embeddings. Thus in order to check that $\imath_\Phi$ respects the relations, it is sufficient to check that this is the case for the composition

$$\imath_L^{\otimes i} : D_Y^{\otimes i} \to D_X^{\otimes i} \hookrightarrow \Omega_L^{\otimes i}\,.$$

On the other hand, the composition of the map $\imath_L^{\otimes i}$ with $\Omega_L^{\otimes i} \to \Omega^i_L$ is exactly the canonical map $D_Y^{\otimes i} \to D^i_Y \hookrightarrow \Omega^i_L$ defined via the non-degenerate morphism $\Phi_Y = \Phi \circ \Phi_X$ and the resulting point $\psi_{\Phi_Y} : L \to \Pi_Y$. This concludes the proof.

Remarks.

1) First, Theorem A$'$ of Mochizuki [Mzk3] shows that also *truncated-*Π_X *versions* of the assertion of the main result above are valid. One can namely replace Δ_X by its central series quotient $\Delta_X^{(n)}$, and consequently Π_X by the corresponding quotient $\Pi_X^{(n)}$ which fits into the exact sequence

$$1 \to \Delta_X^{(n)} \to \Pi_X^{(n)} \to G_k \to 1.$$

Let $n \geq 3$. Then given an open G_k-homomorphism $\Phi^{(n+2)} : \Pi_Y^{(n+2)} \to \Pi_X^{(n+2)}$ there exists a unique dominant k-homomorphism $Y \to X$ such that the canonical open morphism $\Pi_Y^{(n)} \to \Pi_X^{(n)}$ induced by ϕ coincides with $\Phi^{(n)}$ on $\Pi_Y^{(n)}$.

2) Using specializations techniques, Mochizuki proves the relative Hom-form of the birational anabelian Conjecture for finitely generated fields over sub-p-adic fields k as follows:

Theorem (Mochizuki [Mzk3], Theorem B).
Let $K|k$, $L|k$ be regular function fields. Then every open G_k-homomorphism $\Phi : G_K \to G_L$ is defined functorially by a unique k-embedding of fields $L \to K$.

I would like to remark that using techniques developed in order to prove pro-ℓ birational type results, see Part III of these notes, one can sharpen the above result and show the following:

Theorem (Corry–Pop [C–P]). *Every open Π_k-homomorphism $\Phi : \Pi_K \to \Pi_L$ is defined by a unique k-embedding $L \to K$ in a functorial way.*

3) Mochizuki also shows that *hyperbolically fibered surfaces* are anabelian. And moreover, the Isom-form of an anabelian Conjecture for fibered surfaces is true. Here, a hyperbolically fibered surface X is the complement of an étale divisor in a smooth proper family $\tilde{X} \to X_1$ of hyperbolic complete curves over a hyperbolic base curve X_1. The result is:

Theorem (Mochizuki [Mzk3], Theorem D).
Let $Y \to k$ and $X \to k$ be geometrically integral hyperbolically fibered surfaces over a sub-p-adic field k. Then every G_k-isomorphism $\Phi : \pi_1(Y) \to \pi_1(X)$ is defined by a unique k-isomorphism $\phi : Y \to X$ in a functorial way.

Note that in the Theorem above the *full fundamental group* π_1 is needed. It is maybe useful to remark that a naive Hom-form of the above Theorem is not true. Indeed, let k be an infinite base field. Then using general hyperplane arguments, one can show that for every smooth quasi-projective k-variety $X \subseteq \mathbb{P}^N$, there exist smooth k-curves $Y \subseteq X$ obtained from $X \to k$ by intersections with general hyperplanes such that the canonical map $\pi_1^{\mathrm{t}}(Y) \to \pi_1^{\mathrm{t}}(X)$ is surjective. In a second step, one can realize $\pi_1^{\mathrm{t}}(Y)$ in many ways as quotients of fundamental groups $\pi_1^{\mathrm{t}}(Z) \to \pi_1^{\mathrm{t}}(Y)$ for several smooth k-varieties (which can be chosen to be projective, if X is complete), e.g., $Z = Y \times \ldots \times Y$ finitely many times. Finally, the composition

$$\pi_1^{\mathrm{t}}(Z) \to \pi_1^{\mathrm{t}}(Y) \to \pi_1^{\mathrm{t}}(X)$$

is a surjective G_k-morphism, but by its construction, it does not originate from a dominant k-rational map.

III) *Jakob Stix' results concerning hyperbolic curves in positive characteristic*

The results of Stix deal with *hyperbolic non-constant curves* over finitely generated infinite fields k of positive characteristic (but apply as well to such fields of characteristic zero, where the results are already known). Recall that given a curve $X \to k$ with $\mathrm{char}(k) = p > 0$, one says that X is *potentially isotrivial,* if there exists a finite étale cover $X' \to X$ such that X' is defined over a finite field. One can show that if X is hyperbolic, then X is potentially isotrivial if and only if there exists a finite field extension $k'|k$ such that the base change $X' = X \times_k k'$ is defined over a finite field. Further, recall that for a curve $X \to k$ as above, we denote by $X^{\mathrm{i}} \to k^{\mathrm{i}}$ the maximal purely inseparable cover of $X \to k$. And for every integer n we denote by $X^{\mathrm{i}}(n) \to k^{\mathrm{i}}$ the relative Frobenius n-twist.

Theorem (Stix [St1], [St2]).
Let $X \to k$ be a non potentially isotrivial hyperbolic curve over a finitely generated infinite field k with $\mathrm{char}(k) = p > 0$*. Then one can recover $X^{\mathrm{i}} \to k^{\mathrm{i}}$ from $\pi_1^{\mathrm{t}}(X) \to G_k$ in a functorial way.*

Moreover, the relative Isom-form *of the anabelian Conjecture for hyperbolic curves over k is true in the following sense: Let $Y \to k$ be some hyperbolic curve, and let a G_k-isomorphism $\Phi : \pi_1^{\mathrm{t}}(X) \to \pi_1^{\mathrm{t}}(Y)$ be given. Then there exists a unique n and a k^{i}-isomorphism $\phi : X^{\mathrm{i}}(n) \to Y^{\mathrm{i}}$ defining Φ.*

The strategy of proof is as follows:

Let k be a finitely generated infinite field, and $X \to k$ a hyperbolic curve over k. In the notations from the "standard reduction technique", let $X_S \to S$ be a smooth surjective family of hyperbolic curves whose generic fiber is $X \to k$. The idea is as follows:

<u>Case 1.</u> $X \to k$ is an affine hyperbolic curve.

By shrinking S if necessary, we can suppose that $X \to k$ has good reduction at all closed points $s \in S$. From $\pi_1^{\mathrm{t}}(X) \to G_k$ one recovers the local projections $\Phi_s : \pi_1^{\mathrm{t}}(X_s) \to G_{\kappa(s)}$ for all closed points $s \in S$. By Tamagawa's theorem, we can recover the isomorphy type of $X_s^{\mathrm{i}} \to \kappa(s)$ *up to Frobenius twists.* In particular, let $\Phi : \pi_1^{\mathrm{t}}(X) \to \pi_1^{\mathrm{t}}(Y)$ be a G_k-isomorphism, where $Y \to k$ is some hyperbolic curve over k. By the "standard specialization technique", we obtain

$\kappa(s)$-isomorphisms of some relative Frobenius twists of the special fibers, say $\phi_s : X^{\rm i}_s(n_s) \to Y^{\rm i}_s$ defining Φ_s. Unfortunately, the usual globalization techniques work only under the hypothesis the Frobenius twists n_s are constant, say equal to n, on a non-empty open of S (and then they turn to be constant on the whole S). If this is the case, then the local isomorphisms ϕ_s originate indeed from a unique global $k^{\rm i}$-isomorphism $\phi : X^{\rm i}(n) \to Y^{\rm i}$, which defines the given G_k-isomorphism $\Phi : \pi_1^{\rm t}(X) \to \pi_1^{\rm t}(X)$.

Here is the way Stix shows that the exponents n_s are indeed constant: First, by replacing X by a properly chosen tame étale cover, we can suppose that the smooth completion X_0 of X is hyperbolic too. Next we fix some $m > 2$ relatively prime to $p = \mathrm{char}(k)$, and replace k by its finite extension over which the m-torsion of Jac_{X_0} becomes rational. And choose an m-level structure on X_0 by fixing an isomorphism

$$\varphi_{X,m} : {}_m\mathrm{Jac}_{X_0} = \pi_1^{\rm ab}(\overline{X}_0)/m \to (\mathbb{Z}/m)^{2g}$$

Then X_0 endowed with $\varphi_{X,m}$ is classified by a k-rational point $\psi_X : k \to \mathcal{M}_g[m]$. Moreover, using the "standard reduction technique", the level structure $\varphi_{X,m}$ gives rise via the specialization homomorphisms $\mathrm{sp}_s : \pi_1(\overline{X}_0) \to \pi_1(\overline{X}_{0,s})$ canonically to level structures

$$\varphi_{X_s,m} : {}_m\mathrm{Jac}_{X_{0,s}} = \pi_1^{\rm ab}(\overline{X}_{0,s})/m \to (\mathbb{Z}/m)^{2g}.$$

And this happens in such a way that $\psi_X : k \to \mathcal{M}_g[m]$ defined above becomes the generic fiber of a morphism $\psi_{X_S} : S \to \mathcal{M}_g[m]$ whose special fibers classify the curves $X_s \to \kappa(s)$ endowed with the level structures $\varphi_{X_s,m}$.

Now let us come back to the G_k-isomorphism $\Phi : \pi_1^{\rm t}(X) \to \pi_1^{\rm t}(Y)$. Clearly, Φ transports the m-level structure $\varphi_{X,m}$ of X_0 to an m-level structure $\varphi_{Y,m}$ for Y_0. And the local $G_{\kappa(s)}$-isomorphisms $\Phi_s : \pi_1^{\rm t}(X_s) \to \pi_1^{\rm t}(Y_s)$ transport the m-level structures $\varphi_{X_s,m}$ to m-level structures $\varphi_{Y_s,m}$ which are compatible with the specialization morphisms $\mathrm{sp}_s : \pi_1(\overline{Y}_0) \to \pi_1(\overline{Y}_{0,s})$.

Now let us suppose that there exist some exponents n_s and $\kappa(s)$-isomorphisms $\phi_s : X^{\rm i}_s(n_s) \to Y^{\rm i}_s$ which define the G_κ-isomorphisms $\Phi_s : \pi_1^{\rm t}(X_s) \to \pi_1^{\rm t}(Y_s)$. Then ϕ_s prolongs to an $\kappa(s)$-isomorphism $\phi_{0,s} : X_{0,s}(n_s) \to Y_{0,s}$. Next remark that $X_{0,s}$ and its relative Frobenius twists endowed with the same m-level structure $\varphi_{X_s,m} = \varphi_{X_s(n_s),m}$ factor through the same closed point of $\mathcal{M}_g[m]$. Thus we have: The classifying morphisms $\psi_{X_S} : S \to \mathcal{M}_g[n]$ for $X_{0,S}$ and $\psi_{Y_S} : S \to \mathcal{M}_g[n]$ for $Y_{0,S}$ defined above coincide (topologically) on the closed points $s \in S$.

In order to conclude, Stix proves the following:

Proposition (Stix [St1]).

Let S and $\mathcal{M}$ be irreducible $\mathbb{Z}$-varieties. Let $f, g : S \to \mathcal{M}$ be two morphisms which coincide topologically on the closed points of S. Suppose that $f \neq g$. Then

S is defined over $\mathbb{F}_p$ for some p, and f and g differ by a power of Frobenius, which is unique if f is not constant.

Thus applying the Proposition above we conclude that the classifying morphisms ϕ_X and ϕ_Y differ by a power Frob^n of Frobenius. In particular, fiber wise the same is the case. From this one finally deduces that $\Phi : \pi_1^{\mathrm{t}}(X) \to \pi_1^{\mathrm{t}}(Y)$ is defined by some k-isomorphisms $\phi : X^{\mathrm{i}}(n) \to Y^{\mathrm{i}}$ for some integer n.

This completes the proof of the case when X is an affine hyperbolic curve.

Case 2) $X \to k$ is a complete hyperbolic curve.

Let us try to mimic MOCHIZUKI's strategy from the case of complete hyperbolic curves over finitely generated fields of characteristic zero. Then we run immediately into the following difficulty: If k has positive characteristic $p > 0$, there is no obvious reason that some finite properly chosen étale (Galois) covers $X' \to X$ have bad reduction at points $s \in S$ where X has good reduction. Note that the "trick" used by MOCHIZUKI in [Mzk2] in the case $\mathrm{char}(k) = 0$ does definitely not work in positive characteristic. (This follows from Grothendieck's Specialization Theorem: Let $X_{R_s} \to R_s$ be a complete smooth curve, and let $X' \to X$ be an étale Galois cover whose geometric part has degree prime to p. Then X' has potentially good reduction.)

In order to avoid this difficulty, one can nevertheless use the Raynaud, Pop–Saidi, Tamagawa Theorem, see Part III) of these notes. A consequence of that result is the following: Let a closed point $s \in S$ be given. Then there exists a finite étale cover $X^{(s)} \to X$ whose geometric part is a cyclic étale cover of $\overline{X}$ of degree prime to p having the property: Any finite étale cover $X' \to X^{(s)}$ whose geometric part factors through the maximal p-elementary étale cover of $\overline{X}^{(s)}$ *does not have potentially good reduction.* With this input, STIX uses the theory of log-étale fundamental groups in order to conclude the proof in the same style as MOCHIZUKI [Mzk1], but using the methods developed to treat the case of affine hyperbolic curves.

IV) *Mochizuki's cuspidalization results and Applications*

As we have seen so far, the information about "points" and how this information is encoded in Galois theory plays a crucial role in the strategies to tackle anabelian type assertions, both in the birational and in the curve case. The difficulty of proving the anabelian conjecture for complete curves lies exactly in the impossibility of having obvious candidates for "points" which are encoded in the $\pi_1(X)$, because the inertia groups at closed points $x \in X$ are all trivial in $\pi_1(X)$. MOCHIZUKI [Mzk5] had the idea how to detect tame inertia type information at closed points $x \in X$ in a functorial way together with the Galois action of the Frobenius on them. He calls this procedure `cuspidaliazation`. Here is a short introduction to this topic: Let X be a proper hyperbolic curve over a field k which is either finite or a finite extension of $\mathbb{Q}_p$ for some $p > 0$. Further let $U \subseteq X$ be an open sub-scheme of X. Then one has a canonical surjective projection $\pi_1(U) \to \pi_1(X)$ which is compatible with the projections $\pi_1(U) \to G_k$ and $\pi_1(X)$, hence defines a surjective map of the geometric fundamental groups $\pi_1(\overline{U}) \to \pi_1(\overline{X})$.

1) *Abelian cuspidalization*: Let $\pi_1^{\mathrm{c.ab}}(\overline{U})$ be the maximal quotient of $\pi_1(\overline{U})$ which is a central prime to p extension of $\pi_1(\overline{X})$, i.e., $\pi_1^{\mathrm{c.ab}}(\overline{U})$ is the maximal quotient of $\pi_1(\overline{U})$ such that the kernel of $\pi_1(\overline{U}) \to \pi_1^{\mathrm{c.ab}}(\overline{U})$ is contained in the kernel of $\pi_1(\overline{U}) \to \pi_1(\overline{X})$ and the kernel of $\pi_1^{\mathrm{c.ab}}(\overline{U}) \to \pi_1(\overline{X})$ is central in $\pi_1^{\mathrm{c.ab}}(\overline{U})$ and has order prime to the characteristic. We notice that if $x \in X\backslash U$ is a "cuspidal point" of U, i.e., a closed point of X which does not lie in U, and $T_x^{\mathrm{t}} \subset \pi_1(\overline{U})$ is the tame part of an inertia group above x, then T_x^{t} is mapped isomorphically onto its image $T_x^{\mathrm{c.ab}} \subset \pi_1^{\mathrm{c.ab}}(\overline{U})$. More precisely, one has for all cuspidal points x of U the following: $T_x^{\mathrm{t}} \cong \widehat{\mathbb{Z}}' \cong T_x^{\mathrm{c.ab}}$ and the groups $T_x^{\mathrm{c.ab}}$ generate the kernel of $\pi_1^{\mathrm{c.ab}}(\overline{U}) \to \pi_1(\overline{X})$ with a unique relation. Finally, we notice that the kernel of $\pi_1(\overline{U}) \to \pi_1^{\mathrm{c.ab}}(\overline{U})$ is a normal subgroup in $\pi_1(U)$, thus $\pi_1^{\mathrm{c.ab}}(\overline{U})$ fits into an exact sequence of the form $1 \to \pi_1^{\mathrm{c.ab}}(\overline{U}) \to \pi_1^{\mathrm{c.ab}}(U) \to G_k \to 1$, and $\pi_1(U) \to \pi_1(X)$ gives rise to a canonical surjective projection $\pi_1^{\mathrm{c.ab}}(U) \to \pi_1(X)$.

2) *Pro-ℓ cuspidalization*: In the above context, let $\ell \neq \mathrm{char}$ be a fixed prime number. Then one can consider as above the $\pi_1^{\mathrm{c.}\ell}(\overline{U})$ be the maximal quotient of $\pi_1(\overline{U})$ which is a pro-ℓ extension of $\pi_1(\overline{X})$, i.e., $\pi_1^{\mathrm{c.}\ell}(\overline{U})$ is the maximal quotient of $\pi_1(\overline{U})$ such that the kernel of $\pi_1(\overline{U}) \to \pi_1^{\mathrm{c.}\ell}(\overline{U})$ is contained in the kernel of $\pi_1(\overline{U}) \to \pi_1(\overline{X})$, and the kernel of $\pi_1^{\mathrm{c.}\ell}(\overline{U}) \to \pi_1(\overline{X})$ is a pro-ℓ group. As above, we notice that if $x \in X\backslash U$ is a "cuspidal point" of U, and $T_x^{\ell} \subset \pi_1(\overline{U})$ is a Sylow ℓ-group of a tame inertia group above x, then T_x^{ℓ} is mapped isomorphically onto its image $T_x^{\mathrm{c.}\ell} \subset \pi_1^{\mathrm{c.}\ell}(\overline{U})$. More precisely, one has for all cuspidal points x of U the following: $T_x^{t} \cong \mathbb{Z}_\ell \cong T_x^{\mathrm{c.}\ell}$, and there are properly chosen groups T_x^{ℓ} such that their images $T_x^{\mathrm{c.}\ell}$ generate the kernel of $\pi_1^{\mathrm{c.}\ell}(\overline{U}) \to \pi_1(\overline{X})$ with a unique relation. As above, the kernel of $\pi_1(\overline{U}) \to \pi_1^{\mathrm{c.}\ell}(\overline{U})$ is a normal subgroup in $\pi_1(U)$, thus $\pi_1^{\mathrm{c.}\ell}(\overline{U})$ fits into an exact sequence of the form $1 \to \pi_1^{\mathrm{c.}\ell}(\overline{U}) \to \pi_1^{\mathrm{c.}\ell}(U) \to G_k \to 1$, and $\pi_1(U) \to \pi_1(X)$ gives rise to a canonical surjective projection $\pi_1^{\mathrm{c.ab}}(U) \to \pi_1(X)$.

Theorem (See Mochizuki [Mzk5]).

1) *Let $U \subset X$ be a Zariski open sub-scheme as above. Then there are group theoretical recipes to recover the canonical projection $\pi_1^{\mathrm{c.ab}}(U) \to \pi_1(X)$ from $\pi_1(X)$, and the same holds for the projection $\pi_1^{\mathrm{c.}\ell}(U_x) \to \pi_1(X)$ for U_x of the form $U_x := X\backslash\{x\}$ with $x \in X$ a closed point of X.*

2) *Moreover, the group theoretical recipes above are invariant under isomorphisms. Precisely, let $Y \to l$ be a hyperbolic curve with l either finite or a finite extension of some $\mathbb{Q}_q$ for some prime number q such that one has an isomorphism $\Phi : \pi_1(X) \to \pi_1(Y)$. Then the following hold:*

a) Φ *is compatible with the projections* $\pi_1(X) \to G_k$, $\pi_1(Y) \to G_l$ *thus defines an isomorphism* $G_k \to G_l$, *and* k *and* l *have the same* (*residual*) *characteristic.*

b) *For every* $U \subseteq X$ *there exists* $V \subseteq Y$ *such that* $\pi_1(X) \to \pi_1(Y)$ *can be lifted to an isomorphism* $\pi_1^{\mathrm{c.ab}}(U) \to \pi_1^{\mathrm{c.ab}}(V)$ *which maps inertia isomorphically onto inertia. A similar more technical assertion holds for* $\pi_1^{\mathrm{c.}\ell}(U_x)$.

As applications, MOCHIZUKI [Mzk5] showed that the Isom-form of the anabelian conjecture for hyperbolic *complete* curves over finite fields holds, thus extending the results by STIX mentioned previously, and thus reproving the Isom-form of the anabelian conjecture for complete curves in general via STIX' specialization result.

Nevertheless, meanwhile there is a much stronger result concerning the Isom-form of the anabelian conjecture for hyperbolic (complete) curves by SAIDI–TAMAGAWA [S–T], which is based on MOCHIZUKI'S cuspidalization, and sounds as follows: Let Σ be any non-empty set of rational prime numbers. Then for a geometrically integral curve $X \to k$ over some base field k, we denote by $\pi_1^{\Sigma}(\overline{X})$ the pro-Σ completion of the geometric fundamental group of X. Note that the kernel of $\pi_1(\overline{X}) \to \pi_1^{\Sigma}(\overline{X})$ is characteristic in $\pi_1(\overline{X})$, thus this kernel is normal in $\pi_1(X)$. In particular, $\pi_1^{\Sigma}(\overline{X})$ fits canonically into an exact sequence

$$1 \to \pi_1^{\Sigma}(\overline{X}) \to \pi_1^{(\Sigma)}(X) \to G_k \to 1,$$

and we will say that $\pi_1^{(\Sigma)}(X)$ is the `geometrically pro-`Σ fundamental group of $X \to k$. Note that the geometrically pro-Σ fundamental group $\pi_1^{(\Sigma)}$ is one of the several possible `geometrically pro-`$\mathcal{C}$ completions $\pi_1^{(\mathcal{C})}$ of $\pi_1(X)$.

Theorem (See SAIDI–TAMAGAWA [S–T]). *Let* Σ, Ξ *be sets of rational prime numbers, with* Σ *consisting of all but maybe finitely many prime numbers. Let* X *and* Y *be hyperbolic curves over finite fields, and* $\pi_1^{(\Sigma)}(X)$ *and* $\pi_1^{(\Xi)}(Y)$ *be their geometrically pro-*Σ*, respectively pro-*Ξ*, fundamental groups. Then every isomorphism* $\Phi : \pi_1^{(\Sigma)}(X) \to \pi_1^{(\Xi)}(Y)$ *originates from geometry as predicted by the anabelian conjectures for curves, and moreover, if such an isomorphism* Φ *exists, then* $\Sigma = \Xi$.

A further very exciting development resulting from MOCHIZUKI's cuspidalization theory are applications to the *absolute Isom-form* of the anabelian conjectures over sub-p-adic fields, in particular over finite extensions $k|\mathbb{Q}_p$ of $\mathbb{Q}_p$. See MOCHIZUKI [Mzk6], etc., for more about this.

C) The Section Conjectures

Let $X_0 \to k$ be an arbitrary irreducible k-variety, and $X \subset X_0$ an open k-subvariety. Let $x \in X_0$ be a regular k^{i}-rational point of X_0. Then choosing a system of regular parameters $(t_1, \dots, t_d)$ at x, we can construct —by the standard procedure— a valuation v_x of the function field $k(X)$ of X with value group $v_x(K^\times) = \mathbb{Z}^d$ ordered lexicographically, and residue field $k(X)v_x = \kappa(x)$, thus a subfield of k^{i}. Let v be a prolongation of v_x to $k(X)^{\mathrm{s}}$, and let $T_v \subset Z_v$ be the inertia, respectively decomposition group of $v|v_x$ in G_K. By general valuation theory, see e.g., Kuhlmann–Pank–Roquette [K–P–R], one has: *T_v has complements G_v in Z_v.* And clearly, since $k(X)v_x = \kappa(x) \subset k^{\mathrm{i}}$, under the canonical exact sequence

$$1 \to T_v \to Z_v \to G_{k(X)v_x} = G_k \to 1\,,$$

every complement G_v is mapped isomorphically onto $G_k = G_{k^{\mathrm{i}}}$. Therefore, the canonical projection

$$(*) \qquad pr_{k(X)} : G_{k(X)} \to G_k$$

has sections $s_v : G_k \to G_v \subset G_{k(X)}$ constructed as shown above.

Moreover, let us recall that under the canonical projection $G_{k(X)} \to \pi_1(X)$, the decomposition group Z_v is mapped onto the decomposition group Z_x of v in $\pi_1(X)$, and T_v is mapped onto the inertia group T_x of v in $\pi_1(X)$. And finally, any complement G_v of T_v is mapped isomorphically onto a complement G_x of T_x in Z_x. Clearly $G_x \to G_k$ isomorphically, thus

$$(**) \qquad pr_X : \pi_1(X) \to G_k$$

has a section $s_x : G_k \to G_x \subset \pi_1(X)$ defined via the k^{i}-rational point $x \in X(k^{\mathrm{i}})$. One has the following possibilities:

a) Suppose that $x \in X$. Then $T_x = \{1\}$, as the étale covers of X are not ramified over x. Therefore, $Z_x = G_x$. And in this case the sections of pr_X of the form above build a full conjugacy class of sections.

b) Next let $x \in (X_0 \backslash X)(k^{\mathrm{i}})$. Then $T_x \neq \{1\}$ and $G_x \neq Z_x$ in general. Therefore, for a given k^{i}-rational point x, there might exist several complements G_x of T_x in Z_x, thus a "bouquet" of sections of $pr_X : \pi_1(X) \to G_k$, *which are not conjugate* inside $\pi_1(X)$. Such sections are called *sections at infinity* (for the variety X), and the "bouquet" of conjugacy classes of sections is the non-commutative continuous cohomology pointed set $\mathrm{H}^1_{\mathrm{cont}}(G_k, T_x)$ defined via the split exact sequence $1 \to T_x \longrightarrow Z_x \xrightarrow{pr_X} G_k \to 1$.

Nevertheless, in the case X is a curve and $\mathrm{char}(k) = 0$, one has: T_x is an abelian group and $T_x \cong \widehat{\mathbb{Z}}(1)$ as G_k-modules. In particular, the space of sections $\mathrm{H}^1_{\mathrm{cont}}(G_k, T_x)$ is $\mathrm{H}^1_{\mathrm{cont}}(G_k, T_x) \cong \widehat{k^\times}$ by Kummer Theory, where the last group is the adic completion of the multiplicative group $k^\times$ of the field k.

Grothendieck's section Conjecture asserts roughly that under certain "anabelian hypotheses" all the sections of pr_X arise in the way described above. Precisely, recall that a curve $X \to k$ is `non-isotrivial`, if it has no finite cover which is defined over a finite field.

Section Conjectures. *Let k be a finitely generated infinite field and $X \to k$ be a hyperbolic curve which is non-isotrivial. Let $X_0 \supseteq X$ be the smooth completion of X. In the notations from above, the following hold:*

(1) Birational SC: *The sections of $pr_{k(X)}$ arise from k^{i}-rational points of X_0 as indicated above.*

(2) Curve SC: *The sections of pr_X arise from k^{i}-rational points of X_0 of X as indicated above.*

We notice that the Curve SC implies the Birational SC. Indeed, the Birational SC can be easily derived from the Curve SC by starting with the complete geometrically integral smooth curve X_0 and taking limits over a system $X_i \subset X_0$ of Zariski open neighborhoods of the generic point $\eta_{\in} X_0$. But the Birational SC can be formulated for other "interesting" Galois field extensions $\tilde{K}|K$ of $K := k(X_0)$ as follows: First, recall that there exists a canonical bijection between the k-places v of $K|k$ and the closed points x of X_0, by interpreting each such closed point as a Weil prime divisor of X_0. If v and x and correspond to each other, then the corresponding residue fields are equal: $\kappa(x) = \kappa(v)$. Hence x is a k^{i}-rational point if and only if v is k^{i}-rational place of $K|k$. Let $\tilde{K}|K$ be some Galois extension, and let $\tilde{G}_K := \mathrm{Gal}(\tilde{K}|K)$ denote the Galois group of $\tilde{K}|K$. Further let $\tilde{k} := \overline{k} \cap \tilde{K}$ be the "constants" of $\tilde{K}$, and set $\tilde{G}_k := \mathrm{Gal}(\tilde{k}|k)$. Hence one has a canonical exact sequence

$$1 \to \mathrm{Gal}(\tilde{K}|K\tilde{k}) \longrightarrow \mathrm{Gal}(\tilde{K}|K) \xrightarrow{\tilde{p}_K} \mathrm{Gal}(\tilde{k}|k) \to 1\,.$$

For a k^{i}-rational point x of X and its k^{i}-rational place v of K, let $\tilde{v}$ be a prolongation to $\tilde{K}$, and $T_{\tilde{v}} \subseteq Z_{\tilde{v}}$ be the inertia, respectively decomposition, groups of $\tilde{v}|v$, and $G_{\tilde{v}} := \mathrm{Aut}(\tilde{K}\tilde{v}\,|\,Kv)$ the residual automorphism group. By general Hilbert decomposition theory one has a canonical exact sequence

$$(*) \qquad 1 \to T_{\tilde{v}} \to Z_{\tilde{v}} \to G_{\tilde{v}} \to 1\,.$$

We remark that in general $\tilde{k} \subset \tilde{K}\tilde{v}$ can be a strict inclusion, hence in particular, the canonical projection $\tilde{G}_v \to \tilde{G}_k$ is not an isomorphism, even if v is a k^{i}-rational place of $K|k$. Further, the above exact sequence $(*)$ is not necessarily split. The conclusion is that in general a k^{i}-rational point x of X does not necessarily give rise to a section of $\tilde{p}_K : \mathrm{Gal}(\tilde{K}|K) \to \mathrm{Gal}(\tilde{k}|k)$.

Nevertheless, if $\tilde{k} = \overline{k}$ is an algebraic closure of k, then $\tilde{K}\tilde{v} = \overline{k}$, and if v is a k^{i}-rational place of $K|k$, then $G_{\tilde{v}} = G_k$, and the exact sequence $(*)$ is split.

Thus if $\tilde{k} = \overline{k}$, every k^{i}-rational point x of X gives rise via its k^{i}-rational place v to a "bouquet" of conjugacy classes of sections $s_x : G_k \to \mathrm{Gal}(\tilde{K}|K)$ of the projection $\tilde{p}_K : \mathrm{Gal}(\tilde{K}|K) \to G_k$ which consists of the conjugacy classes in the non-commutative continuous cohomology pointed set $\mathrm{H}^1_{\mathrm{cont}}(G_k, T_{\tilde{v}})$ defined via the split exact sequence $(*)$ above.

There are several variants of the the above conjectures, from which we mention here the following:

(**Birational**) p**-adic Section Conjecture**. *The above* (*Birational*) *Section Conjecture holds over a base field* k *which is a finite extension* $k|\mathbb{Q}_p$ *of* $\mathbb{Q}_p$.

It seems that the initial intention of Grothendieck concerning the Curve SC was to make it part of a strategy to prove the Mordell Conjecture, now Faltings' Theorem (which was not yet proven when Grothendieck made the conjectures). Unfortunately, until now the precise relation between the Curve SC and (an effective) Mordell Conjecture is not clarified yet. But there is recent progress on this question by Minhyong Kim [K1], where a new proof of Siegel's Theorem (on the finiteness of integer points on affine curves of genus one) is given using motivic fundamental group techniques), see also [K2], [K3]. In Kim [K4] he expands this kind of ideas to curves X of arbitrary genus (defined over number fields k). Using pro-linear completions of the geometric fundamental group of X (precisely: pro-unipotent completions to suitably chosen primes), one attaches to X a `Selmer variety`, which is a "non-commutative group like" object (commutative in the case of curves of genus one) that carries information on the sections s of the canonical projection $pr_X : \pi_1(X) \to G_k$. Conjecturally, the Selmer variety has nice properties, and Minhyong Kim designs in [K4] an algorithm (of p-adic nature) which –under the conjectural properties of the Selmer varieties– produces the rational points of the curve in discussion, and more impressively, the effectiveness of the algorithm is guaranteed by the validity of the section conjecture. This is a very promising strategy to finally prove an *effective Mordell* using Grothendieck's section conjecture –at least in the case k is a number field. This moves the Curve SC to the center of a very intensive research.

The first known examples of curves over number fields that satisfy the section conjecture were probably given in Stix [St3] and later by Harari–Szamuely [H–Sz], and Stix [St4]. More recently, Hain [Hn] proved the Curve SC for the generic curve of genus $g \geq 5$. Unfortunately, all these examples are *no sections examples* in the sense that there are no sections of pr_X, and hence no rational points. But as we mentioned above, the ostensibly dull case of curves with neither sections nor points is exactly the crucial class of examples.

Concerning the Curve SC over number fields, I would also like to mention the very recent negative result by Hoshi [Ho], which asserts that the `geometrically pro-`p form of the conjecture does not hold over $k := \mathbb{Q}(\zeta_p)$ for p a regular prime and X_0 the Fermat curve given by $X_0^p + X_1^p + X_2^p \subset \mathbb{P}^2_k$.

Nevertheless, the Curve SC is widely open, and so is the weaker birational SC. One of the strongest –unfortunately conditional– result concerning the birational SC is the following:

Theorem (See ESNAULT–WITTENBERG [E–W]). *Let $K = k(X)$ be the function field of a complete curve X over a number field k, and suppose that the Tate–Shafarevich group of the Jaconian of X is finite. Let $\mathcal{K}$ be the maximal abelian extension of $K\overline{k}$, and suppose that the canonical projection $pr_{\mathcal{K}} : \mathrm{Gal}(\mathcal{K}|K) \to G_k$ has a section $s_{\mathcal{K}}$. Then the index of X, i.e., the g.c.d. of the degrees $d_x := [\kappa(x) : k]$ of all the closed points $x \in X$ is 1.*

The p-adic section conjectures has recently moved into the focus of several investigations, as pieces of evidence for the p-adic section conjecture emerged in recent years. Among the results are the following, see also POP [P8]:

Theorem (See KOENIGSMANN [Ko3]).
Let $k|\mathbb{Q}_p$ be a finite extension, and $K|k$ an arbitrary regular field extension. Then for every section $s : G_k \to G_K$ of the canonical projection $pr_K : G_K \to G_k$ one has: The fixed field $K^{(s)}$ of $s(G_k)$ in K^{a} is p-adically closed. Moreover, if v_s is the valuation of $K^{(s)}$ defining it as a p-adically closed field, then $Kv_s = k$.

In particular, if $k = k(X)$ is the function field of a complete k-variety, then every section $s : G_k \to G_{k(X)}$ is defined by a k-rational point $x_s \in X(k)$. The point x_s is exactly the center of v_s on the complete k-variety X.

This is so far the best un-conditional result concerning the (birational) Section Conjecture we have. But it is not at all clear how to "globalize" such p-adic results in order to get the birational Section Conjecture over number fields. There is nevertheless an unexpected sharpening of the p-adic Birational SC as follows: Let $K|\mathbb{Q}_p$ be a finite field extension which contains the p^{th} roots of unity. Then in the notations from above, let $K' \subset K''$ be the maximal elementary $\mathbb{Z}/p$-abelian, respectively $\mathbb{Z}/p$-metabelian extensions of K, and $pr : \mathrm{Gal}(K'|K) \to \mathrm{Gal}(k'|k)$ the canonical projection. Finally, we say that a section of pr is `liftable`, if it can be lifted to a section of $pr : \mathrm{Gal}(K''|K) \to \mathrm{Gal}(k''|k)$.

Theorem (See POP [P9]). *In the above notations, there is a canonical bijection between the "bouquets" of liftable sections of $pr : \mathrm{Gal}(K'|K) \to \mathrm{Gal}(k'|k)$ and the k-rational points $X_0(k)$ of X_0.*

Concerning the p-adic Curve SC, there are several partial results, like MOCHIZUKI's [Mz5] concerning `cuspidal sections` of curves which cover $\mathbb{P}^1\backslash\{0, 1, \infty\}$ as well as MOCHIZUKI's [Mzk6], etc., concerning –among other things– the relation between the p-adic Curve SC and the absolute anabelian conjecture for curves; and "conditional results" proved by SAIDI [Sa2], where so called `good sections` are defined and it is shown that they originate indeed from points, using techniques inspired by the (proof of the) above result POP [P9]. (In all these stories, MOCHIZUKI's cuspidalization plays heavily into the game). But so far, it seems that the strongest unconditional result towards a proof of the p-adic Curve SC seems to be the following: Let $k|\mathbb{Q}_p$ be a finite field extension, X/k be a hyperbolic curve, and $K := k(X)$. Further let $\tilde{K} = k(\tilde{X})$ be the function field of the pro-étale universal cover of X, thus $\mathrm{Gal}(\tilde{K}|K) = \pi_1(X)$.

Theorem (See POP–STIX [P–St]). *For every section $s : G_k \to \pi_1(X)$ of the canonical projection $pr_X : \pi_1(X) \to G_k$ there exists a valuation w of $k(X)$ and a prolongation $\tilde{w}$ to $\tilde{K}|K$ such that $\mathrm{im}(s) \subset Z_{\tilde{w}}$, where $Z_{\tilde{w}}$ is the decomposition group of $\tilde{w}|w$.*

One of the main technical result in the proof is STIX' theorem [St4] asserting that the existence of a section of the canonical projection $pr_X : \pi_1(X) \to G_k$ implies that the `index` of X is a power of p. (Recall that the index of a curve is the g.c.d. of the degrees of all its closed points. In particular, if the curve has a rational point, than the index of the curve is 1.) This makes possible to apply a machinery similar to the one used in the birational case, namely the Tate, Roquette, Lichtenbaum Local-Global Principle for $\mathrm{Br}(X)$ and its generalization by POP [P8], and reduce the problem to a fixed point question about actions of groups on graphs. One concludes by using a concrete structural assertion about the log-étale fundamental group of X as developed by MOCHIZUKI [Mz1] and STIX [St5] together with TAMAGAWA's main result from [T6].

PART III: Beyond the arithmetical action

It is/was a widespread belief that the reason for the existence of anabelian schemes is strong interaction between the arithmetic and a rich algebraic fundamental group, and that this interaction makes étale fundamental groups so rigid, that the only way isomorphisms, respectively open homomorphisms, can occur is the geometrical one. (So to say, morphisms between étale fundamental groups which do not have a reason to exist, do not exist indeed...)

On the other hand, some developments from the 1990's showed evidence for very strong anabelian phenomena for curves and higher dimensional varieties over *algebraically closed fields,* thus in a total absence of a Galois action of the base field. We mention here the following:

a) *Bogomolov's birational anabelian program* (see [Bo])

Let ℓ be a fixed rational prime number. For algebraically closed base fields k of characteristic $\neq \ell$, we consider integral k-varieties $X \to k$, with function field $K := k(X)$. It turns out that there is a major difference between the cases $\dim(X) = 1$ and $\dim(X) > 1$. Indeed, if $\dim(X) = 1$, then the absolute Galois group G_K is profinite free on $|k|$ generators. This is the so called *Geometric case of a Conjecture of Shafarevich,* proved by HARBATER [Ha2], and POP [Po]. On the other hand, if $d = \dim(X) > 1$, then G_K is very complicated (in particular, having $\mathrm{cd}_\ell G_K = d$, etc.).

The guess of BOGOMOLOV [Bo] is that if $\dim(X) > 1$, then the Galois group G_K should be so rich as to encode the birational class of X up to purely inseparable covers and Frobenius twists. More precisely, BOGOMOLOV proposes and gives evidence for the following: In the context above, let $G_K(\ell)$ be the maximal pro-ℓ quotient of G_K, i.e., $G_K(\ell)$ is the Galois group of the maximal Galois pro-ℓ sub-extension $K(\ell)$ of $K^{\mathrm{s}}|K$. Further let $G_K^{(n)}$ denote the n^{th} ℓ^∞-factor in the central series of $G_K(\ell)$, and $K^{(n)}$ be the corresponding fixed fields inside $K(\ell)$. Hence setting $\Pi_K^{(n)} := G_K(\ell)/G^{(n)}$ we have: $\Pi_K := \Pi_K^{(2)}$ is the Galois group of the maximal prol-ℓ abelian sub-extension $K^{(2)} = K^{\ell,\mathrm{ab}}$ of $K(\ell)$, and $\Pi_K^{\mathrm{c}} := \Pi_K^{(3)}$ is the Galois group of the maximal pro-ℓ abelian-by-central sub-extension $K^{(3)}$ of $K(\ell)$. Now BOGOMOLOV [Bo] claim is that in fact the isomorphy type of the function $K|k$ is encoded in Π_K^{c}. (Note that in BOGOMOLOV [Bo] the notation for $\Pi_K^{\mathrm{c}} := G_K(\ell)/G_K^{(3)}$ is $\mathrm{PGal}_K^{\mathrm{c}}$.) The starting point in this development was BOGOMOLOV's theory of *liftable commuting pairs,* see a result mentioned below concerning this.

b) *Tamagawa's Theorem concerning* $\mathbb{P}^1_{k_0}\backslash\{0, 1, \infty, x_1, \dots, x_n\}$

In the mid 1990's TAMAGAWA gave evidence for the fact that some curves over the algebraic closure $k_0 = \overline{\mathbb{F}}_p$ are weakly anabelian, i.e., their isomorphy type as a scheme can be recovered from π_1 or even π_1^{t}. The first precursor of this fact is TAMAGAWA's result that given a smooth curve $X \to k$, the type (g, r) of the curve is encoded in the algebraic fundamental group $\pi_1(\overline{X})$; and moreover, the canonical projections $\pi_1(\overline{X}) \to \pi_1^{\mathrm{t}}(\overline{X}) \to \pi_1(X_0)$ are encoded in the algebraic fundamental group of X. This answered a question raised by HARBATER. And finally, TAMAGAWA showed the following:

Theorem (See TAMAGAWA [T2]). *Let $U = \mathbb{P}^1_{k_0}\backslash\{0, 1, \infty, x_1, \dots, x_n\}$ be an affine open. Then the isomorphy type of U as a scheme can be recovered from $\pi_1^{\mathrm{t}}(U)$. Moreover, if X is any other curve over some algebraically closed field k, and $\pi_1(X) \cong \pi_1(U)$, then $k = k_0$, and $X \cong U$ as schemes.*

The kind of results above show that one can expect anabelian phenomena over algebraically closed base fields, thus in a complete absence of arithmetical Galois action. This kind of anabelian phenomena go *beyond Grothendieck's anabelian Geometry.* Here is a short list of the kind of such anabelian results.

A) Small Galois groups and valuations

Let ℓ be a fixed prime number. We consider fields K of characteristic $\neq \ell$, such that $\mu_\ell \subset K$. We denote by $K(\ell)$ the maximal Galois pro-ℓ extension of K in some fixed algebraic closure K^{a} of K, and denote by $G_K(\ell)$ the Galois group of $K(\ell)|K$. In order to avoid complications arising from orderings in case $\ell = 2$, we will also suppose that $\mu_4 \subset K$ if $\ell = 2$.

In the above context, let v be a non-trivial valuation of $K(\ell)$ such that value group vK is not ℓ-divisible and the residue field Kv has characteristic $\neq \ell$. Let $V_v \subseteq T_v \subseteq Z_v$ be respectively the ramification, the inertia, and the decomposition groups of v in $G_K(\ell)$. Then by the Hilbert decomposition theory for valuations one has, see e.g., [BOU]: $V_v = \{1\}$, as $\mathrm{char}(Kv) \neq \ell$. Thus $T_v = T_v/V_v$ is an Abelian pro-ℓ group. Further, $K(\ell)v = (Kv)(\ell)$, and one has the canonical exact sequence:

$$1 \to T_v \to Z_v \to G_{Kv}(\ell) \to 1.$$

Finally, $vK(\ell)$ is the ℓ-divisible hull of vK. And denoting by $\widehat{vK}$ the ℓ-adic completion of vK, there is an isomorphism of G_{Kv}-modules $T_v \cong \mathrm{Hom}(\widehat{vK}, \mathbb{T}_\ell)$ where $\mathbb{T}_\ell = \varprojlim_m \mu_{\ell^m}$ is the Tate module of Kv. This reduces the problem of describing Z_v to that of describing $Kv(\ell)$. But the essential observation here is that T_v is a non-trivial Abelian normal subgroup of Z_v.

The following result is based on work by Ware [W] if $\ell = 2$, and Koenigsmann [Ko1] if $\ell \neq 2$, see also Efrat [Ef1], [Ef2]. It is the best possible converse to the above description of Z_v:

Theorem (Engler–Koenigsmann [E–K]). *In the above notations let Z be a closed non-procyclic subgroup of $G_K(\ell)$ having a non-trivial Abelian normal subgroup T. Then there exists a valuation w of $K(\ell)$ with the following properties:*

1) *$Z \subseteq Z_w$ and $T \subseteq T_w$.*
2) *The residue field Kw has* $\mathrm{char}(Kw) \neq \ell$.

The proof of the Theorem above is based an a fine analysis of the multiplicative structure of fields with *very small pro-ℓ Galois group.* We will say namely that K has a very small pro-ℓ Galois group, if $K(\ell)$ is non pro-cyclic, but fits into an exact sequence of the form $0 \to \mathbb{Z}_\ell \to K(\ell) \to \mathbb{Z}_\ell \to 0$. In such a case one simply can write down the valuation ring of a valuation w on K satisfying the properties (i), (ii) above, see loc.cit. The rest is just valuation theory techniques.

The above assertion concerning fields with very small pro-ℓ Galois group is somehow parallel to the Bogomolov's theory of liftable commuting pairs mentioned above, first mentioned in Bogomolov [Bo], and finally proved in [B–T1]:

Theorem (Bogomolov–Tschinkel [B–T1]).

Suppose that K contains an algebraically closed subfield. Let $\Gamma \cong \mathbb{Z}_\ell \times \mathbb{Z}_\ell$ be a closed subgroup of Π_K having an abelian preimage under $\Pi^{\mathrm{c}}_K \to \Pi_K$. Then there exists a non-trivial valuation w of K such that denoting by $T_w \subseteq Z_w \subseteq \Pi_K$ the inertia/decomposition group of w in Π_K, the following hold:

1) *$\Gamma \subseteq Z_w$ and $\Gamma \cap T_w \neq 1$.*
2) *The residue field Kw has* $\mathrm{char}(Kw) \neq \ell$.

The proof relies on a very ingenious idea of Bogomolov to compare maps between affine geometries and projective geometries. The two kind of geometries arise as follows: First, by Kummer Theory one has a canonical map

$$K^{\times} \xrightarrow{\jmath} \mathrm{Hom}_{\mathrm{cont}}(\Pi_K, \mathbb{Z}_\ell)$$

which is trivial on $k^{\times}$, as k is algebraically closed. This allows us to interpret $\jmath$ as a map from the projectivization $K^{\times}/k^{\times}$ of the k-vector space $(K, +)$ to the "affine" space on the right, which is $\mathrm{Hom}_{\mathrm{cont}}(\Pi_K, \mathbb{Z}_\ell)$, or even $\mathrm{Hom}_{\mathrm{cont}}(\Pi_K, \mathbb{F}_\ell)$. And in particular, if Π_K is very small, then on the right we do really have an affine geometry. Finally, since such maps between projective and affine geometries are of very special shape, Bogomolov–Tschinkel show that a liftable commuting subgroup $\Gamma \cong \mathbb{Z}_\ell \times \mathbb{Z}_\ell$ of Π_K must contain an element σ which —by duality— defines a *flag function* on $K^{\times}$. Strictly speaking, this means that σ is an inertia element to a valuation w with the claimed properties.

It is interesting to remark that as a by-product of the theory of very small pro-ℓ Galois groups, one obtains a p-adic analog of the Artin–Schreier Theorem for the Galois characterization of the real closed fields. The result is:

Theorem (See Pop [P8], Koenigsmann [Ko1], Efrat [Ef1]).
Let K be a field having G_K isomorphic to some open subgroup of $G_{\mathbb{Q}_p}$. Then K is p-adically closed, i.e., K is Henselian with respect to a valuation v having divisible value group, and residue field Kv contained and relatively algebraically closed in some finite extension $k|\mathbb{Q}_p$ of $\mathbb{Q}_p$.

The proof of this assertion has two main parts: In a first approximation, one uses the techniques mentioned above as developed by Ware, Koeningsmann, Efrat in order to show that the fixed field of every Sylow ℓ-group of G_K carries a Henselian valuation. Using *valuation* theoretical and *Galois* theoretical techniques, one proves that actually K itself carries a non-trivial Henselian valuation v having residual characterisitc $\mathrm{char}(Kv) > 0$. One concludes by applying an *arithmetical* type result by Pop [P8], Theorem E9, and showing that the valuation is actually a p-adic valuation.

B) Variation of fundamental groups in families of curves

As we have seen at the beginning of Part III, one might/should expect strong anabelian phenomena for curves (maybe even more general varieties) over algebraically closed fields of positive characteristic. Maybe a good hint in that direction is the fact that *we do not have a description of the algebraic fundamental group of any potentially hyperbolic curve.* Indeed, if we can recover X from its fundamental group in a functorial way, then the fundamental group of X must encode "moduli" of X, thus an information of a completely different nature than crude profinite group theory. See TAMAGAWA [T3], [T4] for more about this conjectural world.

Now let me explain the best result we have so far for the variation of the geometric fundamental groups in families of complete curves. Recall that for a complete smooth connected curve X of genus $g \geq 2$ over a field of characteristic 0 one has $\pi_1(\overline{X}) \cong \widehat{\Pi}_g$, thus $\pi_1(\overline{X})$ depends on g only. As mentioned above, in positive characteristic $\pi_1(\overline{X})$ is unknown, and it *depends on the isomorphy type* of $\overline{X}$. By Grothendieck's Specialization Theorem, $\pi_1(\overline{X})$ is a quotient of $\widehat{\Gamma}_g$, thus it is topologically finitely generated. In particular, $\pi_1(\overline{X})$ is completely determined by its set of their finite quotients. (Terminology: $\pi_1(\overline{X})$ is a *Pfaffian group.*)

Let $\mathcal{M}_g \to \mathbb{F}_p$ be the coarse moduli space of proper and smooth curves of genus g in characteristic p. One knows that $\mathcal{M}_g$ is a quasi-projective and geometrically irreducible variety. And if k is an algebraically closed field of characteristic p, then $\mathcal{M}_g(k)$ classifies the isomorphism classes of curves of genus g over k. For $\overline{x} \in \mathcal{M}_g(k)$ let $C_{\overline{x}} \to k$ be a curve classified by $\overline{x}$, and let $x \in \mathcal{M}_g$ be such that $\overline{x} : k \to \mathcal{M}_g$ factors through x. We set

$$\pi_1(x) := \pi_1(C_{\overline{x}}),$$

and remark that the structure of $\pi_1(x)$ as a profinite group depends on x only, and not on the concrete geometric point $\overline{x} \in \mathcal{M}_g(k)$ used to define it. In particular, the fundamental group functor gives rise to a map as follows:

$$\pi_1 : \mathcal{M}_g \to \mathcal{G}, \quad x \to \pi_1(x).$$

To finish our preparation we notice that for points $x, y \in \mathcal{M}_g$ such that x is a specialization of y, by Grothendieck's Specialization Theorem there exists a surjective continuous homomorphism $\mathrm{sp} : \pi_1(y) \to \pi_1(x)$. In particular, if η is the generic point of $\mathcal{M}_g$, then C_η is the generic curve of genus g; and every point x of $\mathcal{M}_g$ is a specialization of η. For every $x \in \mathcal{M}_g$, there is a surjective homomorphism $\mathrm{sp}_x : \pi_1(\eta) \to \pi_1(x)$ which is determined up to Galois-conjugacy by the choice of the local ring of x in the algebraic closure of $\kappa(\eta)$. For every $x \in \mathcal{M}_g$ we fix such a map once for all; in particular, if x is a specialization of y, there exists a specialization homomorphism $\mathrm{sp}_{y,x} : \pi_1(y) \to \pi_1(x)$ such that $\mathrm{sp}_{y,x} \circ \mathrm{sp}_y = \mathrm{sp}_x$.

Theorem (RAYNAUD [R2], POP–SAIDI [P–S], TAMAGAWA [T5]).
For all points $s \neq x$ in $\mathcal{M}_g$ with s closed and specialization of x, the specialization homomorphism $\mathrm{sp}_{x,s} : \pi_1(x) \to \pi_1(s)$ is not an isomorphism.

More precisely, there exist cyclic étale covers of X_x of order prime to p, which do not have good reduction under the specialization $x \mapsto s$.

This gives an answer to a question raised by HARBATER:

Corollary. *There is no non-empty open subset $U \subset \mathcal{M}_g$ such that the isomorphy type of the geometric fundamental group $\pi_1(x)$ is constant on U.*

Concerning the proof of the above Theorem: In the case $g = 2$ the above Theorem was proved by RAYNAUD, by introducing a new kind of *theta divisor* (called by now the Raynaud theta divisor). Using this tool, he showed that given a projective curve $X \to k_0$ of genus 2, there exist only finitely many curves $X' \to k_0$ with $\pi_1(X) \cong \pi_1(X')$, see RAYNAUD [R2]. Around the same time POP–SAIDI proposed a way of generalizing RAYNAUD's result to all genera, by combining the theory of Raynaud's theta divisor with the results by HRUSHOVSKI [Hr] on the geometric case of the Manin-Mumford Conjecture as follows: First suppose that $g = 2$. Then for points $x_0 \neq x_1$ in $\mathcal{M}_g$ such that x_0 is a specialization of x_1, it turns out that Raynaud's Result follows from: *If x_0 is a closed point, then sp_{x_1,x_0} is not an isomorphism.* POP–SAIDI showed in [P–S] that this is the case for arbitrary genera $g > 1$, provided x_0 has some special properties, see loc.cit. Finally, TAMAGAWA [T5] elaborating on the method proposed in [P–S] showed that sp_{x_1,x_0} is not an isomorphism, provided $x \neq x_0$ and x_0 is a closed point.

C) Pro-ℓ abelian-by-central birational anabelian Geometry

Here I want to, first, make precise the assertion one expects to prove under BOGOMOLOV's [Bo] hypotheses mentioned at the beginning of Part III and to report on some recent results/developments. Recall that for a fixed prime number ℓ, we denote by $G_K(\ell) \to \Pi_K^{\mathrm{c}} \to \Pi_K$ the Galois groups of a maximal pro-ℓ Galois extension of $K(\ell)|K$, respectively of the maximal abelian-by-central, respectively abelian, sub-extensions $K^{(3)}|K$ and $K^{(2)}|K$ of $K(\ell)|K$. In the case $K = k(X)$ is the function field of a variety over k, the more precise conjecture one can/should make is the following:

Conjecture AbC. *In the above notations, there exists is a group theoretical recipe by which one can recover in a functorial way the isomorphy type of $K|k$ from Π_K^{c}, provided* $\mathrm{td}(K|k) > 1$. *Moreover, if $L|l$ is a further function field over an algebraically closed field l, and $\Phi : \Pi_K^{\mathrm{c}} \to \Pi_L^{\mathrm{c}}$ is an isomorphism, then $L^{\mathrm{i}} \cong K^{\mathrm{i}}$.*

A sketch of a strategy to tackle the Conjecture AbC above from pro-ℓ Galois information, at least in the case k is an algebraic closure of a finite field, was presented in POP [P3]. It has as starting point the following idea: Let $\widehat{K^\times}$ be the ℓ-adic completion of the multiplicative group $K^\times$ of $K|k$.† Since the cyclotomic character of K is trivial, one can identify the ℓ-adic Tate module $\mathbb{T}_{K,\ell}$ of K with $\mathbb{Z}_\ell$ (non-canonically), and let $\imath_K : \mathbb{T}_{K,\ell} \to \mathbb{Z}_\ell$ be a fixed identification. Via Kummer Theory, one has isomorphisms of ℓ-adically complete groups:

$$\widehat{K^\times} = \mathrm{Hom}_{\mathrm{cont}}(\Pi_K, \mathbb{T}_{K,\ell}) \xrightarrow{\imath_K} \mathrm{Hom}_{\mathrm{cont}}(\Pi_K, \mathbb{Z}_\ell),$$

† Recall that for an abelian group A, its ℓ-adic completion is $\widehat{A} := \varprojlim_e A/\ell^e$.

i.e., $\widehat{K^\times}$ can be recovered from Π_K, hence as well from $\Pi^{\rm c}_K$ via the canonical projection $\Pi^{\rm c}_K \to \Pi_K$. On the other hand, since $k^\times$ is divisible, $\widehat{K^\times}$ equals the ℓ-adic completion of the free abelian group $K^\times/k^\times$. Now the idea of recovering $K|k$ is as follows:

a) First, give a recipe to recover the image $\jmath_K(K^\times) = K^\times/k^\times$ of the ℓ-adic completion functor $\jmath_K : K^\times \to K^\times/k^\times \subset \widehat{K^\times}$ inside the "known" ℓ-adically complete group $\widehat{K^\times} = \mathrm{Hom}_{\mathrm{cont}}(\Pi_K, \mathbb{Z}_\ell)$.

b) Second, interpreting $K^\times/k^\times =: \mathcal{P}(K)$ as the projectivization of the (infinite) dimensional k-vector space $(K,+)$, give a recipe to recover the projective lines $\mathfrak{l}_{x,y} := (kx+ky)^\times/k^\times$ inside $\mathcal{P}(K)$, where $x, y \in K$ are k-linearly independent.

c) Third, apply the *Fundamental Theorem of Projective Geometries,* see e.g., Artin [Ar], and deduce that $K|k$ can be recovered from $\mathcal{P}(K)$ endowed with all the lines $\mathfrak{l}_{x,y}$.

d) Finally, show that the recipes above are functorial, i.e., they are invariant under isomorphisms of profinite groups $\Pi_K \to \Pi_L$ which are abelianizations of isomorphisms $\Pi^{\rm c}_K \to \Pi^{\rm c}_L$. In particular, such isomorphisms $\Pi_K \to \Pi_L$ originate actually from geometry.

The strategy from Pop [P3] to tackle the above problems a), b), c), d), above is in principle similar to the strategies (initiated by Neukirch and Uchida) to tackle Grothendieck's anabelian conjectures. It has two main parts, namely a *local theory,* in which one has to recover the prime divisors v of $K|k$ as well as the so called generalized prime divisors of $K|k$; and a *global theory* in which one recovers $\mathcal{P}(K) = K^\times/k^\times$ inside $\widehat{K^\times}$ and finally the so called rational projections $\Pi_K \to \Pi_{k(t)}$ defined by "generic" functions $t \in K$, i.e., having the property that $k(t)$ is relatively algebraically closed in K. See Pop [P5], Introduction, for more about this. And a quite surprising new but very basic fact of the above strategy is the following fact, see Pop [P10], Introduction, for definitions:

Inertia vs Frobenius. *The set of all the* (*tame*) *inertia elements in G_K is topologically closed in G_K. Further, the tame quasi divisorial inertia elements are dense in the set of all the tame inertia elements.*

This is in contrast to the behavior of the set of all the *Frobenius elements* $\mathfrak{Frob}_{\mathcal{X}}$ of models $\mathcal{X}$ of finitely generated fields $\mathcal{K}$, because by the Chebotarev Density Theorem, $\mathfrak{Frob}_{\mathcal{X}}$ are dense in the absolute Galois group $G_{\mathcal{K}}$.

A first application of the above strategy was to prove a stronger form of the *full pro-ℓ* Conjecture AbC in the case k is an algebraic closure of a finite field, see Pop [P4]. Finally, the above strategy led to a positive answer to Conjecture AbC in the case k is an algebraic closure a finite field as follows: First, Bogomolov–Tschinkel [B–T2] proved that in the case k is an algebraic closure of a finite field, the function fields $K|k$ of transcendence degree $\mathrm{td}(K|k) = 2$ can be recovered from $\Pi^{\rm c}_K$ as predicted by Conjecture AbC. The strongest assertion concerning the case k is an algebraic closure of a finite field was proved in Pop [P6] and is the following:

Theorem (See Pop [P6]). *In the above notations, the following hold:*

1) *There exists a group theoretical recipe by which one can recover the transcendence degree* $\mathrm{td}(K|k) = \dim(X)$ *of* $K|k$ *and the fact that* k *is an algebraic closure of a finite field from* Π^{c}_K. *Further suppose that* $\mathrm{td}(K|k) > 1$ *and that* k *is an algebraic closure of a finite field. Then there exists a group theoretical recipe which recovers* $K|k$ *from* Π^{c}_K *in a functorial way.*
2) *The recipes above are invariant under isomorphisms as follows: Let* $L|k$ *be a further function field with* l *algebraically closed, and* $\Phi : \Pi_K \to \Pi_L$ *be a continuous isomorphism which can be lifted to an abstract isomorphism* $\Pi^{\mathrm{c}}_K \to \Pi^{\mathrm{c}}_L$. *Then there exists* $\epsilon \in \mathbb{Z}_\ell^\times$ *and an isomorphism of field extensions* $\phi : L^{\mathrm{i}}|l \to K^{\mathrm{i}}|k$ *such that* $\epsilon \cdot \Phi$ *is induced via Kummer Theory by* ϕ. *Moreover,* ϕ *is unique up to Frobenius twists and* ϵ *is unique up to multiplication by* p*-powers, where* p *is the characteristic.*

The Theorem above is a far reaching extension of Grothendieck's birational conjecture in positive characteristic, as it implies Grothendieck's birational conjecture if $\mathrm{td}(K) > 1$; moreover, it implies the *geometrically pro-ℓ* form of Grothendieck's birational anabelian Conjecture. But I should mention right away that the arithmetical pro-ℓ form of Grothendieck's birational conjecture is completely open at this moment.

D) The Ihara/Oda–Matsumoto Conjecture

The Ihara question from the 1980's, which in the 1990's became a conjecture by Oda–Matsumoto, for short I/OM, is about giving a topological/combinatorial description of the absolute Galois group of the rational numbers. Let me explain the question/conjecture I/OM in detail:

Let $k_0 \subset \mathbb{C}$ be a fixed finitely generated field, e.g., a number field, and k the algebraic closure of k_0 inside $\mathbb{C}$. For every geometrically integral k_0-variety X, let $\overline{X} := X \times_{k_0} k$ be the base change to k, and let $\overline{\pi}(X) := \pi_1(\overline{X}, *)$ denote the algebraic fundamental group of X, where $*$ is a geometric point of X (which we will not mention anymore in order to simplify notations) defined by some fixed algebraic closure $\overline{k}|k$, thus defining the absolute Galois group G_{k_0} of k_0 as well. Via the exact sequence of étale fundamental groups

$$1 \to \overline{\pi}(X) \to \pi_1(X) \to G_{k_0} \to 1,$$

one gets a representation $\rho_X : G_{k_0} \to \mathrm{Out}(\overline{\pi}(X))$. By the functoriality of the étale fundamental group functor, the collection of all the representations $(\rho_X)_X$, is compatible with k_0-morphisms of geometrically connected k_0-varieties.

In particular, for every small category $\mathcal{V}$ of geometrically integral varieties over k_0, the `algebraic fundamental group functor` $\overline{\pi} : \mathcal{V} \to \mathcal{G}$, $X \mapsto \overline{\pi}(X)$, of $\mathcal{V}$ gives rise to a representation

$$\imath_{\mathcal{V}} : G_{k_0} \to \mathrm{Aut}(\overline{\pi}_{\mathcal{V}}),$$

where $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ is the automorphism group of $\overline{\pi}_{\mathcal{V}}$. In down to earth terms, the elements $\sigma \in \mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ are the families $\sigma = (\sigma_X)_{X\in\mathcal{V}}$, $\sigma_X \in \mathrm{Out}(\overline{\pi}_1(X))$, which are compatible with $\overline{\pi}_1(f) : \overline{\pi}_1(X) \to \overline{\pi}_1(Y)$ for all morphisms $f : X \to Y$ in $\mathcal{V}$.

Recall that $\overline{\pi}(X)$ is nothing but the profinite completion of the topological fundamental group $\pi_1^{\mathrm{top}}(X(\mathbb{C}), *)$ of the "good" topological space $X(\mathbb{C})$, thus $\overline{\pi}(X)$ is an object of topological/combinatorial nature. Following GROTHENDIECK, one should give a new description of G_{k_0} by finding categories $\mathcal{V}$ for which $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ has a "nice" description, and $\imath_{\mathcal{V}}$ is an isomorphism. If so, then via the isomorphism $\imath_{\mathcal{V}}$, we would have a new non-tautological description of G_{k_0}. For $k_0 = \mathbb{Q}$, GROTHENDIECK suggested to study $\overline{\pi}_{\mathcal{V}}$, for $\mathcal{V}$ sub-categories of the *Teichmüller modular tower* $\mathcal{T}$ of all the moduli spaces $M_{g,n}$. For instance, if $\mathcal{V}_0 = \{M_{04}, M_{05}\}$ endowed with "connecting homomorphisms," $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}_0})$ is the famous *Grothendieck–Teichmüller group* $\widehat{GT}$, which was intensively studied first by DRINFEL'D [Dr], IHARA [I1], [I2], [I3], DELIGNE [De], and lately by several others, e.g. HAIN–MATSUMOTO [H–M], HARBATER–SCHNEPS [H–Sch], IHARA–MATSUMOTO [I–M], LOCHAK–SCHNEPS [L–Sch], NAKAMURA–SCHNEPS [N–Sch], and many others.

We know little about the nature of $\imath_{\mathcal{V}}$ in general, and the situation is quite mysterious. Concerning the injectivity, DRINFEL'D remarked that using Belyi's Theorem [Be] it follows that $\imath_{\mathcal{V}}$ is injective, provided $C := \mathbb{P}^1\backslash\{0, 1, \infty\}$ lies in $\mathcal{V}$. Further, VOEVODSKY showed that the same is true if $C \in \mathcal{V}$ with X some affine open of a curve of genus one. It is conjectured though that $\imath_{\mathcal{V}}$ should be injective as soon as $\mathcal{V}$ contains at least one hyperbolic curve (and it seems that HOSHI–MOCHIZUKI have announced a proof of this fact), but so far the following is the best result one has in this direction:

Theorem (See MATSUMOTO [Ma]). *Suppose that $\mathcal{V}$ contains some affine hyperbolic curve C. Then the representation $\imath_{\mathcal{V}} : G_{k_0} \to \mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ is injective.*

The question about the surjectivity of the representation $\imath_{\mathcal{V}}$ is of a completely different nature and less understood, and IHARA asked in the 1980's whether $\imath_{\mathcal{V}}$ is onto, thus an isomorphism, provided $k_0 = \mathbb{Q}$ and $\mathcal{V}$ is the category of <u>all</u> the k_0-varieties. Finally, based on some "motivic evidence," ODA–MATSUMOTO conjectured in the 1990's that the answer to Ihara's Question should be positive. The author showed (unpublished), that the answer to I/OM is positive —for more general fields, thus giving a positive answer to I/OM. See also ANDRÉ [An], where the author defines `tempered fundamental groups`, introduces a p-adic tempered variant $\widehat{GT}_p$ of $\widehat{GT}$, and (re)proves the I/OM over p-adic fields.

We now formulate the **pro-ℓ abelian-by-central I/OM**, which implies the usual I/OM: Let $\overline{\pi}(X) \longrightarrow \Pi^{\mathrm{c}}_X \longrightarrow \Pi_X$ be the pro-ℓ abelian-by central, respectively pro-ℓ abelian, quotients of $\overline{\pi}$. Since the kernels of the above group homomorphisms are characteristic subgroups of $\overline{\pi}(X)$ and Π^{c}_X, respectively, the above homomorphisms give rise to homomorphisms $\mathrm{Out}(\overline{\pi}(X)) \to \mathrm{Out}(\Pi^{\mathrm{c}}_X) \to \mathrm{Out}(\Pi_X)$. Further, $\mathbb{Z}_\ell^\times$ acts by multiplication on Π_X and this multiplication can be lifted to an action of $\mathbb{Z}_\ell^\times$ on Π^{c}_X. For every category $\mathcal{V}$ as above, one has a corresponding morphism $\overline{\pi}_{\mathcal{V}} \to \Pi^{\mathrm{c}}_{\mathcal{V}} \to \Pi_{\mathcal{V}}$ of functors, which by the discussion above gives rise to a group homomorphism $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}}) \to \mathrm{Aut}(\Pi^{\mathrm{c}}_{\mathcal{V}}) \to \mathrm{Aut}(\Pi_{\mathcal{V}})$. Let $\mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}})$ be the image of $\mathrm{Aut}(\Pi^{\mathrm{c}}_{\mathcal{V}})$ in $\mathrm{Aut}(\Pi_{\mathcal{V}})$ modulo the action of $\mathbb{Z}_\ell^\times$. In other words, the elements of $\mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}})$ are exactly the $\mathbb{Z}_\ell^\times$ equivalence classes of automorphisms of $\Pi_{\mathcal{V}}$ which can be lifted to automorphisms of $\Pi^{\mathrm{c}}_{\mathcal{V}}$. As above, we get a group homomorphism

$$\imath^{\mathrm{c}}_{\mathcal{V}} : G_{k_0} \to \mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}}).$$

Pro-ℓ abelian-by-central I/OM. *The homomorphism $\imath^{\mathrm{c}}_{\mathcal{V}}$ is an isomorphism in the case $k_0 = \mathbb{Q}$ and $\mathcal{V}$ is the category of all the geometrically integral $\mathbb{Q}$-varieties.*

We will actually prove a stronger/more precise assertion than the above pro-ℓ abelian-by-central I/OM which evolves as follows: Let $U_0 := \mathbb{P}^1\backslash\{0, 1, \infty\} \times k_0$ be the `tripod` over k_0 (terminology by Mochizuki–Hoshi). For every geometrically integral k-variety X, and a basis $\{U_i\}_i$ of Zariski (affine) open neighborhoods of the generic point η_X of X, we consider the category $\mathcal{V}_X = \{U_i\}_i \cup \{U_0\}$ with morphisms the inclusions $U_{i''} \hookrightarrow U_{i'}$ and the dominant k-morphisms $\varphi_i : U_i \to U_0$. In particular, $\mathrm{Aut}(\mathcal{V}_X) = 1$ is trivial, and thus the automphism group of $\Pi_{\mathcal{V}_X}$ are the systems $(\sigma_i)_i$ with $\sigma_i \in \mathrm{Aut}^{\mathrm{c}}(\Pi_{U_i})$ and $\sigma_0 \in \mathrm{Aut}^{\mathrm{c}}(\Pi_{U_0})$ such that the following diagrams are commutative, when defined:

$$\begin{array}{ccc} \Pi_{U_j} & \xrightarrow{\sigma_j} & \Pi_{U_j} \\ \downarrow & & \downarrow \\ \Pi_{U_i} & \xrightarrow{\sigma_i} & \Pi_{U_i} \end{array} \qquad\qquad \begin{array}{ccc} \Pi_{U_i} & \xrightarrow{\sigma_i} & \Pi_{U_i} \\ \downarrow & & \downarrow \\ \Pi_{U_0} & \xrightarrow{\sigma_0} & \Pi_{U_0} \end{array}$$

Note that every k_0 embedding $k_0(t) \hookrightarrow k_0(X)$ originates from some dominant k_0 morphism $\varphi_j : U_i \to U_0$ for U_j sufficiently small. Therefore, every $(\sigma_i)_i$ gives rise to an automorphism $\sigma \in \mathrm{Aut}^{\mathrm{c}}(\Pi_K)$ which is compatible with the projections $p_\imath : \Pi_K \to \Pi_{U_0}$ defined by all embeddings $\imath : k_0(t) \hookrightarrow k_0(X)$, i.e., one has:

$$p_\imath \circ \sigma = \sigma \circ p_\imath.$$

This suggests that a possible way to prove the pro-ℓ abelian-by-central I/OM is to prove its *birational form* for each category $\mathcal{V}_X$ with $\dim(X)$ sufficiently large, which is the stronger assertion that every $\sigma \in \mathrm{Aut}^{\mathrm{c}}(\Pi_K)$ which is compatible with all the projections $p_\imath$ originates from Gal_k. Before announcing the second main result of this manuscript, which answers the above question, let $\mathrm{Aut}_k(K) \subset \mathrm{Aut}(K)$ be the group of all the k-automorphisms of K, respectively all the field automorphisms of K. Note that since $k \subset K$ is the unique maximal algebraically closed subfield in K, every $\phi \in \mathrm{Aut}(K)$ maps k isomorphically onto itself, hence $\mathrm{Aut}(K)$ acts on k. Let $k_K \subset k_0$ be the corresponding fixed field.

Theorem (See Pop [P7]). *In the above notations, the following hold:*

1) *If* $\dim(X) > \dim(k_0) + 1$*, one has a canonical exact sequence of the form:*

$$1 \to \mathrm{Aut}_k(K) \to \mathrm{Aut}^{\mathrm{c}}(\Pi_K) \to \mathrm{Aut}_{k_K}(k) \to 1.$$

Thus if $k_K = k_0$ *and* $\mathrm{Aut}_k(K) = 1$*, then* $\imath^{\mathrm{c}}_K : G_{k_0} \to \mathrm{Aut}^{\mathrm{c}}(\Pi_K)$ *is an isomorphism, thus the pro-ℓ abelian-by-central* I/OM *holds for the category* $\mathcal{V}_X$*.*

2) *Let* k_0 *be arbitrary and* $\dim(X) > 1$*. Then the canonical representation*

$$\imath^{\mathrm{c}}_{\mathcal{V}_X} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}_X})$$

is an isomorphism. In particular, the pro-ℓ abelian-by-central I/OM *holds for the category* $\mathcal{V}_X$*, thus the classical* I/OM *holds too.*

Some major open Questions/Problems:

Q1: Let k be an algebraically closed field of positive characteristic. Can one recover $\mathrm{td}(k)$ from $\pi_1(\mathbb{A}^1_k)$?

Q2: For k as above, give a non tautological description of of $\pi_1(\mathbb{A}^1_k)$.

Q3: For k as above, give a non tautological description of $\pi_1(X)$ and/or $\pi_1^{\mathrm{t}}(X)$ for some hyperbolic curve $X \to k$.

Q4: Give an algebraic proof of the fact that $\pi_1(\mathbb{P}^1_{\mathbb{C}} \backslash \{0, 1, \infty\})$ is generated by inertia elements c_0, c_1, c_∞ over $0, 1, \infty$ with a single relation $c_0 c_1 c_\infty = 1$.

Q5: Give a proof of the Hom-form of Grothendieck's birational anabelian Conjecture in positive characteristic.

Q6: Prove the geometrically pro-ℓ Isom-form and/or the Hom-form of the anabelian Conjecture for curves in positive characteristic.

Q7: Let $x_1 \neq x_0$ in $\mathcal{M}_g \to \mathbb{F}_p$ be such that x_0 is a specialization of x_1. Show that $\mathrm{sp}_{x_1 x_0} : \pi_1(x_1) \to \pi_1(x_0)$ is not an isomorphism.

Q8: Prove the Section Conjecture, say over number fields and/or p-adic fields.

Q9: Prove the AbC Conjecture over arbitrary algebraically closed base fields k.

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