LARGE FIELDS IN DIFFERENTIAL GALOIS THEORY

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Abstract. We solve the inverse differential Galois problem over differential fields with a large field of constants of infinite transcendence degree over \( \mathbb{Q} \). More generally, we show that over such a field, every split differential embedding problem can be solved. In particular, we solve the inverse differential Galois problem and all split differential embedding problems over \( \mathbb{Q}_p(x) \).

Introduction

Large fields play a central role in field arithmetic and modern Galois theory, providing an especially fruitful context for investigating rational points and extensions of function fields of varieties. In this paper we study differential Galois theory over this class of fields.

Differential Galois theory, the analog of Galois theory for linear differential equations, had long considered only algebraically closed fields of constants; but more recently other constant fields have been considered (e.g. see [AM05], [And01], [BHH16], [ChvdP13], [Dyc08], [LSP17]). Results on the inverse differential Galois problem, asking which linear algebraic groups over the constants can arise as differential Galois groups, have all involved constant fields that happen to be large. In this paper, we prove the following result (see Theorem 3.2):

**Theorem A.** If \( k \) is any large field of infinite transcendence degree over \( \mathbb{Q} \), then every linear algebraic group over \( k \) is a differential Galois group over the field \( k(x) \) with derivation \( d/dx \).

In differential Galois theory (as in usual Galois theory), authors have considered embedding problems, which ask whether an extension with a Galois group \( H \) can be embedded into one with group \( G \), where \( H \) is a quotient of \( G \). (For example, see [MvdP03], [Hrt05], [Obe03], [Ern14], [BHHW16], [BHH17].) In order to guarantee solutions, it is generally necessary to assume that the extension is split (i.e., \( G \to H \) has a section). In this paper we prove the following result about split embedding problems over large fields (see Theorem 4.3):

**Theorem B.** If \( k \) is a large field of infinite transcendence degree over \( \mathbb{Q} \), then every split differential embedding problem over \( k(x) \) with derivation \( d/dx \) has a proper solution.

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The class of large fields, which was introduced by Pop [Pop96], includes in particular \( \mathbb{R} \), \( \mathbb{Q}_p \), \( k((t)) \), \( k((s,t)) \), algebraically closed fields, and pseudo-algebraically closed fields. Results about this class of fields thus have broad applications. Moreover, in usual Galois theory, the key properties of large fields turn out to be just what is needed in order to carry out proofs of the inverse problem over function fields of arithmetic curves and to prove that finite split embedding problems over such function fields have proper solutions. We refer the reader to [Pop14] for a further discussion.

Theorem A generalizes a number of known results on the differential inverse Galois problem (e.g., in the cases of \( k \) being algebraically closed, or real, or a field of Laurent series in one variable), as well as yielding other results (e.g., the cases of PAC fields, Laurent series in more than one variable, and the \( p \)-adics). Moreover, we generalize this result further from \( k(x) \) to all differential fields with field of constants \( k \) that are finitely generated over \( k \) (Corollary 3.4). In particular, we solve the inverse differential Galois problem over \( \mathbb{Q}_p(x) \) and more generally, over all differential fields that are finitely generated over \( \mathbb{Q}_p \) and have field of constants \( \mathbb{Q}_p \).

In fact, our proof shows somewhat more. Given a differential field \( k_0 \) of characteristic zero, and a linear algebraic group \( G \) over \( k_0 \), there exists an integer \( n \) such that for any large overfield \( k/k_0 \) of transcendence degree at least \( n \), there is a Picard-Vessiot ring over \( k(x) \) with differential Galois group \( G_k \) (see Theorem 3.2(a)). A similar assertion holds in the situation of Theorem B (see Theorem 4.3(a)). In both cases, the integer \( n \) could in principle be computed from the input data.

Theorems A and B provide differential analogs of results in usual Galois theory about large fields; and our strategy here, like the one there, relies on reducing to the case of Laurent series fields. On the one hand, Laurent series fields are large; on the other hand, any large field \( k \) is existentially closed in the Laurent series field \( k((t)) \). In usual Galois theory, the proof that every finite split embedding problem for \( k(x) \) has a proper solution if \( k \) is large (see [Pop96, Main Theorem A], [HJ98, Theorem C], [HS05, Theorem 4.3]) involved first proving such a result for large fields of the form \( k = k_0((t)) \); and a result in that case (see [Pop96, Lemma 1.4], [HJ98, Proposition B], [HS05, Theorem 4.1]) can be proven by means of patching, due to such fields being complete. In the current manuscript (where we restrict to fields of characteristic zero, as is common in differential Galois theory), we build on [BHH17, Theorem 4.2], where it was shown that proper solutions exist to every split differential embedding problem over \( k_0((t))(x) \) that is induced from a split embedding problem over \( k_0(x) \). (That assertion in turn built on results in [BHH16] and [BHH16], which relied on patching methods.) Since Laurent series fields are large, the main result in this current paper also yields a new result over Laurent series fields, namely that in [BHH17] the hypothesis on the embedding problem being induced from \( k_0(x) \) can be dropped.

As in the case of embedding problems over large fields in usual Galois theory, it is necessary in our main result to assume that the embedding problem is split. In usual Galois theory, this is because in order for all finite embedding problems over \( k(x) \) to have proper solutions, it is necessary by [Ser02, I.3.4, Proposition 16] for \( k(x) \) to have cohomological dimension at most one; and hence for \( k \) to be separably closed (not merely large). In differential Galois theory, every finite regular Galois extension of \( k(x) \) is a Picard-Vessiot ring for a
finite constant group, and so the same reason applies. On the other hand, in usual Galois
theory, every finite embedding problem over \( k(x) \) (even if not split) has a proper solution if
\( k \) is algebraically closed, and in fact has many such solutions in a precise sense; this implies
that the absolute Galois group of \( k(x) \) is free of rank \( \text{card}(k) \) (see \cite{Pop95} and \cite{Har95}).
In the differential situation, it was shown in \cite{BHHW16} Theorem 3.7 that all differential
embedding problems over \( \mathbb{C}(x) \) have proper solutions. The main theorem of the current
paper combined with Proposition 3.6 of \cite{BHHW16} implies that for any algebraically closed
field \( k \) of infinite transcendence degree over \( \mathbb{Q} \), every differential embedding problem over
\( k(x) \) has a proper solution (Corollary 4.5).

This manuscript is organized as follows. Section 1 begins with a short summary of ex-
amples and properties of large fields. Proposition 1.3 in that section, which was proven by
Arno Fehm, states that the function field of a smooth connected variety over a subfield of a
large field can be embedded into that large field under certain hypotheses. This proposition
and its corollary are key to reducing to the case of Laurent series fields in Sections 3 and 4.
Section 2 reviews Picard-Vessiot theory over arbitrary fields of constants and discusses what
it means for a Picard-Vessiot ring to descend to a subfield of the given differential field.
Sections 3 and 4 give the solutions to differential inverse problem and differential embedding
problems, respectively. In each case, the main ingredient is a proposition proving that all
input data is defined over a rational function field over a finitely generated subfield of the
field of constants.

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1. Embeddings into large fields

The aim of this section is to prove that certain subfields of the Laurent series field \( k((t)) \)
\( k \) can be embedded into \( k \) if \( k \) is a large field. We begin by recalling the definition and
basic properties of large fields. For field extensions \( l/k \), we write \( \text{td}(l/k) \) to abbreviate the
transcendence degree.

**Definition 1.1.** A field \( k \) is **large** if for every smooth \( k \)-curve the existence of one \( k \)-rational
point implies the existence of infinitely many such points.

Basic examples of large fields include algebraically closed fields (or more generally, PAC
fields) and fields that are complete with respect to a nontrival absolute value (see e.g. \cite{Pop14}).
In particular, \( \mathbb{C}, \mathbb{R}, \mathbb{Q}_p \) for \( p \) a prime, and the Laurent series field \( k_0((t)) \) over an arbitrary
field \( k_0 \) are all large.

More generally, fraction fields of domains that are Henselian with respect to a non-trivial
ideal are large by \cite{Pop10}, Theorem 1.1. This includes Henselian valued fields (with re-
spect to non-trivial valuations), for example Puiseaux series fields, as well as fraction fields
\( k_0((t_1, \ldots, t_n)) \) of power series rings in several variables. In Remark 1.5 we give more exam-
pies of large fields.

There are a number of characterizations of large fields (see \cite{Pop14}). We list some of them
here for ease of citation.

**Proposition 1.2.** A field \( k \) is large if and only if it satisfies one of the following properties:
(a) Every smooth $k$-curve with a rational point has $\text{card}(k)$ rational points.
(b) Every smooth $k$-variety $X$ satisfies either $X(k) = \emptyset$ or $X(k)$ is dense in $X$.
(c) The field $k$ is existentially closed in its Laurent series field $k((t))$.

In particular, we will use that if $k$ is a large field and $X/k$ is a smooth variety which has a $k((t))$-point, then $X$ has a $k$-point.

The following result was proven in [Feh11]; see Theorem 1 and Lemma 4 there. Below we give another proof, using a different strategy.

**Proposition 1.3.** Let $k$ be a large field, $l \subseteq k$ be a subfield, and $V$ be a smooth connected $l$-variety with function field $L = l(V)$ and $V(k)$ non-empty. Suppose that $\dim(l/k) \geq \dim(V)$. Then the canonical embedding of fields $l \hookrightarrow k$ can be prolonged to an embedding of fields $L \hookrightarrow k$. Equivalently, there exist $k$-rational points dominating the generic point of $V$.

**Proof.** Since $V$ is smooth and connected, it is also integral. Hence the given $k$-rational point is contained in a nonempty (dense) affine open subvariety which is smooth and integral, and we may replace $V$ by that subvariety (which we again call $V$). Let $R := l[V]$ be its coordinate ring; then $L = \text{Frac}(R)$. Given any $k$-point of $\text{Spec}(R)$ (i.e., a point $x \in \text{Spec}(R)$ together with an $l$-algebra map $\iota : \kappa(x) \hookrightarrow k$), let $d_x := \text{td}(\kappa(x)/l)$. Choose $(x, \iota)$ as above such that $d_x$ is maximal; hence $d_x \leq \dim(V)$. It suffices to show that $d_x = \dim(V)$, since then $x$ is the generic point of $V$.

Suppose to the contrary that $d_x < \dim(V)$. Let $u := (u_1, \ldots, u_r)$ be a system of functions in $R$ such that its image $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_r)$ under the reduction map $	ilde{R} \to \kappa(x)$ is a transcendence basis of $\kappa(x)$ over $l$. The composition $l[u] \to R \to \tilde{R} = \kappa(x)$ is injective, hence $l[u] \cap I_x = \{0\}$, where $I_x \triangleleft R$ is the prime ideal defining $x$. Let $l_1 := l(u) = \text{Frac}(l[u])$ and $R_1 := R \otimes_{l[u]} l_1$. The $l$-embedding $R \hookrightarrow R_1$ defines a dominant morphism of schemes $V_1 := \text{Spec} R_1 \hookrightarrow \text{Spec} R = V$, with $V_1$ a smooth $l_1$-variety. Since $\kappa(x)$ is an algebraic field extension of $l_1$, $x \in V$ is the image of a closed point of $V_1$. Hence $\iota : \kappa(x) \hookrightarrow k$ defines a $k$-point $x_1 \in V_1(k)$. Let $\tilde{l}$ be the algebraic closure of $l_1$ in $k$. Since $\text{td}(l_1/l) < \text{td}(L/l) = \dim(V) \leq \text{td}(k/l)$, it follows that $\tilde{l}$ is strictly contained in $k$. Hence by Theorem 3.1, 2) from [Pop4], $V_1$ has a $k$-point that is not an $l_1$-point. The associated point $z \in V_1 = \text{Spec}(R_1)$ is equipped with an $l_1$-embedding $z : \kappa(z) \hookrightarrow k$ whose image is thus not algebraic over $l_1$. Viewing $z$ as a point of $V$ via $V_1 \hookrightarrow V$, we obtain a contradiction because

$$d_z = \text{td}(\kappa(z)/l) = \text{td}(\kappa(z)/l_1) + \text{td}(l_1/l) > \text{td}(l_1/l) = \text{td}(\kappa(x)/l) = d_x.$$ 

\[\square\]

**Corollary 1.4.** Let $k$ be a large field, $k_0 \subseteq k$ and $k_1 \subseteq k_0((t))$ be subfields with $k_0 \subseteq k_1$, $\text{td}(k_1/k_0) \leq \text{td}(k/k_0)$ and $k_1/k_0$ finitely generated. Then there exists a $k_0$-embedding $k_1 \hookrightarrow k$.

In particular, if $k_0 \subseteq k$ are fields such that $k$ is large and $\text{td}(k/k_0)$ is infinite, then for every finitely generated field extension $k_1/k_0$ with $k_1 \subseteq k_0((t))$ there is a $k_0$-embedding $k_1 \hookrightarrow k$.

**Proof.** Let $k_1$ be as in the statement of the corollary. Since $K_0 := k_0((t))$ is separably generated over $k_0$ and $k_0$ is relatively algebraically closed in $K_0$ (that is, $K_0/k_0$ is a regular field extension), it follows that $k_1$ is separably generated over $k_0$ and $k_0$ is relatively algebraically closed in $k_1$ as well. Equivalently, there exists a geometrically integral smooth $k_0$-variety $V$ with $k_0(V) = k_1$. For such a $V$, $V(k_1)$ is non-empty (because it contains the generic point of $V$) and thus $V(K_0)$ is non-empty as well since $K_0 \supseteq k_1$. Therefore, so is $V(K)$, since
\( K = k((t)) \supseteq k_0((t)) = K_0 \). By Proposition 1.2(c), \( k \) is existentially closed in \( K = k((t)) \); and so \( V(k) \) is also non-empty. An application of Proposition 1.3 yields a \( k_0 \)-embedding \( k_1 \hookrightarrow k \) (with \( l \) of loc.cit. replaced by \( k_0 \)). \qed

We conclude this section by providing some examples of large fields of prescribed transcendence degree:

**Remark 1.5.** Let \( \kappa \) be a Henselian valued field with respect to a non-trivial valuation \( \upsilon \). Every relatively algebraically closed subfield \( k \subseteq \kappa \) is henselian with respect to the restriction \( \upsilon|_k \) of \( \upsilon \) to \( k \). Thus if \( \upsilon|_k \) is non-trivial, then \( k \) is large. This gives a recipe to construct large subfields \( k \subseteq \kappa \) of any positive transcendence degree \( d \) bounded by that of \( \kappa \).

Explicit examples are \((d = 0)\) the algebraic \( p \)-adics (i.e., the relative algebraic closure of \( \mathbb{Q} \) in \( \mathbb{Q}_p \)) and \((d = 1)\) the algebraic Laurent series over \( \mathbb{Q} \) (i.e., the algebraic closure of \( \mathbb{Q}(t) \) in \( \mathbb{Q}((t)) \)). An example of countable infinite transcendence degree is the algebraic closure of \( \mathbb{Q}(t, x_1, x_2, \ldots) \) in \( \mathbb{Q}(x_1, x_2, \ldots)((t)) \). By taking a bigger set of variables \( x_i \), we obtain large fields of any given uncountable transcendence degree.

### 2. Picard-Vessiot theory

Let \( F \) be a differential field of characteristic zero with field of constants \( K \). Classically, differential Galois theory over \( F \) is set up under the assumption that \( K \) is algebraically closed; we refer to [vdPS03] for this case. We start this section by recapitulating differential Galois theory over differential fields with arbitrary fields of constants. Details can be found in [Dyc08] and [BHH16].

If \( R \) is a differential ring, we let \( C_R \) denote its ring of constants. The field of constants \( K = C_F \) is relatively algebraically closed in \( F \). Consider a matrix \( A \in F^{n \times n} \) and the corresponding linear differential equation \( \partial(y) = Ay \). A **fundamental solution matrix** for this equation is a matrix \( Y \in \text{GL}_n(R) \) with entries in some differential ring extension \( R/F \) such that \( \partial(Y) = A \cdot Y \); i.e., the columns of the matrix \( Y \) form a fundamental set of solutions. A **Picard-Vessiot ring** for \( \partial(y) = Ay \) is a simple differential ring extension \( R/F \) with \( C_R = K \) such that \( R \) is generated by the entries of a fundamental solution matrix \( Y \in \text{GL}_n(R) \) together with the inverse of its determinant. For short, we write \( R = F[Y, \det(Y)^{-1}] \). It follows from the differential simplicity that \( R \) is an integral domain with \( C_{\text{Frac}(R)} = C_F \):

**Lemma 2.1.** Let \( R \) be a simple differential ring containing \( \mathbb{Q} \). Then \( R \) is an integral domain, its field of constants is a field and \( \text{Frac}(R) \) has the same field of constants.

**Proof.** As in [vdPS03] Lemma 1.17.1], it can be shown that every zero divisor is nilpotent and that the radical ideal is a differential ideal (see also [Dyc08] Lemma 2.2]). Hence \( R \) is an integral domain. If \( x \in \text{Frac}(R) \) is constant, then \( I = \{a \in R \mid ax \in R\} \) is a non-zero differential ideal in \( R \) and thus \( 1 \in I \) and \( x \in R \). Hence \( C_{\text{Frac}(R)} = C_R \) and in particular, \( C_R \) is a field.

Conversely, if \( R \) is generated by a fundamental solution matrix, then differential simplicity follows from being an integral domain together with \( C_{\text{Frac}(R)} = F \). If \( C_F \) is algebraically closed, this is a well-known criterion. For arbitrary fields of constants it was proven in [Dyc08] Cor. 2.7]:
Proposition 2.2. Let $L/F$ be an extension of differential fields with $C_L = C_F$ and consider a matrix $A \in F^{n \times n}$. Assume that there exists a matrix $Y \in \text{GL}_n(L)$ with $\partial(Y) = AY$. Then $R = [Y, \det(Y)^{-1}] \subseteq L$ is a Picard-Vessiot ring for the differential equation $\partial(y) = Ay$.

The differential Galois group $\text{Aut}^\partial(R/F)$ of a Picard-Vessiot ring $R/F$ is defined as the functor $G$ from the category of $K$-algebras to the category of groups, defined by $G(S) := \text{Aut}^\partial(R \otimes_K S/F \otimes_K S)$, the group of $(F \otimes_K S)$-linear automorphisms of $(R \otimes_K S)$. Here, we equip the $K$-algebra $S$ with the trivial derivation. It can be shown that $\text{Aut}^\partial(R/F)$ is represented by the $K$-Hopf algebra $C_{R \otimes_K F} = K[\{Y^{-1} \otimes Y\}, \det(Y^{-1} \otimes Y)^{-1}]$, where $(Y^{-1} \otimes Y)$ is an abbreviation for the matrix product $(Y^{-1} \otimes 1) \cdot (1 \otimes Y)$. Hence the functor $\text{Aut}^\partial(R/F)$ is an affine group scheme of finite type over $K$, and thus (since $\text{char}(K) = 0$) a linear algebraic group over $K$. Note that if $K$ is algebraically closed, then $\text{Aut}^\partial(R/F)$ is determined by its $K$-rational points $\text{Aut}^\partial(R/F)(K) = \text{Aut}^\partial(R/F)$, the group of $F$-linear differential automorphisms of $R$, which is classically the definition of the differential Galois group. However, if $K$ is not algebraically closed, then $\text{Aut}^\partial(R/F)$ does not contain enough information on $\text{Aut}^\partial(R/F)$ in general.

In the remainder of this section, we study the behavior of Picard-Vessiot rings under extensions of the constants. The following is a well-known statement in differential algebra; see for example [Mau10] Lemma 10.7] for a proof.

Lemma 2.3. Let $R$ be a simple differential ring with field of constants $K$ and let $S$ be a $K$-algebra which we equip with the trivial derivation. Then there is a bijection between the differential ideals in $R \otimes_K S$ and the ideals in $S$, given by $I \mapsto I \cap S$ for differential ideals in $R \otimes_K S$ and $J \mapsto R \otimes_K J$ for ideals in $S$. In particular, if $S$ is a field then $R \otimes_K S$ is a simple differential ring.

If $G$ is a linear algebraic group over a field $K$ and $K'/K$ is a field extension, we let $G_{K'}$ denote the base change of $G$ from $K$ to $K'$. If $K'/K$ is algebraic and $R/F$ is a Picard-Vessiot ring with differential Galois group $G$, then $F' = F \otimes_K K'$ is a differential field extension of $F$ and $R \otimes_K K'$ is a Picard-Vessiot ring over $F'$ with differential Galois group $G_{K'}$. Indeed, since $K$ is algebraically closed in $F$, $F \otimes_K K'$ is an integral domain and as $K'/K$ is algebraic, it is an algebraic field extension of $F$; thus the derivation extends uniquely to $F'$ (with field of constants $C_{F'} = K'$). Moreover, $R \otimes_K K'$ is generated over $F'$ by the same fundamental solution matrix as $R/F$, $C_{R \otimes_K K'} = K \otimes_K K' = C_{F'}$ and $R \otimes_K K'$ is a simple differential ring by Lemma 2.3. Finally, $\text{Aut}^\partial(R \otimes_K K'/F') = \text{Aut}^\partial(R/F)_{K'}$ is immediate from the definition.

If $K'/K$ is a non-algebraic field extension, $F \otimes_K K'$ is not a field but merely an integral domain and it is slightly more complicated to extend the constants from $K$ to $K'$. We consider the differential field extension $F' = \text{Frac}(F \otimes_K K')$ of $F$. The following proposition shows that if $R/F$ is a Picard-Vessiot ring with differential Galois group $G$, $R \otimes_F F'$ is a Picard-Vessiot ring over $F'$ with differential Galois group $G_{K'}$. Note that this generalizes the construction in the case when $K'/K$ is algebraic: If $K'/K$ is algebraic, then $\text{Frac}(F \otimes_K K') = F \otimes_K K'$ and $R \otimes_F F' \cong R \otimes_K K'$.

Proposition 2.4. Let $F$ be a field of characteristic zero with field of constants $K$ and let $R/F$ be a Picard-Vessiot ring with differential Galois group $G$. Let $K'/K$ be a field extension
and define $F' = \frac{\text{Frac}(F \otimes_K K')}{F'}$ and $R' = R \otimes_F F'$. Then $F'$ is a differential field extension of $F$ with $C_{F'} = K'$ and $R'$ is a Picard-Vessiot ring over $F'$ with Galois group $G_{K'}$.

**Proof.** The derivation on $F$ extends canonically to $F \otimes_K K'$ and hence to $F'$ by considering elements in $K'$ as constants. By Lemma 2.3, $F \otimes_K K'$ is a simple differential ring and thus we can apply Lemma 2.1 to obtain $C_{F'} = C_{F \otimes_K K'} = K'$.

Since $R/F$ is a Picard-Vessiot ring, there exists a differential equation $\partial(y) = Ay$ over $F$ and a fundamental solution matrix $Y \in \text{GL}_n(R)$ with $R = F[Y, \det(Y)^{-1}]$. We identify $R$ with a subring of $R'$ and obtain $R' = F'[Y, \det(Y)^{-1}]$. Let $S$ denote the set of non-zero elements in $F \otimes_K K'$. Then $R' = R \otimes_F F' = R \otimes_F S^{-1}(F \otimes_K K') = S^{-1}(R \otimes_F (F \otimes_K K'))$.

Hence

$$R' = S^{-1}(R \otimes_K K') \text{ and } \text{Frac}(R') = \text{Frac}(R \otimes_K K'),$$

where we identified $R \otimes_K K'$ with the subring $R \otimes_F (F \otimes_K K')$ of $R'$. As $R$ is simple, $R \otimes_K K'$ is simple by Lemma 2.3 and has field of constants $K'$. It follows from Lemma 2.1 that $\text{Frac}(R') = \text{Frac}(R \otimes_K K')$ has field of constants $K' = C_{F'}$. Therefore, $R'$ is a Picard-Vessiot ring over $F'$ by Proposition 2.2 (applied to $L = \text{Frac}(R')$).

Let $G'$ denote the differential Galois group of $R'/F'$. We claim that $G' = G_{K'}$. For every $K'$-algebra $S$, there is an injective group homomorphism

$$G_{K'}(S) = \text{Aut}^0(R \otimes_K S/F \otimes_K S) \to G'(S) = \text{Aut}^0(R' \otimes_{K'} S/F' \otimes_{K'} S)$$

using that $R' \otimes_{K'} S$ is a localization of $R \otimes_K S$. Conversely, every $\gamma \in G'(S)$ restricts to an injective differential homomorphism $R' \otimes_{K'} S \to R' \otimes_K S$. The matrix $B = Y^{-1} \gamma(Y) \in \text{GL}_n(R' \otimes_K S)$ has constant entries and is thus contained in $\text{GL}_n(S)$. Therefore, $\gamma(Y) = YB$ is contained in $R \otimes_K S$. Since $R = F[Y, \det(Y)^{-1}]$, we conclude that $\gamma(R \otimes_K S) = R \otimes_K S$. Thus $\gamma$ restricts to an element in $G_{K'}(S)$. Hence the homomorphism $G_{K'}(S) \to G'(S)$ is a bijection and it defines an isomorphism of linear algebraic groups $G_{K'} \to G'$.

We record a special case here for later use.

**Corollary 2.5.** Let $k(x)$ be a rational function field of characteristic zero equipped with the derivation $d/dx$ and let $K/k$ be a field extension. If $R/k(x)$ is a Picard-Vessiot ring with differential Galois group $G$ then $R \otimes_{k(x)} K(x)$ is a Picard-Vessiot ring over $K(x)$ with differential Galois group $G_K$.

**Proof.** Since $K(x) = \text{Frac}(k(x) \otimes_k K)$, the claim follows from Proposition 2.4. □

**Definition 2.6.** For $F$ and $F'$ as in Proposition 2.4 we say that a Picard-Vessiot ring $R'/F'$ descends to a Picard-Vessiot ring over $F$ if there exists a Picard-Vessiot ring $R/F$ together with an $F'$-linear differential isomorphism $R \otimes_F F' \cong R'$.

In particular, a Picard-Vessiot ring $R$ over $K(x)$ descends to a Picard-Vessiot ring over $k(x)$ if there exists a Picard-Vessiot ring $R_0/k(x)$ together with a $K(x)$-linear differential isomorphism $R \cong R_0 \otimes_{k(x)} K(x)$. □
3. The inverse differential Galois problem

The aim of the next proposition is to show that the data associated to a Picard-Vessiot ring over a rational function field over some field \( k \) is in fact already given over the rational function field over a finitely generated subfield of \( k \). This is technical but not surprising since all related objects (the linear algebraic group as well as the Picard-Vessiot ring itself) are finitely generated. An analogous result for embedding problems can be found in the next section (Proposition 4.2 below).

**Proposition 3.1.** Let \( F = K(x) \) be a rational function field of characteristic zero with derivation \( \partial = d/dx \) and let \( R/F \) be a Picard-Vessiot ring with differential Galois group \( G \). Let further \( k_0 \subseteq K \) be a subfield and let \( G_0 \) be a linear algebraic group over \( k_0 \) with \( (G_0)_K = G \). Then there is a finitely generated field extension \( k_1/k_0 \) with \( k_1 \subseteq K \) such that \( R/K(x) \) descends to a Picard-Vessiot ring \( R_1/k_1(x) \) with differential Galois group \( (G_0)_{k_1} \).

**Proof.** As \( R \) is a finitely generated \( F \)-algebra, we can write \( R \) as a quotient of a polynomial ring \( F[X_1, \ldots, X_r] \) by an ideal \( J \). We fix generators \( g_1, \ldots, g_m \) of \( J \):

\[
R = K(x)[X_1, \ldots, X_r]/(g_1, \ldots, g_m).
\]

We fix an extension of \( \partial \) from \( F \) to \( F[X_1, \ldots, X_r] \) such that this derivation induces the given derivation on \( R \). In particular, \( J \) is a differential ideal in \( K(x)[X_1, \ldots, X_r] \). We can now choose a finitely generated field extension \( k/k_0 \) with \( k \subseteq K \) such that

1. \( g_i \in k(x)[X_1, \ldots, X_r] \) for all \( i = 1, \ldots, m \), and
2. \( \partial(X_i) \in k(x)[X_1, \ldots, X_r] \) for all \( i = 1, \ldots, r \) and
3. \( R = K(x)[Y, \det(Y)^{-1}] \) for a fundamental solution matrix \( Y \in \text{GL}_n(R) \) with the property that all entries of \( Y \) have representatives in \( k(x)[X_1, \ldots, X_r] \), and
4. the element in \( R \) represented by \( X_i \) can be written as a polynomial expression over \( k(x) \) in the entries of \( Y \) and \( \det(Y)^{-1} \) for all \( i = 1, \ldots, r \).

Property (2) implies that \( k(x)[X_1, \ldots, X_r] \) is a differential subring of \( K(x)[X_1, \ldots, X_r] \). Set \( I = J \cap k(x)[X_1, \ldots, X_r] \). Then \( I \) is a differential ideal in \( k(x)[X_1, \ldots, X_r] \) and it contains \( g_1, \ldots, g_m \) by (1). As \( K(x)/k(x) \) is faithfully flat, \( I \) is thus generated by \( g_1, \ldots, g_m \). We define \( R_1 = k(x)[X_1, \ldots, X_r]/I \). Hence

\[
R_1 = k(x)[X_1, \ldots, X_r]/(g_1, \ldots, g_m)
\]

is a differential ring and as \( K(x) \) is flat over \( k(x) \), there is a \( K(x) \)-linear isomorphism of differential rings

\[
R_1 \otimes_{k(x)} K(x) \cong R.
\]

Let \( c \in C_{R_1} \). As \( C_R = K \), there exists an \( a \in K \) such that we have \( c \otimes 1 = 1 \otimes a \) in \( R_1 \otimes_{k(x)} K(x) \). Thus \( a \in k(x) \) and \( c = a \in k \). Hence \( C_{R_1} = k \).

Next, consider a non-zero differential ideal \( I_1 \subseteq R_1 \). Then \( J_1 = I_1 \otimes_{k(x)} K(x) \) is a non-zero differential ideal in \( R_1 \otimes_{k(x)} K(x) \cong R \), and as \( R \) is a simple differential ring, we conclude \( 1 \in J_1 \). As \( K(x)/k(x) \) is faithfully flat, \( R_1 \otimes_{k(x)} K(x) \) is faithfully flat over \( R_1 \) and therefore \( I_1 = J_1 \cap R_1 \). Hence \( 1 \in I_1 \) and we conclude that \( R_1 \) is a simple differential ring.

Finally, (3) implies that the matrix \( Y \) has entries in the subring \( R_1 \) of \( R \). Its determinant \( \det(Y) \in R_1 \) is a unit when considered as an element in \( R_1 \otimes_{k(x)} K(x) \) and thus \( \det(Y) \)
is invertible in $R_1$, so $Y \in \text{GL}_n(R_1)$. Set $A = \partial(Y)Y^{-1}$. As $Y$ is a fundamental solution matrix for $R/K(x)$, $A$ has entries in $K(x)$. On the other hand, $Y \in \text{GL}_n(R_1)$ implies that the entries of $A$ are contained in $R_1$. Hence $A$ has entries in $R_1 \cap K(x) = k(x)$ and thus $Y$ is a fundamental solution matrix for a differential equation over $k(x)$. Furthermore, $R_1 = k(x)[Y, \det(Y)^{-1}]$ by (4). Hence $R_1$ is a Picard-Vessiot ring over $k(x)$.

Let $G_1$ be the differential Galois group of $R_1/k(x)$. Then $G_1$ is a linear algebraic group over $k$ and $(G_1)_k = G$ by Corollary 2.5. Therefore, $(G_1)_k = ((G_0)_k)_k$, and hence there exists a finite extension $k_1/k$ with $(G_1)_{k_1} = (G_0)_{k_1}$ and we conclude that $R$ descends to the Picard-Vessiot ring $R_1 \otimes_{k(x)} k_1(x)$ over $k_1(x)$ with differential Galois group $(G_0)_{k_1}$ by Corollary 2.5.

**Theorem 3.2.**

(a) Let $k_0$ be a field of characteristic zero, and let $G$ be a linear algebraic group over $k_0$. Then there exists a constant $c_G \in \mathbb{N}$, depending only on $G$, with the following property: For all large fields $k$ with $k_0 \subseteq k$ and $\text{td}(k/k_0) \geq c_G$; $G_k$ is a differential Galois group over $(k(x), \frac{d}{dx})$.

(b) If $k$ is a large field of infinite transcendence degree over $\mathbb{Q}$, then every linear algebraic $k$-group is a differential Galois group over $k(x)$ endowed with $\partial = d/dx$.

**Proof.** Let $K := k_0((t))$ be the Laurent series field over $k_0$. Then $\partial = d/dx$ extends from $k(x)$ to $K(x)$ and by [BHH16] Thm. 4.5, there exists a Picard-Vessiot ring $R/K(x)$ with differential Galois group $G_K$. Then by Proposition 3.1 there exists a finitely generated field extension $k_1/k_0$ with $k_1 \subseteq K$ such that $R/K(x)$ descends to a Picard-Vessiot ring $R_1/k_1(x)$ with differential Galois group $G_{k_1}$. Set $c_G := \text{td}(k_1/k_0)$.

Let $k$ be a large field with $k_0 \subseteq k$ and $\text{td}(k/k_0) \geq c_G$. Then by Corollary 1.4 there exists a $k_0$-embedding $k_1 \hookrightarrow k$. To conclude the proof of (a), we can now base change $R_1$ to $R_1 \otimes_{k_1(x)} k(x)$, and obtain a Picard-Vessiot ring over $k(x)$ with differential Galois group $(G_{k_1})_k = G_k$ by Corollary 2.5.

The proof of assertion (b) follows easily from (a), by noticing that every linear algebraic $k$-group $G$ descends to a subfield $k_0 \subseteq k$, which is finitely generated over $\mathbb{Q}$. □

**Remark 3.3.** In principle, the bound $c_G$ in Theorem 3.2 above can be computed from the input data.

By [BHH16] Cor. 4.14 (this is an adaption of a trick due to Kovacic), this result extends from the rational function field $k(x)$ to all finitely generated field extensions with arbitrary derivations that have field of constants $k$:

**Corollary 3.4.** Let $k$ be large field of infinite transcendence degree over $\mathbb{Q}$. Let $F$ be a differential field with a non-trivial derivation and field of constants $k$. If $F/k$ is finitely generated, then every linear algebraic group over $k$ is a differential Galois group over $F$.

This result in particular applies if the field of constants $k$ is $\mathbb{Q}_p$ (or, more generally, a Henselian valued field of infinite transcendence degree) or if $k$ is the fraction field $k_0((t_1, \ldots, t_n))$ of a power series ring in several variables.
4. Differential embedding problems

In this section, we solve split differential embedding problems over \( k(x) \) for large fields \( k \) of infinite transcendence degree. To this end, we work with differential torsors, which were introduced in [BHHW16]. Let \( F \) be a differential field of characteristic zero with field of constants \( K \) and let \( G \) be a linear algebraic group over \( K \). We equip its coordinate ring \( K[G] \) with the trivial derivation, hence \( F[G_F] = F \otimes_K K[G] \) is a differential ring extension of \( F \). We write \( F'[G] = F[G_F] \). A differential \( G_F \)-torsor is a \( G_F \)-torsor \( X = \text{Spec}(R) \) such that \( R \) is a differential ring extension of \( F \) and such that the co-action \( \rho: R \to R \otimes_F F'[G] \) is a differential homomorphism. A morphism of differential \( G_F \)-torsors \( \varphi: X \to Y \) is a morphism of \( G_F \)-torsors (i.e., a \( G_F \)-equivariant morphism of varieties) such that the corresponding homomorphism \( F[Y] \to F[X] \) is a differential homomorphism.

If \( \text{Spec}(R) \) is a differential \( G_F \)-torsor and \( H \) is a closed subgroup of \( G \), the ring of invariants is defined as \( R^{H_F} = \{ r \in R \mid \rho(r) = r \otimes 1 \} \). If \( N \) is a normal closed subgroup of \( G \), then \( \text{Spec}(R^{N_F}) \) is a differential \((G/N)_F\)-torsor and the co-action \( R^{N_F} \to R^{N_F} \otimes_F F'[G/N] = R^{N_F} \otimes_F F[G]^N \) is obtained from restricting the co-action \( \rho: R \to R \otimes_F F'[G] \) (see Prop. 1.17 together with Prop. A.6(b) in [BHHW16]).

By Kolchin’s theorem, if \( R/F \) is a Picard-Vessiot ring with differential Galois group \( G \), then \( \text{Spec}(R) \) is a \( G_F \)-torsor. The co-action \( \rho: R \to R \otimes_F F'[G] \) can be described explicitly as follows. Let \( Y \in \text{GL}_n(R) \) be a fundamental solution matrix, i.e., \( R = F[Y, \det(Y)^{-1}] \). Recall that \( K[G] = C_{R \otimes_F R} \) is generated by the entries of the matrix \( Y^{-1} \otimes Y \) and its inverse. Then \( \rho \) is determined by setting \( \rho(Y) = Y \otimes (Y^{-1} \otimes Y) \). Conversely, if \( X = \text{Spec}(R) \) is a differential \( G_F \)-torsor with the property that \( R \) is a simple differential ring and \( C_R = K \), then \( R \) is a Picard-Vessiot ring over \( F \) with differential Galois group \( G \) ([BHHW16 Prop. 1.12]).

**Lemma 4.1.** Let \( K/k \) be a field extension in characteristic zero and let \( F_1 \) be a differential field with field of constants \( k \). We equip \( K \) with the trivial derivation and set \( F = \text{Frac}(F_1 \otimes_k K) \). Let further \( G \) be a linear algebraic group over \( k \). Assume that we are given a Picard-Vessiot ring \( R/F \) with differential Galois group \( G_K \) which descends to a Picard-Vessiot ring \( R_1/F_1 \) with differential Galois group \( G \). Then the following holds.

(a) The co-action \( \rho: R \to R \otimes_F F'[G] \) restricts to the co-action \( \rho_1: R_1 \to R_1 \otimes_{F_1} F_1[G] \).

(b) For every closed subgroup \( H \) of \( G \), the isomorphism \( R_1 \otimes_{F_1} F_1 \cong R \) restricts to an isomorphism \( R_1^{H_{F_1}} \otimes_{F_1} F_1 \cong R^{H_F} \).

**Proof.** Let \( Y \in \text{GL}_n(R_1) \) be a fundamental solution matrix, i.e., \( R_1 = F_1[Y, \det(Y)^{-1}] \). As \( R \) descends to \( R_1 \), there is a differential isomorphism \( R_1 \otimes_{F_1} F_1 \cong R \) over \( F_1 \). Hence after identifying \( R_1 \) with a subring of \( R \), we obtain an equality \( R = F[Y, \det(Y)^{-1}] \). Define \( Z = Y^{-1} \otimes Y \in \text{GL}_n(R \otimes_F R) \subseteq \text{GL}_n(R \otimes_F R) \). Recall that \( F_1[G] = F_1[Z, \det(Z)^{-1}] \) and the co-action \( \rho_1: R_1 \to R_1 \otimes_{F_1} F_1[G] \) is given by \( Y \mapsto Y \otimes Z \). Similarly, the co-action \( \rho: R \to R \otimes_F F'[G] \) is given by \( Y \mapsto Y \otimes Z \). Hence \( \rho = \rho_1 \otimes_{F_1} F \) and (a) follows.

The \( H \)-invariants are defined as \( R^H = \{ f \in R \mid \rho(f) = f \otimes 1 \} \) and so the equality \( \rho = \rho_1 \otimes_{F_1} F \) implies (b). \( \square \)
A split differential embedding problem \((N \ltimes H, S)\) over \(F\) consists of a semidirect product \(N \ltimes H\) of linear algebraic groups over \(K\) together with a Picard-Vessiot ring \(S/F\) with differential Galois group \(H\). A proper solution of \((N \ltimes H, S)\) is a Picard-Vessiot ring \(R/F\) with differential Galois group \(N \ltimes H\) and an embedding of differential rings \(S \subseteq R\) such that the following diagram commutes:

\[
\begin{array}{ccc}
N \ltimes H & \xrightarrow{\cong} & H \\
\downarrow & & \downarrow \\
\text{Aut}^\partial(R/F) & \xrightarrow{\text{res}} & \text{Aut}^\partial(S/F)
\end{array}
\]

Equivalently, \(R\) is a Picard-Vessiot ring with differential Galois group \(N \ltimes H\) such that there exists an isomorphism of differential \(H_F\)-torsors \(\text{Spec}(S) \cong \text{Spec}(R^{N_F})\) ([BHHW16 Lemma 2.8]).

**Proposition 4.2.** Let \(F = K(x)\) be a rational function field of characteristic zero with derivation \(\partial = d/dx\) and let \(k_0 \subseteq K\) be a subfield. Let \((N_0 \ltimes H_0, S_0)\) be a split differential embedding problem over \(k_0(x)\) and assume that there exists a proper solution \(R\) of the induced differential embedding problem \(((N_0)_K \ltimes (H_0)_K, S_0 \otimes_{k_0(x)} K(x))\) over \(K(x)\). Then there exists a finitely generated field extension \(k_1/k_0\) with \(k_1 \subseteq K\) such that the following holds: \(R/K(x)\) descends to a Picard-Vessiot ring \(R_1/k_1(x)\) that is a proper solution of the split differential embedding problem \(((N_0)_{k_1} \ltimes (H_0)_{k_1}, S_0 \otimes_{k_0(x)} k_1(x))\) over \(k_1(x)\).

**Proof.** We define \(N = (N_0)_K, H = (H_0)_K, S = S_0 \otimes_{k_0(x)} K(x)\) and further \(G = N \ltimes H\) and \(G_0 = N_0 \ltimes H_0\), hence \((G_0)_K = G\). By Proposition 3.1 there exists a finitely generated extension \(k_1/k_0\) with \(k_1 \subseteq K\) such that \(R\) descends to a Picard-Vessiot ring \(R_1/k_1(x)\) with differential Galois group \((G_0)_{k_1}\). Therefore, we can write \(R = K(x)[X_1, \ldots, X_r]/I\) and \(R_1 = k_1(x)[X_1, \ldots, X_r]/I_1\), for some polynomial ring \(K(x)[X_1, \ldots, X_r]\) with a suitable derivation that restricts to \(k_1(x)[X_1, \ldots, X_r]\) and some differential ideal \(I\) that is generated by its contraction \(I_1 = I \cap k_1(x)[X_1, \ldots, X_r]\). Similarly, we can write \(S_0 = k_0(x)[Y_1, \ldots, Y_s]/J_0\), \(S = K(x)[Y_1, \ldots, Y_s]/J\) with \(J = J_0 \otimes_{k_0(x)} K(x)\). We define \(S_1 = S_0 \otimes_{k_0(x)} k_1(x)\). Then \(S_1 = k_1(x)[Y_1, \ldots, Y_s]/J_1\) with \(J_1 = J_0 \otimes_{k_0(x)} k_1(x)\). Since \(K(x)/k_1(x)\) is faithfully flat, \(J_1 = J \cap k_1(x)[Y_1, \ldots, Y_s]\). Let

\[
\varphi: S \rightarrow R^{N_{K(x)}}
\]

be the given isomorphism of \(H_{K(x)}\)-torsors. After passing from \(k_1\) to a finitely generated extension, we may assume that

1. \(\varphi\) maps the elements in \(S = K(x)[Y_1, \ldots, Y_s]/J\) represented by \(Y_1, \ldots, Y_s\) to elements in \(R = K(x)[X_1, \ldots, X_r]/I\) that are represented by elements in \(k_1(x)[X_1, \ldots, X_r]\)
2. \(R^{N_{K(x)}}\) is generated as a \(K(x)\)-algebra by finitely many elements \(\alpha_1, \ldots, \alpha_m \in R = K(x)[X_1, \ldots, X_r]/I\) with the property that all \(\alpha_1, \ldots, \alpha_m\) are represented by elements in \(k_1(x)[X_1, \ldots, X_r]\)
3. for \(i = 1, \ldots, m, \alpha_i = \varphi(\beta_i)\) for an element \(\beta_i \in S = K(x)[Y_1, \ldots, Y_s]/J\) that is represented by an element in \(k_1(x)[Y_1, \ldots, Y_s]\).
For the sake of simplicity, we will write expressions such as $N_{k_1(x)}$, $H_{k_1(x)}$ meaning $(N_0)_{k_1(x)}$, $(H_0)_{k_1(x)}$. We will also write expressions such as $k_1[G]$, $k_1[H]$ meaning $k_1[G_0]$ and $k_1[H_0]$, respectively.

Property (1) implies $\varphi(S_1) \subseteq R_1 \cap R^{N_{k_1(x)}}$ and as $R_1 \cap R^{N_{k_1(x)}} = R_1^{N_{k_1(x)}}$ by Lemma 4.1(a) we conclude that $\varphi$ restricts to an injective differential homomorphism

$$\varphi_1 : S_1 \to R_1^{N_{k_1(x)}}.$$ 

It remains to show that $\varphi_1$ is an isomorphism of $H_{k_1(x)}$-torsors.

We claim that $R_1^{N_{k_1(x)}} = k_1[\alpha_1, \ldots, \alpha_m]$. Since $R_1 \cap R^{N_{k_1(x)}} = R_1^{N_{k_1(x)}}$, Property (2) implies that $\alpha_i$ is contained in $R_1^{N_{k_1(x)}}$ for all $i$, and hence $R_1^{N_{k_1(x)}} \supseteq k_1[\alpha_1, \ldots, \alpha_m]$. On the other hand, $\alpha_1, \ldots, \alpha_m$ generate $R^{N_{k_1(x)}}$, that is,

$$R^{N_{k_1(x)}} = k_1[\alpha_1, \ldots, \alpha_m] \otimes_{k_1(x)} K(x).$$

By Lemma 4.1(b) we also have an equality $R^{N_{k_1(x)}} = R_1^{N_{k_1(x)}} \otimes_{k_1(x)} K(x)$ and thus

$$R_1^{N_{k_1(x)}} \otimes_{k_1(x)} K(x) = k_1[\alpha_1, \ldots, \alpha_m] \otimes_{k_1(x)} K(x)$$

and we conclude

$$R_1^{N_{k_1(x)}} = k_1[\alpha_1, \ldots, \alpha_m].$$

Therefore, Property (3) implies that $\varphi_1$ is surjective. Finally, since $\varphi$ is $H_{k_1(x)}$-equivariant, we conclude that its restriction is $H_{k_1(x)}$-equivariant, where we use Lemma 4.1(a) together with the fact that the co-action of $H_{k_1(x)}$ on $R^{N_{k_1(x)}}$ is given by restricting $R \to R \otimes_F F[G]$ to $R^{N_F} \to R^{N_F} \otimes_F F[G]^{N_F} = R^{N_F} \otimes_F F[H]$. \hfill \Box

**Theorem 4.3** (Main theorem).

(a) Let $k_0$ be a field of characteristic zero, and let $E = (N_0 \times H_0, S_0)$ be a split differential embedding problem over $(k_0(x), \frac{d}{dx})$. Then there is a constant $c_E \in \mathbb{N}$, depending only on $E$, with the following property: For all large fields $k$ with $k_0 \subseteq k$ and $\text{td}(k/k_0) \geq c_E$, the induced differential embedding problem $((N_0)_k \times (H_0)_k, S_0 \otimes_{k_0(x)} k(x))$ over the differential field $(k(x), \frac{d}{dx})$ has a proper solution.

(b) If $k$ is a large field of infinite transcendence degree over $\mathbb{Q}$, then every split differential embedding problem over the differential field $(k(x), \frac{d}{dx})$ has a proper solution.

**Proof.** Set $G_0 = N_0 \rtimes H_0$. We define $K = k_0((t))$ and endow $K(x)$ with the derivation $d/dx$. Then $\hat{S} = S_0 \otimes_{k_0(x)} K(x)$ is a Picard-Vessiot ring over $k(x)$ with differential Galois group $(H_0)_K$ by Corollary 2.5. By BHH17, the split embedding problem $((N_0)_K \times (H_0)_K, \hat{S})$ has a proper solution, i.e., there exists a Picard-Vessiot ring $\hat{R}/K(x)$ with differential Galois group $(G_0)_K$ such that $\hat{R}(N_0)_K(x)$ and $\hat{S}$ are isomorphic as differential $H_{k_1(x)}$-torsors.

Then by Proposition 4.2 there exists a finitely generated field extension $k_1/k_0$ with $k_1 \subseteq K = k_0((t))$ with the property that $\hat{R}$ descends to a Picard-Vessiot ring $R_1/k_1(x)$ with differential Galois group $(G_0)_{k_1}$, and such that $R_1^{(N_0)_{k_1(x)}}$ and $S_0 \otimes_{k_0(x)} k_1(x)$ are isomorphic as differential $(H_0)_{k_1(x)}$-torsors. Set $c_E := \text{td}(k_1/k_0)$.

Now suppose that $k$ is a large field with $k_0 \subseteq k$ and $\text{td}(k/k_0) \geq c_E$. Set $N = (N_0)_k$, $H = (H_0)_k$, $G = (G_0)_k$ and $S = S_0 \otimes_{k_0(x)} k(x)$. We claim that the embedding problem
\((N \times H, S)\) over \(k(x)\) has a proper solution. By Corollary 4.4 there exists a \(k_0\)-embedding \(k_1 \hookrightarrow k\) and hence we can define \(R = R_1 \otimes_{k_1(x)} k(x)\). Then \(R\) is a Picard-Vessiot ring over \(k(x)\) with differential Galois group \((G_0)_{k_1} = (G_0)_k = G\) by Corollary 2.5. The isomorphism \(R_{1(N_0)k_1(x)} \cong S_0 \otimes_{k_0(x)} k_1(x)\) of differential \((H_0)_{k_1(x)}\)-torsors gives rise to an isomorphism \(R_{N_{k(x)}} \cong S_0 \otimes_{k_0(x)} k(x)\) of differential \(H_{k(x)}\)-torsors by base change from \(k_1(x)\) to \(k(x)\), where the equality \(R_{1(N_0)k_1(x)} \otimes_{k_1(x)} k(x) = R_{N_{k(x)}}\) follows from Lemma 4.1(b) and \(H_{k(x)}\)-equivariance follows from Lemma 4.1(a). As \(S_0 \otimes_{k_0(x)} k(x) = S\), we obtain an isomorphism of \(H_{k(x)}\)-torsors \(R_{N_{k(x)}} \cong S\). Hence \(R\) solves the embedding problem \((N \times H, S)\) over \(k(x)\) which concludes the proof of (a).

Assertion (b) follows from (a) as follows: Let \((N \times H, S)\) be a split differential embedding problem over \(k(x)\), i.e., \(G = N \times H\) is a linear algebraic group over \(k\) and \(S/K(x)\) is a given Picard-Vessiot ring with differential Galois group \(H\). We fix a finitely generated field extension \(k_0/\mathbb{Q}\) with \(k_0 \subseteq k\) such that \(G\) and its structure of a semidirect product descends to a linear algebraic group \(G_0 = N_0 \times H_0\) over \(k_0\). By Proposition 3.1, we may in addition choose \(k_0\) such that \(S\) descends to a Picard-Vessiot ring \(S_0\) over \(k_0(x)\) with differential Galois group \(H_0\), i.e., \(S_0 \otimes_{k_0(x)} k(x) \cong S\). We conclude the proof by applying part (a) of the theorem.

Remark 4.4. In principle, the bound \(c_\mathcal{E}\) in Theorem 4.3(a) can be computed from the input data.

Corollary 4.5. Let \(k\) be an algebraically closed field of infinite transcendence degree over \(\mathbb{Q}\). Then every differential embedding problem defined over \(k(x)\) has a proper solution.

Proof. According to \cite{BHHW16} Proposition 3.6], if \(F\) is a one-variable differential function field over an algebraically closed field of constants \(k\), and if every split differential embedding problem over \(F\) has a proper solution, then every differential embedding problem over \(F\) has a proper solution. Using this, the corollary then follows immediately from Theorem 4.3.

References


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