ON THE BIRATIONAL ANABELIAN PROGRAM INITIATED BY BOGOMOLOV (I)

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1. Introduction

Let ℓ be a fixed rational prime number. Consider function fields $K|k$ over algebraically closed fields k of characteristic $\neq \ell$. For each such a function field $K|k$, let $\Pi^{\rm c}_K := {\rm Gal}(K''|K)$ be the Galois group of a maximal pro-ℓ abelian-by-central Galois extension $K''|K$, and $\Pi_K = {\rm Gal}(K'|K)$ be the Galois group of the maximal pro-ℓ abelian sub-extension $K'|K$ of $K''|K$. At the beginning of the 1990's, Bogomolov [1] initiated a program which has as ultimate goal to recover function fields $K|k$ with ${\rm td}(K|k) > 1$ as above from $\Pi^{\rm c}_K$. (Note that Bogomolov denotes $\Pi^{\rm c}_K$ by ${\rm PGal}^c_K$.) If successful, this program would go far beyond Grothendieck's birational anabelian conjectures,[1] as k being algebraically closed, there is no arithmetical action in the game. The program is far from being completed, and the present manuscript settles that program in the case k is an algebraic closure of a finite field.

Theorem I. *Let $K|k$ be a function field with ${\rm td}(K|k) > 1$ and k an algebraic closure of a finite field of characteristic $\neq \ell$. Then the following hold:*

1) *There exists a group theoretical recipe which recovers $K|k$ from $\Pi^{\rm c}_K$.*

2) *The above group theoretical recipe is functorial in the following sense: Let $L|l$ be any function field with l an algebraically closed field, and let $\Phi : \Pi_K \to \Pi_L$ be the abelianization of some isomorphism $\Phi^{\rm c} : \Pi^{\rm c}_K \to \Pi^{\rm c}_L$. Then denoting by $L^{\rm i}$ and $K^{\rm i}$ the perfect closures, there exist an isomorphism of extensions of fields $\imath : L^{\rm i}|l \to K^{\rm i}|k$ and an ℓ-adic unit $\epsilon \in \mathbb{Z}^\times_\ell$ such that $\epsilon \cdot \Phi$ is induced by $\imath$. Moreover, $\imath$ is unique up to Frobenius twists, and ϵ is unique up to multiplication by p-powers, where $p = {\rm char}(k)$.*

3) *For a function field $L|l$ as above, let ${\rm Isom}^{\rm F}(L, K)$ be the set of isomorphisms of field extensions $\imath : L^{\rm i}|l \to K^{\rm i}|k$ up to Frobenius twists, and let ${\rm Isom}^{\rm c}(\Pi_K, \Pi_L)$ be the set of abelianizations of continuous group isomorphisms $\Pi^{\rm c}_K \to \Pi^{\rm c}_L$, up to multiplication by ℓ-adic units $\epsilon \in \mathbb{Z}^\times_\ell$. Then there is a canonical bijection*

$$ {\rm Isom}^{\rm F}(L, K) \to {\rm Isom}^{\rm c}(\Pi_K, \Pi_L). $$

For a sketch of a strategy to prove the above Theorem I see Pop [14], Introduction. The Main Theorem of loc.cit. reduces the proof of the above Theorem I to recovering the *total*

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[1] See Grothendieck [7], [8], and well as Szamuely [18] and Faltings [5] for more on this.

decomposition graph $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ of $K|k$ together with its *rational quotients,* and showing that the group theoretical recipes to do so are *invariant* under group isomorphisms.

Before going into the details of the proof, let me mention the following:

• First, the condition $\mathrm{td}(K|k) > 1$ is necessary in order to recover $K|k$ from its Galois theory. Indeed, if $\mathrm{td}(K|k) = 1$, then the absolute Galois group G_K of K is profinite free of rank $|K|$ (by a result of Harbater and Pop), thus the only information G_K –hence also the only information $\Pi^{\rm c}_K$– encodes is $|K|$.

• In the same way, one cannot expect to have a pure Galois theoretical *Hom-form* of the above Theorem I. Indeed, for $K|k$ be as in the Theorem I, and $u \in K$ a non-constant function, let κ_u be the relative algebraic closure of $k(u)$ in K. Then the canonical projection $G_K \to G_{\kappa_u}$ is surjective, and $\kappa_u|k$ is a function field with $\mathrm{td}(\kappa_u|k) = 1$ and κ_u countable. Thus as indicate above, G_{κ_u} is profinite free on countably many generators. On the other hand, since K is countable, G_K is countably generated. Therefore, there are many onto projections $G_{\kappa_u} \to G_K$, which do not originate from geometry. We conclude that there are many non-geometric onto group projections $G_K \to G_{\kappa_u} \to G_K$, thus also $\Pi^{\rm c}_K \to \Pi^{\rm c}_K$ ones.

• The Main Theorem above implies corresponding assertions in the arithmetical situation.

Historical note: The first attempt to give a recipe to recover $K|k$ from $\Pi^{\rm c}_K$ was made in Bogomolov [1]. Although loc.cit. is too sketchy in order to be sure what the author precisely proposes, a thorough inspection shows that loc.cit. provides a fundamental tool for recovering inertia elements of valuations v of K (which nevertheless may be non-trivial on k). This is Bogomolov's theory of *commuting liftable pairs.* On the other hand, there are serious technical issues and difficulties when one wants to complete proofs along the lines suggested in loc.cit.

A sketch of a viable global theory —at least in the case k *is an algebraic closure of a finite field,* was proposed in the notes of my MSRI Talk of 1999, see Pop [11]. It was followed by Pop [12], where several technical details from Pop [11] were worked out, especially concerning the global theory, and Pop [13] where the *full pro-ℓ variant* of the above Theorem I was proved. Finally, Pop [14] sums up in a systematic way the results scattered in the above manuscripts and reduces the problem of recovering $K|k$ from $\Pi^{\rm c}_K$ to recovering the total decomposition graph $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ of $K|k$ and its rational quotients.

Finally, for k *an algebraic closure of a finite field,* let me mention the following:

• Pop [13] proves the full pro-ℓ variant of the above Theorem I, and the conclusion is correspondingly stronger, namely that the isomorphism Φ itself is defined by an embedding of function fields $\phi : L^{\rm i}|l \to K^{\rm i}k$. The full $G_K(\ell)$ and not just its quotient $\Pi^{\rm c}_K$ was used there, because the "right" local theory, now developed in Pop [15], was not available back then. On the other hand, Pop [13] contains all the ideas as well as all the technical ingredients of the proof of Theorem I above (but maybe in a less elaborate form).

• Bogomolov–Tschinkel [2], [3] deal with the case $K = k(X)$ with $X \to k$ a projective, smooth surface. In the initial variant of their manuscript [2], they considered only the case when $\pi_1(X)$ is finite, and proved that if $\Pi^{\rm c}_K$ and $\Pi^{\rm c}_L$ are isomorphic, then $K|k$ and $L|l$ are isomorphic up to pure inseparable closures, provided k and l are algebraic closures of finite fields of char $\neq 2$. Nevertheless, in the published version [3] of their earlier manuscript [2], they announce their main result for surfaces in a form almost identical with the above

Theorem I and use a strategy of proof which is in many ways very similar to the one announced in Pop [11], used in Pop [13] and used here.

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Reconstruction Process:

Given: Π^{c}_K *and the abelianization* $\Phi : \Pi_K \to \Pi_L$ *of an isomorphism* $\Phi^{\mathrm{c}} : \Pi^{\mathrm{c}}_K \to \Pi^{\mathrm{c}}_L$

- $\Pi^{\mathrm{c}}_K \implies \mathrm{td}(K|k)$ [Characterization Theorem (1)]
  $\implies \mathrm{td}(K|k) = \mathrm{td}(L|l)$
- $\Pi^{\mathrm{c}}_K \implies$ quasi divisorial subgroups of Π_K [Characterization Theorem (2)]
  $\implies \mathfrak{In}.\mathfrak{tm}.\mathfrak{q}.\mathfrak{div}(K)$ [Characterization Theorem (3)]
  $\implies \mathfrak{In}.\mathfrak{tm}(K)$ and $\Phi\big(\mathfrak{In}.\mathfrak{tm}(K)\big) = \mathfrak{In}.\mathfrak{tm}(L)$ [Fact 3.3 (1) and (3)]
  $\implies$ flags of generalized quasi divisorial subgroups of Π_K [Proposition 3.5 (1)]
  $\implies \Phi(\text{flags gen.q.p.div.sgr.}\,\Pi_K) = \text{flags gen.q.p.div.sgr.}\,\Pi_L$ [Proposition 3.5 (3)]

If k a.c.f.f. (is an algebraic closure of a finite field), then k has only the trivial valuation. Thus every flag of generalized quasi divisorial subgroups is actually a flag of generalized divisorial subgroups. Conclude that:

- $\Pi^{\mathrm{c}}_K + k$ a.c.f.f. $\implies$ the flags of generalized divisorial subgroups of Π_K
  $\implies \mathcal{G}_{\mathcal{D}^{\mathrm{tot}}_K}$ (Discussion preceding Remark/Definition 3.1)
- $\Pi^{\mathrm{c}}_K +$ quasi divisorial subgroups of $\Pi_K \implies$ whether k a.c.f.f. [Proposition 4.4 (1)]
- k a.c.f.f. $\implies$ l a.c.f.f. and $\Phi(\mathcal{G}_{\mathcal{D}^{\mathrm{tot}}_K}) = \mathcal{G}_{\mathcal{D}^{\mathrm{tot}}_L}$ [Proposition 4.4 (3)]
- $\Pi^{\mathrm{c}}_K + k$ a.c.f.f. $\implies$ rational quotients [Proposition 5.3 (1)]
  $\implies \Phi$ compatible with rational quotients [Proposition 5.3 (2)]

Therefore:

- $\Pi^{\mathrm{c}}_K + k$ a.c.f.f. $\implies \mathcal{G}_{\mathcal{D}^{\mathrm{tot}}_K}$ + the rational quotients
  $\implies K|k$ [Theorem 2.1 (1)]
  $\implies \epsilon \cdot \Phi$ is geometric for some $\epsilon \in \mathbb{Z}^\times_\ell$ [Theorem 2.1 (2)]

2. Recalling some facts from Pop [14]

- Prime divisor graphs (See Pop [14], Section 3 for more details)

Recall that a prime divisor of a function field $K|k$ is a discrete valuation v of K whose valuation ring is the local ring $\mathcal{O}_{X,x_1}$ of the generic point x_1 of some Weil prime divisor of some normal model $X \to k$ of $K|k$. If so, then the residue field Kv of v is the function field $Kv = \kappa(x_1)$, thus $\mathrm{td}(Kv|k) = \mathrm{td}(K|k) - 1$. A valuation $\mathfrak{v}$ of $K|k$ is called a prime r-divisor if it satisfies the following equivalent conditions:

i) $\mathfrak{v}$ is trivial on k, the residue field has $\mathrm{td}(K\mathfrak{v}|k) = \mathrm{td}(K|k) - r$, and there exists a chain of valuations $\mathfrak{v}_1 < \cdots < \mathfrak{v}_r := \mathfrak{v}$.

ii) $\mathfrak{v}$ is the valuation theoretical composition $\mathfrak{v} = v_r \circ \cdots \circ v_1$, where v_1 is a prime divisor of K, and inductively, v_{i+1} is a prime divisor of the residue function field $\kappa(v_i)\,|\,k$.

A sequence of prime divisors $(v_r, \dots, v_1)$ as above, will be called an Parshin r-chain of $K|k$. By definition, the trivial valuation will be considered a generalized prime divisor of rank zero, and the corresponding Parshin chain is the trivial Parshin chain. Finally note that $r \leq \mathrm{td}(K|k)$, and that in the above notations, one has $\mathfrak{v}_i = v_i \circ \cdots \circ v_1$ for all $i \geq 1$.

We define the total prime divisor graph $\mathcal{D}^{\mathrm{tot}}_K$ of K to be the half-oriented graph as follows:

a) The vertices of $\mathcal{D}^{\mathrm{tot}}_K$ are the residue fields $K\mathfrak{v}$ of all the generalized prime divisors $\mathfrak{v}$ of $K|k$ viewed as distinct function fields.

b) For given $\mathfrak{v} = v_r \circ \cdots \circ v_1$ and $\mathfrak{w} = w_s \circ \cdots \circ w_1$, the edges from $K\mathfrak{v}$ to $K\mathfrak{w}$ are as follows:

  i) If $\mathfrak{v} = \mathfrak{w}$, then the trivial valuation $\mathfrak{v}/\mathfrak{w} = \mathfrak{w}/\mathfrak{v}$ of $K\mathfrak{v} = K\mathfrak{w}$ is the only edge from $K\mathfrak{v} = K\mathfrak{w}$ to itself; and it is by definition a non-oriented edge.

  ii) If $K\mathfrak{v} \neq K\mathfrak{w}$, then the set of edges from $K\mathfrak{v}$ to $K\mathfrak{w}$ is non-empty iff $s = r + 1$ and $v_i = w_i$ for $1 \leq i \leq r$; and if so, then $w_s = \mathfrak{w}/\mathfrak{v}$ is the only edge from $K\mathfrak{v}$ to $K\mathfrak{w}$, and it is by definition an oriented edge.

1) *Embeddings.* Let $L|l \hookrightarrow K|k$ be an embedding of function fields which maps l isomorphically onto k. Then the canonical restriction map of valuations $\mathrm{Val}_K \to \mathrm{Val}_L$, $v \mapsto v|_L$, gives rise to a surjective morphism of the total prime divisor graphs $\varphi_\imath : \mathcal{D}^{\mathrm{tot}}_K \to \mathcal{D}^{\mathrm{tot}}_L$.

2) *Restrictions.* Given a generalized prime divisor $\mathfrak{v}$ of $K|k$, let $\mathcal{D}^{\mathrm{tot}}_{\mathfrak{v}}$ be the set of all the generalized prime divisors $\mathfrak{w}$ of $K|k$ with $\mathfrak{v} \leq \mathfrak{w}$. Then the map $\mathcal{D}^{\mathrm{tot}}_{\mathfrak{v}} \to \mathcal{D}^{\mathrm{tot}}_{K\mathfrak{v}}$, $\mathfrak{w} \mapsto \mathfrak{w}/\mathfrak{v}$, is an isomorphism of $\mathcal{D}^{\mathrm{tot}}_{\mathfrak{v}}$ onto $\mathcal{D}^{\mathrm{tot}}_{K\mathfrak{v}}$.

• Decomposition graphs (See Pop [14], Section 3 for more details)

Let $K|k$ be as above, and let $\mathbb{T}_{\ell,K} = \varprojlim_e \mu_{\ell^e}$ be the ℓ-adic Tate module of $K^\times$.

For every valuation $*$ of K we denote by $T_* \subseteq Z_*$ the inertia/decomposition groups of $*$ in Π_K, and notice the following, see e.g., Pop [15], Introduction, and the beginning of Section 3 here, for a discussion of these facts: For every prime divisor v of $K|k$ one has $T_v \cong \mathbb{T}_{\ell,K}$, and for every prime r-divisor $\mathfrak{v}$ one has $T_{\mathfrak{v}} \cong \mathbb{T}^r_{\ell,K}$. Further, for generalized prime divisors $\mathfrak{v}$ and $\mathfrak{w}$ one has: $Z_{\mathfrak{v}} \cap Z_{\mathfrak{w}} \neq 1$ if and only if $\mathfrak{v}, \mathfrak{w}$ are not independent as valuations, i.e., $\mathcal{O} := \mathcal{O}_{\mathfrak{v}}\,\mathcal{O}_{\mathfrak{w}} \neq K$; and if so, then $\mathcal{O}$ is the valuation ring of a generalized prime divisor $\mathfrak{u}$ of $K|k$ which turns out to be the unique generalized prime divisor with $T_{\mathfrak{u}} = T_{\mathfrak{v}} \cap T_{\mathfrak{w}}$, and also the unique generalized prime divisor of $K|k$ maximal with the property $Z_{\mathfrak{v}}, Z_{\mathfrak{w}} \subseteq Z_{\mathfrak{u}}$.

In particular, $\mathfrak{v} = \mathfrak{w}$ iff $T_{\mathfrak{v}} = T_{\mathfrak{w}}$ iff $Z_{\mathfrak{v}} = Z_{\mathfrak{w}}$. Further, $\mathfrak{v} < \mathfrak{w}$ iff $T_{\mathfrak{v}} \subset T_{\mathfrak{w}}$ strictly iff $Z_{\mathfrak{v}} \supset Z_{\mathfrak{w}}$ strictly, and $T_{\mathfrak{w}}/T_{\mathfrak{v}} \cong \mathbb{Z}_\ell^{s-r}$ if $\mathfrak{v}$ is a prime r-divisor, and $\mathfrak{w}$ is a prime s-divisor.

We conclude that the partial ordering of the set of all the generalized prime divisors $\mathfrak{v}$ of $K|k$ *is encoded in the set* of their inertia/decomposition groups $T_{\mathfrak{v}} \subseteq Z_{\mathfrak{v}}$. In particular, the existence of the trivial, respectively nontrivial, edge from $K\mathfrak{v}$ to $K\mathfrak{w}$ in $\mathcal{D}^{\mathrm{tot}}_K$ is equivalent to $T_{\mathfrak{v}} = T_{\mathfrak{w}}$, respectively to $T_{\mathfrak{v}} \subset T_{\mathfrak{w}}$ and $T_{\mathfrak{w}}/T_{\mathfrak{v}} \cong \mathbb{Z}_\ell$.

Via the Galois correspondence and the functorial properties of the Hilbert decomposition theory for valuations, we attach to the total prime divisor graph $\mathcal{D}_K^{\mathrm{tot}}$ of $K|k$ a graph $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ whose vertices and edges are in bijection with those of $\mathcal{D}_K^{\mathrm{tot}}$, as follows:

a) The vertices of $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ are the pro-ℓ groups $\Pi_{K\mathfrak{v}}$, viewed as distinct pro-ℓ groups (all $\mathfrak{v}$).

b) If the edge from $K\mathfrak{v}$ to $K\mathfrak{w}$ exists, the corresponding edge from $\Pi_{K\mathfrak{v}}$ to $\Pi_{K\mathfrak{w}}$ is endowed with the pair of groups $T_{\mathfrak{w}/\mathfrak{v}} \subseteq Z_{\mathfrak{w}/\mathfrak{v}}$ viewed as subgroups of $\Pi_{K\mathfrak{v}}$, thus $\Pi_{K\mathfrak{w}} = Z_{\mathfrak{w}/\mathfrak{v}}/T_{\mathfrak{w}/\mathfrak{v}}$.

The graph $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ will be called the total decomposition graph of $K|k$, or of Π_K.

The functorial properties of the graphs of prime divisors translate in the following functorial properties of the decomposition graphs:

1) *Embeddings.* Let $\imath : L|l \hookrightarrow K|k$ be an embedding of function fields which maps l isomorphically onto k. Then the canonical projection homomorphism $\Phi_\imath : \Pi_K \to \Pi_L$ is an open homomorphism, and moreover, for every generalized prime divisor $\mathfrak{v}$ of $K|k$ and its restriction $\mathfrak{v}_L$ to L one has: $\Phi_\imath(Z_\mathfrak{v}) \subseteq Z_{\mathfrak{v}_L}$ is an open subgroup, and $\Phi_\imath(T_\mathfrak{v}) \subseteq T_{\mathfrak{v}_L}$ satisfies: $\Phi_\imath(T_\mathfrak{v}) = 1$ iff $\mathfrak{v}_L$ is the trivial valuation. Therefore, $\Phi_\imath$ gives rise to a morphism of total decomposition graphs, which we denote:

$$\Phi_\imath : \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \to \mathcal{G}_{\mathcal{D}_L^{\mathrm{tot}}}.$$

2) *Restrictions.* Given a generalized prime divisor $\mathfrak{v}$ of $K|k$, let $pr_\mathfrak{v} : Z_\mathfrak{v} \to \Pi_{K\mathfrak{v}}$ be the canonical projection. Then for every $\mathfrak{w} \geq \mathfrak{v}$ we have: $T_\mathfrak{w} \subseteq Z_\mathfrak{w}$ are mapped onto $T_{\mathfrak{w}/\mathfrak{v}} \subseteq Z_{\mathfrak{w}/\mathfrak{v}}$. Therefore, the total decomposition graph of $K\mathfrak{v}|k$ can be recovered from the one of $K|k$ in a canonical way via $pr_\mathfrak{v} : Z_\mathfrak{v} \to \Pi_{K\mathfrak{v}}$.

• Rational quotients (See Pop [14], Section 5 for more details)

Let $K|k$ be a function field as above satisfying $\mathrm{td}(K|k) > 1$. For $u \in K$ non-constant, we denote $\kappa_u := \overline{k(u)} \cap K$. Then $\kappa_u|k$ is the function field of a unique complete (thus projective) normal curve $X_u \to k$. Hence the set of prime divisors of $\kappa_u|k$ is actually in bijection with the (local rings at the) closed points of X_u, thus with the Weil prime divisors of X_u. Therefore, we will denote $\mathcal{D}_{\kappa_u}^{\mathrm{tot}}$ simply by $\mathcal{D}_{\kappa_u}$, and $\mathcal{G}_{\mathcal{D}_{\kappa_u}}$ by $\mathcal{G}_{\kappa_u}$.

Let $\imath_u : \kappa_u \to K$ be the canonical embedding, and $\Phi_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$ the (surjective) canonical projection. Then by the functoriality of embeddings, Φ_{κ_u} gives rise to a morphism of total decomposition graphs $\Phi_{\kappa_u} : \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \to \mathcal{G}_{\kappa_u}$. We say that $\Phi_{\kappa_u} : \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \to \mathcal{G}_{\kappa_u}$ is a rational quotient of $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$, if $\kappa_u = k(u)$, i.e., $k(u)$ is relatively algebraically closed in K, and call such a u a "general element" of K. A "birational Bertini" type argument shows that there are "many" general elements in K, see Lang [9], Ch.VIII, and Pop [14], Fact 5.8: For any given algebraically independent functions $t, t' \in K$, not both inseparable, $t_{a',a} := t/(a't' + a)$ is a general element of K for almost all $a', a \in k$. A set of general elements $\Sigma \subset K$ is a Bertini set if Σ contains almost all elements $t_{a',a}$ for all t, t' as above. We denote by $\mathfrak{A}_K = \{\Phi_{\kappa_x}\}_{\kappa_x}$ the set of all the rational quotients of $K|k$, and consider subsets $\mathfrak{A} \subset \mathfrak{A}_K$ containing all the $\Phi_{\kappa_x} \in \mathfrak{A}$, $x \in \Sigma$, with Σ some Bertini set of general elements, and call them for short, Bertini type sets of rational quotients.

The relation between rational projections and morphisms of geometric decomposition graphs is as follows: Let $\imath : L|l \hookrightarrow K|k$ be an embedding of function fields with $\imath(l) = k$, and $K|\imath(L)$ a separable field extension. Then there exists a Bertini type set $\mathfrak{B} = \{\Phi_{\kappa_y}\}_{\kappa_y}$ for $L|l$ such that $\kappa_x := \imath(\kappa_y)$ is relatively algebraically closed in K for all κ_y. Hence for all $\Phi_{\kappa_y} \in \mathfrak{B}$

and the corresponding $\Phi_{\kappa_x} \in \mathfrak{A}_K$, $\kappa_x := \imath(\kappa_y)$, we get: The isomorphism $\Phi_{\kappa_x\kappa_y} : \mathcal{G}_{\kappa_x} \to \mathcal{G}_{\kappa_y}$ defined by $\imath_{\kappa_x\kappa_y} := \imath|_{\kappa_y}$ satisfies $\Phi_{\kappa_y} \circ \Phi_\imath = \Phi_{\kappa_x\kappa_y} \circ \Phi_{\kappa_x}$ for all $\kappa_y \in \mathfrak{B}$ and the corresponding $\kappa_x \in \mathfrak{A}$. Because of this property, we will say that $\Phi_\imath$ is compatible with the rational quotients.

The first part of the Main Theorem of Pop [14], Introduction, is the following:

Theorem 2.1. *Let $K|k$ and $L|l$ function fields with* $\mathrm{td}(K|k) > 1$. *Let $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ and $\mathcal{H}_{\mathcal{D}_L^{\mathrm{tot}}}$ be their total decomposition graphs, which we endow with Bertini type sets of rational quotients $\mathfrak{A}$, respectively $\mathfrak{B}$. Then the following hold:*

1) *There exists a group theoretical recipe which recovers $K|k$ from $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ endowed with $\mathfrak{A}$.*

2) *Moreover, this recipe is invariant under isomorphisms in the following sense: For every isomorphism of decomposition graphs $\Phi : \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \to \mathcal{H}_{\mathcal{D}_L^{\mathrm{tot}}}$ which is compatible with the sets of rational quotients $\mathfrak{A}$ and $\mathfrak{B}$, there exist an ℓ-adic unit $\epsilon \in \mathbb{Z}_\ell^\times$ and an isomorphism of field extensions $\imath : L^{\mathrm{i}}|l \to K^{\mathrm{i}}|k$ such that $\Phi = \epsilon \cdot \Phi_\imath$, where $\Phi_\imath : \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \to \mathcal{H}_{\mathcal{D}_L^{\mathrm{tot}}}$ is the isomorphism defined by $\imath$ as indicated above. Further, $\imath$ is unique up to Frobenius twists, and ϵ is unique up to multiplication by powers of p, where* $p = \mathrm{char}(k)$.

Clearly, Theorem 2.1 above reduces the proof of Theorem I from the Introduction to recovering the total decomposition graph $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ and its rational quotients from Π_K^{c}, and showing that the group theoretical recipe to do so is invariant under abelianizations $\Pi_K \to \Pi_L$ of isomorphisms $\Pi_K^{\mathrm{c}} \to \Pi_L^{\mathrm{c}}$. Thus Theorem I is reduced to proving the following:

Theorem 2.2. *Let $K|k$ be a function field with* $\mathrm{td}(K|k) > 1$ *and k an algebraic closure of a finite field, and $L|l$ an arbitrary function field with l algebraically closed. Then:*

1) *There exists a group theoretical recipe which recovers $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ from Π_K^{c}. Moreover, if $\Phi : \Pi_K \to \Pi_L$ is the abelianization of an isomorphism $\Phi^{\mathrm{c}} : \Pi_K^{\mathrm{c}} \to \Pi_L^{\mathrm{c}}$, then Φ defines an isomorphism of total decomposition graphs $\Phi : \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \to \mathcal{G}_{\mathcal{D}_L^{\mathrm{tot}}}$.*

2) *There exists a group theoretical recipe which recovers the rational quotients of $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$. Moreover, every isomorphism $\Phi : \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \to \mathcal{G}_{\mathcal{D}_L^{\mathrm{tot}}}$ is compatible with rational quotients.*

3. Recovering the generalized prime divisors

A) *On the space of inertia elements*

Let K be a field of characteristic $\neq \ell$ which contains all the ℓ^∞ roots of unity. As usual, denote by $K'|K$ a maximal abelian pro-ℓ extension of K, and by Π_K its Galois group. For valuations v of K, let $v'|v$ denote prolongations of v to K'. Then the corresponding inertia/decomposition groups $T_{v'|v} \subseteq Z_{v'|v}$ are conjugated under Π_K; hence they are equal, because Π_K is Abelian. We denote these groups by $T_v \subseteq Z_v$, and call them the inertia/decomposition groups at v. The residual extension $K'v'|Kv$ is a maximal abelian pro-ℓ extension of Kv, i.e., $K'|v' = (Kv)'$, and there exists a canonical exact sequence:

$$1 \to T_v \to Z_v \to \Pi_{Kv} \to 1\,.$$

Let $\overline{w}$ be a valuation of Kv, and let $w = \overline{w} \circ v$ be the (valuation theoretical) composition of $\overline{w}$ and v. Then one has $Z_w \subseteq Z_v$ and $T_w \supseteq T_v$, and moreover, $T_w \subseteq Z_w$ are exactly the preimages of $T_{\overline{w}} \subseteq Z_{\overline{w}}$ under the canonical projection $Z_v \to \Pi_{Kv}$, and one gets a commutative exact diagram in which the vertical arrows are surjective:

$$\begin{array}{ccccccc} 1\to & T_v & \to & T_w & \to & T_{\overline{w}} & \to 1 \\ & \| & & \cap & & \cap & \\ 1\to & T_v & \to & Z_w & \to & Z_{\overline{w}} & \to 1 \\ & & & \downarrow & & \downarrow & \\ & & & \Pi_{Kw} & = & \Pi_{(Kv)\overline{w}} & \end{array}$$

Remark/Definition 3.1.

1) An element $\sigma \in \Pi_K$ is called inertia element, if there exists some valuation v on K such that $\sigma \in T_v$. We denote by $\mathfrak{In}(K) \subseteq \Pi_K$ the set of all the inertia elements in Π_K.

An element $\sigma \in \Pi_K$ is called tame inertia element if is inertia element at some v with $\mathrm{char}(Kv) \neq \ell$. We denote by $\mathfrak{In}.\mathfrak{tm}(K) \subseteq \Pi_K$ the set of all the tame inertia elements in Π_K.

2) For valuations v, let $\mathfrak{In}(v) \subseteq \mathfrak{In}(K)$ and $\mathfrak{In}.\mathfrak{tm}(v) \subseteq \mathfrak{In}.\mathfrak{tm}(K)$ be the sets of all the inertia elements, respectively all the tame inertia elements, at valuations $w \geq v$.

3) Let $1 \to T_v \to Z_v \xrightarrow{\pi} \Pi_{Kv} \to 1$ be the canonical exact sequence. Then for $w \geq v$ we have: $T_w \supseteq T_v$, and $T_{w/v} = T_w/T_v$ under the above exact sequence.

4) Therefore, under the projection $\pi : Z_v \to \Pi_{Kv}$ the following hold: $\mathfrak{In}(v)$ is the preimage of $\mathfrak{In}(Kv)$ and $\mathfrak{In}(Kv)$ is the image of $\mathfrak{In}(v)$.

5) Correspondingly, the same holds for the tame inertia: $\mathfrak{In}.\mathfrak{tm}(v)$ is the preimage of $\mathfrak{In}.\mathfrak{tm}(Kv)$ and $\mathfrak{In}.\mathfrak{tm}(Kv)$ is the image of $\mathfrak{In}.\mathfrak{tm}(v)$ under the projection $\pi : Z_v \to \Pi_{Kv}$.

6) The sets $\mathfrak{In}(Kv)$ and $\mathfrak{In}.\mathfrak{tm}(Kv)$ are topologically closed in Π_{Kv}, by Pop [16], Introduction, Theorem A. In particular, $\mathfrak{In}(v), \mathfrak{In}.\mathfrak{tm}(v) \subseteq \Pi_K$ are topologically closed subsets.

B) *Generalized divisorial subgroups*

Let k be an algebraically closed field with $\mathrm{char}(k) \neq \ell$, and $K|k$ a function field with $d := \mathrm{td}(K|k) > 1$. We recall the following, see Pop [15], for more details:

Definition/Remark 3.2. Let $\mathfrak{v}$ be a valuation of K which is not necessarily trivial on k.

1) $\mathfrak{v}$ is called a quasi prime divisor of $K|k$, if it is minimal among the valuations satisfying:
   i) $\mathfrak{v}K/\mathfrak{v}k \cong \mathbb{Z}$ as abstract groups.
   ii) $K\mathfrak{v}\,|\,k\mathfrak{v}$ is a function field with $\mathrm{td}(K\mathfrak{v}\,|\,k\mathfrak{v}) = \mathrm{td}(K|k) - 1$.

2) $\mathfrak{v}$ is called a quasi prime r-divisor of $K|k$ if $\mathfrak{v}$ is the (valuation theoretical) composition $\mathfrak{v} = v_r \circ \cdots \circ v_1$, where v_1 is a quasi prime divisor of $K|k$, and inductively, v_{i+1} is a quasi prime divisor of the residue function field $K\mathfrak{v}_i\,|\,k\mathfrak{v}_i$, where $\mathfrak{v}_i := v_i \circ \cdots \circ v_1$.

Note that the quasi prime divisors of $K|k$ are exactly the quasi prime 1-divisors of $K|k$, and the prime r-divisors of $K|k$ (as defined in Section 2), are exactly the quasi prime r-divisors of $K|k$ which are trivial on k.

We recall one of the main results from Pop [15], Theorem 4.4, which asserts that the inertia/decomposition groups $T_\mathfrak{v} \subset Z_\mathfrak{v}$ of quasi prime r-divisors $\mathfrak{v}$ with $\mathrm{char}(K\mathfrak{v}) \neq \ell$ can be recovered from Π_K^{c} via the canonical projection $\Pi_K^{\mathrm{c}} \to \Pi_K$ as follows:

Characterization Theorem. *Let $K|k$ be a function field over the algebraically closed field k, $\mathrm{char}(k) \neq \ell$. Let $pr : \Pi_K^{\mathrm{c}} \to \Pi_K$ be the canonical projection, and for subgroups T, Z, Δ of Π_K, let T'', Z'', Δ'' denote their preimages in Π_K^{c}. Then the following hold:*

1) *The transcendence degree* $d = \mathrm{td}(K|k)$ *is the maximal integer* d *such that there exist closed subgroups* $\Delta \cong \mathbb{Z}_\ell^d$ *of* Π_K *with* Δ'' *abelian.*

2) *Suppose that* $d := \mathrm{td}(K|k) > 1$. *Let* $T_\alpha \subset Z_\alpha$ *be closed subgroups of* Π_K. *Then* Z_α *endowed with* T_α *is a quasi divisorial subgroup of* Π_K *if and only if* $T_\alpha \subset Z_\alpha$ *are maximal in the set of closed subgroups* $T \subset Z$ *of* Π_K *which satisfy:*

i) Z *contains a closed subgroup* $\Delta \cong \mathbb{Z}_\ell^d$ *such that* Δ'' *is abelian.*

ii) $T \cong \mathbb{Z}_\ell$, *and* T'' *is the center of* Z''.

3) *Finally let* $\mathfrak{In.tm.q.div}(K) \subset \mathfrak{In.tm}(K)$ *be the set of all the inertia elements which are tame inertia elements at quasi divisors of* K. *Then* $\mathfrak{In.tm.q.div}(K) \subset \Pi_K$ *can be recovered from* Π_K^{c} *as being* $\mathfrak{In.tm.q.div}(K) = \cup T_\alpha$, *with* T_α *as at* 2) *above.*

As an immediate corollary of the Characterization Theorem above and applying the main result from Pop [16], Introduction, one has the following:

Fact 3.3. Let $K|k$ be a function field with $\mathrm{td}(K|k) > 1$ and k algebraically closed of characteristic $\neq \ell$, and $L|l$ a further function field with l algebraically closed. Then one has:

1) The set of all the tame inertia elements $\mathfrak{In.tm}(K) \subset \Pi_K$ can be recovered from Π_K^{c} via the Characterization Theorem, as being the topological closure of $\mathfrak{In.tm.q.div}(K)$ in Π_K.

2) Let $\Phi : \Pi_K \to \Pi_L$ be the abelianization of some isomorphism $\Phi^{\mathrm{c}} : \Pi_K^{\mathrm{c}} \to \Pi_L^{\mathrm{c}}$. Then Φ maps the family of all the quasi divisorial subgroups $T_v \subseteq Z_v$ of Π_K bijectively onto the family of all the quasi divisorial subgroups $T_w \subseteq Z_w$ of Π_L of $L|l$, respectively $\mathfrak{In.tm.q.div}(K) \subset \Pi_K$ homeomorphically onto $\mathfrak{In.tm.q.div}(L) \subset \Pi_L$.

Definition 3.4.

1) A flag of generalized (quasi) prime divisors for $K|k$ is any sequence $\mathfrak{v}_1 \leq \cdots \leq \mathfrak{v}_r$ such that each $\mathfrak{v}_m$ is a (quasi) prime m-divisor of $K|k$ for $1 \leq m \leq r$.

2) A flag of generalized (quasi) divisorial subgroups of Π_K, is the sequence $Z_{\mathfrak{v}_1} \geq \cdots \geq Z_{\mathfrak{v}_r}$ of the decomposition groups of a flag of generalized (quasi) prime divisors $\mathfrak{v}_1 \leq \cdots \leq \mathfrak{v}_r$ endowed with the corresponding sequence of inertia groups $T_{\mathfrak{v}_1} \leq \cdots \leq T_{\mathfrak{v}_r}$.

We will next show that Fact 3.3 above (thus the Characterization Theorem) can be used to recover the flags of generalized quasi divisorial subgroups in Π_K from Π_K^{c}.

In particular, if k is an algebraic closure of a finite field, then these will be the *flags of generalized prime divisors* of $K|k$. Indeed, in the case k is an algebraic closure of a finite field, the quasi prime divisors of $K|k$ are precisely the prime divisors of $K|k$, because k has the trivial valuation only.

Proposition 3.5. *Let* $K|k$ *be a function field with* $\mathrm{td}(K|k) > 1$ *and* k *algebraically closed of characteristic* $\neq \ell$, *and* $\Pi_K^{\mathrm{c}} \to \Pi_K$ *be the canonical projection. Let* $L|l$ *be a further function field with* l *algebraically closed. Then in the usual notations, the following hold:*

1) Flags of generalized quasi divisorial subgroups:

For a given sequence $Z_1 \geq \cdots \geq Z_r$ *of closed subgroups of* Π_K *endowed with closed subgroups* $T_1 \leq \cdots \leq T_r$ *such that* $T_m \subseteq Z_m$ *for each* m, *the following are equivalent:*

i) $Z_1 \geq \cdots \geq Z_r$ *endowed with* $T_1 \leq \cdots \leq T_n$ *is a flag of generalized quasi divisorial subgroups of* Π_K.

ii) $Z_1 \geq \cdots \geq Z_r$ *endowed with* $T_1 \leq \cdots \leq T_r$ *are maximal among the subgroups of* Π_K *satisfying for all* m *the following: First,* Z_m *contains* $\Delta \cong \mathbb{Z}_\ell^d$ *whose preimage* Δ'' *in* Π_K^{c} *is an abelian subgroup, and second,* $T_m \subset \mathfrak{In.tm}(K)$*, and* $T_m \cong \mathbb{Z}_\ell^m$*, and the preimage* T_m'' *of* T_m *in* Π_K^{c} *equals the center of the preimage* Z_m'' *of* Z_m *in* Π_K^{c}*.*

2) Flags of (quasi) divisorial subgroups in residue fields:

Let v *be a generalized (quasi) prime divisor of* $K|k$*, and* $1 \to T_v \to Z_v \xrightarrow{\pi} \Pi_{Kv} \to 1$ *be its canonical exact sequence. Then the generalized (quasi) divisorial subgroups of* Π_{Kv} *are precisely the images* $\pi(T), \pi(Z)$ *of the generalized (quasi) divisorial subgroups* T, Z *in* Π_K *which satisfy* $Z \subseteq Z_v$ *and* $T \supseteq T_v$*.*

- *In particular, if* k *is an algebraic closure of a finite field, the total decomposition graph* $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ *of* $K|k$ *can be recovered from* Π_K^{c} *as indicated above.*

3) *Let* $\Phi : \Pi_K \to \Pi_L$ *be the abelianization of an isomorphism* $\Pi_K^{\mathrm{c}} \to \Pi_L^{\mathrm{c}}$*. Then* Φ *maps the flags of generalized quasi divisorial subgroups of* Π_K *bijectively onto those of* Π_L*.*

Proof. To 1): First, the implication i) $\Rightarrow$ ii) follows from the subsection A) above.

The implication ii) $\Rightarrow$ i) is nevertheless more difficult, and assumes familiarity with the results and facts from Pop [15]. We prove the implication by induction on r as follows: By the maximality condition, it follows from the Characterization Theorem above that $T_1 \subset Z_1$ are the inertia/decomposition groups of a quasi prime divisor $\mathfrak{v}_1$ of $K|k$ (which is actually a prime divisor, if k is the algebraic closure of a finite field). Now suppose that $r > 1$, and that the assertion is proved for $Z_1 \geq \cdots \geq Z_{r-1}$ endowed with $T_1 \leq \cdots \leq T_{r-1}$, hence there exists a flag of generalized prime divisors $\mathfrak{v}_1 \leq \cdots \leq \mathfrak{v}_{r-1}$ with $Z_m = Z_{\mathfrak{v}_m}$, $T_m = T_{\mathfrak{v}_m} \cong \mathbb{Z}_\ell^m$ for $1 \leq m \leq r-1$. Next we notice that the hypothesis that T_r'' is the center of Z_r'' is equivalent to the fact that T_r is precisely the set of all the $\sigma \in Z_r$ which have some preimage $\sigma'' \in \Pi_K^{\mathrm{c}}$ which commutes with Z_r''. We have the following two cases:

a) $T_r = Z_r$. Then, first, since $\Delta \cong \mathbb{Z}_\ell^d$ is contained in Z_r, it follows that $r \geq d$, and therefore $r = d$, by the Characterization Theorem, 1), because T_r'' is by hypothesis abelian. Second, since $T_r \subset \mathfrak{In.tm}(K)$ by hypothesis, it follows that Z_r consists of tame inertia elements only. Hence by loc.cit. Proposition 3.5, 1), it follows that there exists a valuation $\mathfrak{v}_r$ of K such that $T_r \subseteq T_{\mathfrak{v}_r}$, and moreover, $\mathfrak{v}_r$ equals its abelian pro-ℓ Λ-core, where Λ is the fixed field of T_r in K'.

b) $T_r \subset Z_r$ strictly. Then setting $Z := Z_r$ and $\Sigma_Z := T_r$, the hypotheses of loc.cit. Proposition 3.5, 2), are satisfied. Hence by loc.cit. there exists a unique valuation $\mathfrak{v}_r$ of $K|k$ such that $Z_r \subseteq Z_{\mathfrak{v}_r}$ and $T_r = T_{\mathfrak{v}_r} \cap Z_r$, and $\mathfrak{v}_r$ equals its pro-ℓ abelian Λ-core, where Λ is the fixed field of T_r in K'.

Claim: $\mathfrak{v}_r$ is a quasi prime r-divisor, and $\mathfrak{v}_{r-1} < \mathfrak{v}_r$.

Indeed, let $\pi : Z_{r-1} = Z_{\mathfrak{v}_{r-1}} \to \Pi_{K\mathfrak{v}_{r-1}}$ be the canonical projection, hence $\ker(\pi) = T_{\mathfrak{v}_{r-1}}$. Since $T_{\mathfrak{v}_{r-1}} = T_{r-1} \cong \mathbb{Z}_\ell^{r-1}$ by hypothesis, and $T_r \cong \mathbb{Z}_\ell^r$, it follows that $\pi(T_r)$ is a non-trivial subgroup of $\Pi_{K\mathfrak{v}_{r-1}}$. Equivalently, if $\mathfrak{v}_{r-1}'$ is a prolongation of $\mathfrak{v}_{r-1}$ to K', then $\Lambda\mathfrak{v}_{r-1}' \subset K'\mathfrak{v}_{r-1}'$ strictly. Therefore, by the definition of the abelian pro-ℓ Λ-core, it follows that $\mathfrak{v}_{r-1}$ equals its abelian pro-ℓ Λ-core. Thus both $\mathfrak{v}_r$ and $\mathfrak{v}_{r-1}$ equal their abelian pro-ℓ Λ-cores. Hence since T_r is contained in both $Z_r \subseteq Z_{\mathfrak{v}_r}$ and $Z_{r-1} = Z_{\mathfrak{v}_{r-1}}$, if follows by loc.cit. Proposition 2.5, that $\mathfrak{v}_{r-1}$ and $\mathfrak{v}_r$ are comparable. And since $T_{\mathfrak{v}_{r-1}} = T_{r-1} < T_r \subseteq T_{\mathfrak{v}_r}$, thus $T_{\mathfrak{v}_{r-1}} \subset T_{\mathfrak{v}_r}$ strictly, it follows by the discussion in subsection A) above that we must have $\mathfrak{v}_{r-1} < \mathfrak{v}_r$.

Thus in particular, we must have $Z_{\mathfrak{v}_r} \subseteq Z_{\mathfrak{v}_{r-1}} = Z_{r-1}$ and in particular, $T_{\mathfrak{v}_r} \subseteq Z_{\mathfrak{v}_r} \subseteq Z_{r-1}$ too. Since furthermore, the preimage $T''_{\mathfrak{v}_r}$ of $T_{\mathfrak{v}_r}$ in Π^{c}_K is always contained in the center of the preimage $Z''_{\mathfrak{v}_r}$ of $Z_{\mathfrak{v}_r}$ in Π^{c}_K, it follows by the maximality condition on T_r, Z_r that we must have $Z_r = Z_{\mathfrak{v}_r}$ and $T_r = T_{\mathfrak{v}_r}$. And finally, since (by hypothesis) $Z_r = Z_{\mathfrak{v}_r}$ contains a subgroup $\Delta \cong \mathbb{Z}^d_\ell$ whose preimage in Π^{c}_K is abelian, it follows by loc.cit. Proposition 4.2, that $\mathrm{td}(K\mathfrak{v}_r|k) = d - r$, and that $\mathfrak{v}_r(K^\times) \cong \mathbb{Z}^r$ as an abstract group. On the other hand, since by the induction hypothesis we have $\mathfrak{v}_{r-1}(K^\times) \cong \mathbb{Z}^{r-1}$ lexicographically ordered, and $\mathfrak{v}_{r-1} < \mathfrak{v}_r$, we see that $\mathfrak{v}_r(K^\times) \cong \mathbb{Z}^r$ lexicographically ordered.

This concludes the proof of the Claim, and of assertion 1).

To 2): This follows immediately from the discussion in subsection A) above and the following remarks: Let $\mathcal{V}_v$ be the set of all the generalized (quasi) prime divisors w of $K|k$ such that $v \le w$. Then $w \mapsto w/v$ gives a bijection from $\mathcal{V}_v$ onto the set of all the generalized prime divisors of the residue function field $Kv\,|\,kv_k$, where $v_k := v|_k$. But then by Remark 3.1, it follows that $Z_w \mapsto Z_{w/v} = Z_w/T_v$ and $T_w \mapsto T_{w/v} = T_w/T_v$, defines the desired bijection.

To 3): This follows immediately from Fact 3.3 and assertions 1) and 2) above. $\square$

4. Recovering the nature of k

In this section we give a recipe to recover from Π^{c}_K the fact that k is an algebraic closure of a finite field (or not so). To have a name, we will call this for short the nature of k. Note that $\mathrm{char}(k)$ is not part of "nature of k".

We begin by mentioning the following obvious consequence of Theorem 3.5, 2), above:

Fact 4.1. In the usual context, let $\mathfrak{v}$ be a generalized prime r-divisor of $K|k$, where $r = \mathrm{td}(K|k) - 1$. Then $K\mathfrak{v}\,|\,k$, is a function field with $\mathrm{td}(K\mathfrak{v}\,|\,k) = 1$, thus $K\mathfrak{v}$ is the function field of a complete smooth curve $X_{\mathfrak{v}} \to k$, and the following hold:

1) For every quasi prime divisor $\mathfrak{w}_{\mathfrak{v}}$ of $K\mathfrak{v}\,|\,k$ we have: $T_{\mathfrak{w}_{\mathfrak{v}}} = Z_{\mathfrak{w}_{\mathfrak{v}}}$, and $T_{\mathfrak{w}_{\mathfrak{v}}} \cong \mathbb{Z}_\ell$. Further, if $\mathfrak{w}_{\mathfrak{v}} \ne \mathfrak{w}'_{\mathfrak{v}}$ are quasi prime divisors of $K\mathfrak{v}\,|\,k$, then $Z_{\mathfrak{w}_{\mathfrak{v}}} \cap Z_{\mathfrak{w}'_{\mathfrak{v}}} = \{1\}$.

2) Let $\pi_{\mathfrak{v}} : Z_{\mathfrak{v}} \to \Pi_{K\mathfrak{v}}$ be the canonical projection. Then the quasi divisorial subgroups $T_{\mathfrak{w}_{\mathfrak{v}}} = Z_{\mathfrak{w}_{\mathfrak{v}}}$ in $\Pi_{K\mathfrak{v}}$ are precisely the images $\pi_{\mathfrak{v}}(T_{\mathfrak{w}}), \pi_{\mathfrak{v}}(Z_{\mathfrak{w}})$ of all the generalized quasi divisorial subgroups $T_{\mathfrak{w}} \subseteq Z_{\mathfrak{w}}$ of Π_K with $T_{\mathfrak{v}} \subset T_{\mathfrak{w}}$ and $Z_{\mathfrak{v}} \supset Z_{\mathfrak{w}}$ strictly.

We next recall briefly some well known facts about Π_L in the case $L|l$ is the function field of a complete smooth curve $C \to l$ with l algebraically closed of $\mathrm{char}(l) \ne \ell$.

Let $C(l) \subset C$ be the set of closed points of C, and let $g = g_C$ be the geometric genus of C. For every $x \in C(l)$, let $T_x \subset \Pi_L$ be the inertia group at x. Then Π_L satisfies the following: There exist $\sigma_1, \dots, \sigma_{2g} \in \Pi_L$ and $\tau_x \in T_x$ such that $(\sigma_i)_i, (\tau_x)_x$ topologically generate Π_L and satisfy a single pro-finite relation, namely $\prod_x \tau_x = 1$.

In particular, if $T_L \subseteq \Pi_L$ is the closed subgroup of Π_L generated by all the τ_x, then Π_L/T_L equals the pro-ℓ abelian quotient $\Pi_1(C) := \pi_1^{\mathrm{ab},\ell}(C)$ of the fundamental group of C. This observation enables us to prove the following:

Lemma 4.2. *Let $l_1 \subseteq l$ be algebraically closed fields of characteristic $\ne \ell$, and $L|l$ be a function field with $\mathrm{td}(L|l) = 1$. Further let $\mathcal{T} := (T_w)_w$ be the set of all the quasi divisorial inertia groups in Π_L (respectively, those for which w is trivial on l_1), and $T_{\mathcal{T}} \subseteq \Pi_L$ be the closed subgroup generated by all the $(T_w)_w$. Then the following are equivalent:*

i) *The base field l is an algebraic closure of a finite field* (*respectively,* $l_1 = l$).

ii) *There exist generators $\tau_w \in T_w$ such that $\prod_w \tau_w = 1$, and this is the only pro-relation the system of generators $(\tau_w)_w$ satisfies.*

Moreover, if the equivalent conditions i), ii) *are satisfied, then $\Pi_L/T_{\mathcal{T}}$ has $2g$ generators, where g is the geometric genus of the smooth complete model $C \to l$ of $L|l$.*

Proof. First, we remark that $\mathrm{td}(L|l) = 1$ implies that every generalized quasi prime r-divisor of $L|l$ is actually a quasi prime divisor, as $1 \leq r \leq \mathrm{td}(L|l) = 1$.

Second, by the very definition of a quasi prime divisor, it follows that any two distinct quasi-prime divisors v, w of $L|l$ are not comparable. Further, if L^T is the inertia field of w, then w equals its abelian pro-ℓ L^T core, see Pop [15], Section 2, B), for definitions and basic facts. Therefore, by loc.cit., Proposition 2.5, it follows that if $w \neq v$ are distinct quasi prime divisors of $L|l$, then $T_w \cap T_v = 1$.

The implication i) $\Rightarrow$ ii): If l is an algebraic closure of a finite field (respectively, $l = l_1$) then all quasi prime divisors w of $L|l$ (respectively those which are trivial on $l_1 = l$), have trivial restriction to l, hence they are actually prime divisors of $L|l$. Then conclude by applying the discussion before the Lemma 4.2.

Next consider the implication ii) $\Rightarrow$ i). By contradiction, suppose that l is not an algebraic closure of a finite field (respectively, that $l_1 \neq l$). Then by general valuation theory, there exists a non-trivial valuation v_l of l (which is trivial on l_1 in the case $l_1 \neq l$). Let $\mathfrak{v}$ be a constant reduction à la Deuring of $L|l$ which prolongs v_l to L, which means that the residue field $L\mathfrak{v}$ is a function field in one variable over lv_l. See e.g. Roquette [17] for more about constant reductions of function fields. Then for every prime divisor $\overline{w}_{\mathfrak{v}}$ of $L\mathfrak{v}\,|lv_l$, the valuation theoretical composition $w_{\mathfrak{v}} := \overline{w}_{\mathfrak{v}} \circ \mathfrak{v}$ is a quasi prime divisor of L (which is trivial on l_1, in the case $l_1 \neq l$). Note that there are infinitely many such quasi prime divisors w of $L|l$, as they are in bijection with the closed points of the complete smooth model $C_{\mathfrak{v}} \to lv_l$ of the function field $L\mathfrak{v}\,|\,lv_l$. Finally let $\mathcal{T}_0$ and $\mathcal{T}_{\mathfrak{v}}$ be the sets of inertia subgroups T_v with v prime divisor of $L|l$, respectively of the form $T_{w_{\mathfrak{v}}}$ with $w_{\mathfrak{v}}$ as defined above, and note that any two distinct such groups have trivial intersection.

In particular, if $(\tau_v)_v, (\tau_{w_{\mathfrak{v}}})_{w_{\mathfrak{v}}}$ are part of the system of generators $(\tau_w)_w$ satisfying condition ii), then there exists at most one topological relation between these elements in Π_L. On the other hand, if $<T_L>$ is the closed subgroup of Π_L generated by $\mathcal{T}_0$, then by hypothesis, there exists at most one topological relation between the images $(\overline{\tau}_{w_{\mathfrak{v}}})_{w_{\mathfrak{v}}}$ of $(\tau_{w_{\mathfrak{v}}})_{w_{\mathfrak{v}}}$ in the quotient group $\Pi_L/<T_L>$. Since the later group is topologically finitely generated, it follows that the infinite system of elements $(\overline{\tau}_{w_{\mathfrak{v}}})_{w_{\mathfrak{v}}}$ must satisfy more than just one topological relation. Contradiction! □

Remark/Definition 4.3. Let G be a free abelian pro-ℓ group endowed with a family of pro-cyclic subgroups $(T_v)_v$, and let $T \subset G$ be the closed subgroup generated by $(T_v)_v$. We say that G endowed with $(T_v)_v$ is curve like [of genus g], if there exist generators τ_v of T_v, which we call distinguished, such that:

i) $\prod_v \tau_v = 1$, and this is the only profinite relation satisfied by $(\tau_v)_v$.

ii) G/T is finitely generated [precisely $G/T \cong \mathbb{Z}_\ell^{2g}$].

In particular, the content of the previous Lemma 4.2 can be expressed in this terminology as follows: Let $(T_v)_v$ be the family of all the quasi divisorial inertia subgroups of Π_L (respectively, of those which are trivial on l_1). Then Π_L endowed with $(T_v)_v$ is complete curve like [of genus g] iff l is an algebraic closure of a finite field (respectively, iff $l_1 = l$) [and if so, then $L|l$ is the function field of a complete smooth curve of genus g.]

Finally recall that for a fixed generalized quasi prime divisor $\mathfrak{v}$ of $K|k$, the inertia groups of the generalized quasi prime divisors $T_{\mathfrak{w}_{\mathfrak{v}}}$ of $K\mathfrak{v}\,|\,k\mathfrak{v}$ are of the form $T_{\mathfrak{w}_{\mathfrak{v}}} = T_{\mathfrak{w}}/T_{\mathfrak{v}}$, where $T_{\mathfrak{w}}$ are all the generalized quasi prime divisorial inertia groups in Π_K with $T_{\mathfrak{v}} \subseteq T_{\mathfrak{w}}$.

Proposition 4.4. *In the above notations, recalling that $d := \mathrm{td}(K|k)$, the following hold:*

1) Finite fields: *The field k is an algebraic closure of a finite field iff for all quasi $(d-1)$ divisorial subgroups $T_{\mathfrak{v}}, Z_{\mathfrak{v}}$ of Π_K, the quotient group $\Pi_{K\mathfrak{v}} = Z_{\mathfrak{v}}/T_{\mathfrak{v}}$ endowed with the set of all its quasi divisorial inertia groups $(T_{\mathfrak{w}_{\mathfrak{v}}})_{\mathfrak{w}_{\mathfrak{v}}}$ is curve like in the above sense.*

2) Residual finite fields: *Let $\mathfrak{v}$ be a quasi r-divisorial subgroup $T_{\mathfrak{v}}, Z_{\mathfrak{v}}$ of Π_K with $r < d$. Then the residue field $k\mathfrak{v}$ is an algebraic closure of a finite field iff for every quasi $(d-1)$ divisorial subgroup $T_{\mathfrak{w}} \subset Z_{\mathfrak{w}}$ of Π_K with $T_{\mathfrak{v}} \subset T_{\mathfrak{w}}$ one has: $\Pi_{K\mathfrak{w}} = Z_{\mathfrak{w}}/T_{\mathfrak{w}}$ endowed with all the groups of the form $(T_{\mathfrak{w}_{\mathfrak{v}}}/T_{\mathfrak{w}})_{\mathfrak{w}_{\mathfrak{v}}}$ with $T_{\mathfrak{w}_{\mathfrak{v}}} \subset T_{\mathfrak{w}}$, is curve like in the above sense.*

3) *Let $\Phi : \Pi_K \to \Pi_L$ be the abelianization of an isomorphism $\Pi_K^{\mathrm{c}} \to \Pi_L^{\mathrm{c}}$. If k is an algebraic closure of a finite field, so is l, and Φ defines an isomorphism $\Phi : \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \to \mathcal{G}_{\mathcal{D}_L^{\mathrm{tot}}}$.*

Proof. Assertion 1) is an immediate consequence of Lemma 4.2, and assertion 2) is simply the relative form of assertion 1). For the proof of assertion 3), recall that by Proposition 3.5, 3), Φ maps the flags of generalized quasi divisorial subgroups of Π_K isomorphically onto those of Π_L. Therefore, by assertion 1), it follows that k is an algebraic closure of a finite finite iff l is so. Now supposing that k is an algebraic closure of a finite field, thus l is so too, their generalized quasi prime divisors are actually generalized prime divisors, etc. $\square$

5. Recovering the rational quotients

Our aim in this section is to show that in the case k is an algebraic closure of a finite field, the rational quotients of $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$, thus those of the geometric decomposition graphs $\mathcal{G}_{\mathcal{D}_K}$ for $K|k$, can be recovered form $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$. We will moreover show that the recipe to recover the rational projections is compatible with isomorphisms of geometric decomposition graphs. Before we move on, we recall that for every abelian group A one has: Denoting $A_{(\ell)} := A \otimes \mathbb{Z}_{(\ell)}$, the kernel of the canonical map $A \to A_{(\ell)}$ is an ℓ-divisible subgroup of A.

A) *Geometric decomposition graphs and rational quotients*

For readers sake, we recall here a few basic facts from Pop [14]. Let $K|k$ be a function field as usual, and D be a set of prime divisors of $K|k$. We denote by $T_D \subseteq \Pi_K$ the closed subgroup of generated by all the T_v, $v \in D$, and set $\Pi_{1,D} := \Pi_K/T_D$, and call it the fundamental group of D. In the case of the set of all the prime divisors D_K of $K|k$, we set $\Pi_{1,K} := \Pi_{1,D_K}$, and call it the birational fundamental group of $K|k$.

We say that D is geometric, if there exists a normal model $X \to k$ of $K|k$ such that $D = D_X$ is the set of Weil prime divisors of X. (Note that if D is geometric, we can always choose X with $D = D_X$ and X quasi projective.) For $D = D_X$ geometric, there exist canonical projections $\Pi_{1,D} \to \Pi_1(X)$ and $\Pi_{1,D} \to \Pi_{1,K}$. We further set $\mathrm{Div}(D) := \mathrm{Div}(X)$ and $\mathfrak{Cl}(D) := \mathfrak{Cl}(X)$, the Weil divisors group, respectively divisor class group, of X. Recall that by Pop [14], Section 3, especially the discussion starting at Remarks 3.7, we have the following: Π_K endowed with $(T_v)_{v\in D}$ gives rise canonically to a long exact sequence of ℓ-adically complete $\mathbb{Z}_\ell$-modules

$$1 \to \widehat{U}_D \longrightarrow \widehat{K^\times} \xrightarrow{\mathrm{div}_D} \widehat{\mathrm{Div}}(D) \longrightarrow \widehat{\mathfrak{Cl}}(D) \to 0,$$

where $\widehat{U}_D$ is the ℓ-adic dual of $\Pi_{1,D}$, and div_D is the (ℓ-adic completion of the) D-divisor map, and the third map is the canonical projection. By loc.cit. Proposition 3.10, the fact that $D = D_X$ is complete regular like, i.e., the facts that $\Pi_{1,K} = \Pi_{1,D}$ and that $\widehat{\mathfrak{Cl}}(D)$ has positive rational rank, are encoded in $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ endowed with $D = D_X$. And by loc.cit. Proposition 3.11, if D is "sufficiently" large, the ℓ-divisible subgroup $\mathfrak{Cl}'(D) \subset \mathfrak{Cl}(D)$ and its preimage $\mathrm{Div}'(D)$ in $\mathrm{Div}(D)$ as well as the preimage $\mathcal{L}_D \subset \widehat{K^\times}$ of $\mathrm{Div}'(D)_{(\ell)}$ in $\widehat{K^\times}$ depend on $K|k$ only, and not on D. Further, $\mathrm{Div}'(D)_{(\ell)} = \mathrm{div}_D(\widehat{K^\times}) \cap \mathrm{Div}(D)_{(\ell)}$ and $\mathrm{div}_D(\mathcal{L}_D)/\mathcal{H}_D(K)_{(\ell)} = \mathfrak{Cl}'(D)_{(\ell)}$, where $\mathcal{H}_D(K) := \mathrm{div}_D(K^\times)$.

We conclude this discussion by recalling that a subgraph $\mathcal{D}_K \subset \mathcal{D}_K^{\mathrm{tot}}$ is called geometric complete regular like for $K|k$, if it satisfies the following:

i) For every vertex $\mathfrak{v}$ of $\mathcal{D}_K$, the set $D_{\mathfrak{v}}$ of non-trivial edges of $\mathcal{D}_K$ originating at $K\mathfrak{v}$ is complete regular like.

ii) For every vertex $K\mathfrak{v}$, the trivial edge at $\mathfrak{v}$ belongs to $\mathcal{D}_K$, and every maximal path of oriented edges has length $d = \mathrm{td}(K|k)$.

The subgraph $\mathcal{G}_{\mathcal{D}_K}$ of $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ defined by a geometric (complete regular like) graph $\mathcal{D}_K$ for $K|k$ is called a geometric (complete regular like) decomposition graph for $K|k$. If $\mathcal{G}_{\mathcal{D}_K}$ is such a graph, and D is the set of non-trivial edges of $\mathcal{D}_K$ originating at K, we denote $\widehat{U}_{\mathcal{D}_K} := \widehat{U}_D$, and $\mathcal{L}_{\mathcal{D}_K} := \mathcal{L}_D$, and call them the global units, respectively the canonical divisorial lattice, of $\mathcal{G}_{\mathcal{D}_K}$. We conclude this subsection with the following fact, which follows immediately from loc.cit., Proposition 3.11 and Fact 7.2:

Lemma 5.1. *Let $\jmath_K : K^\times \to \widehat{K^\times}$ be the ℓ-adic completion functor, and $\mathcal{G}_{\mathcal{D}_K}$ be a geometric complete regular like decomposition graph. Then and the following hold:*

1) $\jmath_K(K^\times) \subset \mathcal{L}_{\mathcal{D}_K}$ and $\jmath_K(K^\times) \cap \widehat{U}_{\mathcal{D}_K} = 1$.

2) If k is an algebraic closure of a finite field, then $\mathcal{L}_{\mathcal{D}_K}/\big(\widehat{U}_{\mathcal{D}_K} \cdot \jmath_K(K^\times)\big)$ is a torsion group.

Proof. To 1): The assertion $\jmath_K(K^\times) \subset \mathcal{L}_{\mathcal{D}_K}$ follows from the fact that the image of $\jmath_K(K^\times)$ in $\mathfrak{Cl}(D)$, hence in $\widehat{\mathfrak{Cl}}'(D)$, is trivial. To prove $\jmath_K(K^\times) \cap \widehat{U}_{\mathcal{D}_K} = 1$, recall that by loc.cit., Fact 7.5, for complete regular like $D = D_X$ one has $\Gamma(X, \mathcal{O}_X)^\times = k^\times$. Hence the kernel of the divisor map $\mathrm{div}_D : K^\times \to \mathrm{Div}(D)$ is $k^\times$ and therefore, $\jmath_K(K^\times) \to \widehat{\mathrm{Div}}(D)$ is injective, etc.

To 2): Since $\mathrm{div}_D : \widehat{K^\times} \to \widehat{\mathrm{Div}}(D)$ gives rise to an isomorphism $\mathcal{L}_{\mathcal{D}_K}/\widehat{U}_{\mathcal{D}_K} \to \mathrm{div}_D(\mathcal{L}_{\mathcal{D}_K})$, one has $\mathcal{L}_{\mathcal{D}_K}/\big(\widehat{U}_{\mathcal{D}_K} \cdot \jmath_K(K^\times)\big)_{(\ell)} \cong (\mathcal{L}_{\mathcal{D}_K}/\widehat{U}_{\mathcal{D}_K})/\jmath_K(K^\times)_{(\ell)} \cong \mathrm{div}_D(\mathcal{L}_{\mathcal{D}_K})/\mathcal{H}_D(K)_{(\ell)} = \mathfrak{Cl}'(D)_{(\ell)}$. Since k is an algebraic closure of a finite field, $\mathfrak{Cl}'(D)$ is a torsion group by Pop [14], Fact 7.2, hence $\mathfrak{Cl}'(D)_{(\ell)}$ is a torsion group too. Thus $\mathcal{L}_{\mathcal{D}_K}/\big(\widehat{U}_{\mathcal{D}_K} \cdot \jmath_K(K^\times)\big)_{(\ell)}$ is a torsion group by the above sequence of isomorphisms. Conclude by taking into account the exact sequence

$$0 \to \mathcal{H}_D(K)_{(\ell)}/\mathcal{H}_D(K) \to \mathcal{L}_{\mathcal{D}_K}/\mathcal{H}_D(K) \to \mathfrak{Cl}'(D)_{(\ell)} \to 0,$$

in which both $\mathcal{H}_D(K)_{(\ell)}/\mathcal{H}_D(K)$ and $\mathfrak{Cl}'(X)_{(\ell)}$ are torsion groups. $\square$

B) *Morphisms and rational quotients*

We recall briefly the notion of an abstract rational quotient of a decomposition graph, see Pop [14], Sections 4, and 5, especially Definition/Remark 4.4 and 4.8 of loc cit.

First, let $\mathcal{G}_\alpha$ be the pair consisting of a pro-ℓ group G_α endowed with a system of pro-cyclic subgroups $(T_{v_\alpha})_{v_\alpha}$ which makes G_α curve like of genus $g = 0$ in the sense of Remark/Definition 4.3 above. We notice that if $T \subset G_\alpha$ is a pro-cyclic closed subgroup of G_α and $T_{v_\alpha} \cap T$ non-trivial, then $T \subseteq T_{v_\alpha}$. In particular, T_{v_α} is the only maximal pro-cyclic subgroup of G_α containing T. (We omit the obvious proof).

We fix a distinguished system of generators $\mathfrak{T} := (\tau_{v_\alpha})_{v_\alpha}$ for $(T_{v_\alpha})_{v_\alpha}$, and notice that if $\mathfrak{T}' := (\tau'_{v_\alpha})_{v_\alpha}$ is another distinguished systems of generators, then there exists a unique ℓ-adic unit $\epsilon \in \mathbb{Z}_\ell^\times$ such that $\mathfrak{T}' = \mathfrak{T}^\epsilon$. Let $\widehat{\mathcal{L}}_{\mathcal{G}_\alpha} := \mathrm{Hom}_{\mathrm{cont}}(G_\alpha, \mathbb{Z}_\ell)$ be the ℓ-adic dual of G_α, and $\mathcal{L}_{\mathfrak{T}} \subset \widehat{\mathcal{L}}_{\mathcal{G}_\alpha}$ be the $\mathbb{Z}_{(\ell)}$-submodule generated by all the functionals $\varphi_{v_\alpha} : G_\alpha \to \mathbb{Z}_\ell$ satisfying $\varphi_{v_\alpha}(\tau_{v'_\alpha}) = \delta_{v_\alpha v'_\alpha}$ with $\delta_{v_\alpha v'_\alpha}$ the Kronecker symbol. Notice that if $\mathfrak{T}' = \mathfrak{T}^\epsilon$ as above, then $\epsilon \cdot \mathcal{L}_{\mathfrak{T}'} = \mathcal{L}_{\mathfrak{T}}$, thus the ℓ-adic equivalence class of the $\mathbb{Z}_{(\ell)}$-modules of the form $\mathcal{L}_{\mathfrak{T}}$ depends on $\mathcal{G}_\alpha$ only, and not on the distinguished system of generators $\mathcal{L}_{\mathfrak{T}}$. We will denote such lattices by $\mathcal{L}_{\mathcal{G}_\alpha}$ and call them the divisorial lattices of $\mathcal{G}_\alpha$. We call the indices v_α the 1-indices of $\mathcal{G}_\alpha$ and consider the continuous homomorphisms $\jmath^{v_\alpha} : \widehat{\mathcal{L}}_{\mathcal{G}_\alpha} \to \mathbb{Z}_\ell$ defined by $\jmath^{v_\alpha}(\varphi_{v'_\alpha}) = \delta_{v_\alpha v'_\alpha}$ and call them the (ℓ-adic completions of the abstract) valuations of $\mathcal{G}_\alpha$.

A morphism $\Phi_\alpha : \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \to \mathcal{G}_\alpha$ is every open group homomorphism $\Phi_\alpha : \Pi_K \to G_\alpha$ which inductively on the transcendence degree satisfies the following: First, for every $T_{v_\alpha} \subset G_\alpha$ as above, the set $\Sigma_{v_\alpha} := \{v \,|\, v \text{ prime divisor of } K|k \text{ and } \Phi_\alpha(T_v) \subseteq T_{v_\alpha} \text{ open}\}$ has a finite non-empty intersection with every complete regular like geometric set of prime divisors of $K|k$. Second, for every 1-index v of $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ the following hold:

i) There exists some $T_{v_\alpha} \subset G_\alpha$ such that $\Phi_\alpha(T_v) \subseteq T_{v_\alpha}$.

ii) If $\Phi_\alpha(T_v) = 1$, then the induced homomorphism $\Phi_{\alpha,v} : \Pi_{Kv} \to G_\alpha$ has open image and defines a morphism of the residual (total) decomposition graph $\Phi_{\alpha,v} : \mathcal{G}_{\mathcal{D}_{Kv}^{\mathrm{tot}}} \to \mathcal{G}_\alpha$.

Note that taking ℓ-adic duals, Φ_α gives rise to a Kummer homomorphism $\hat{\phi}_\alpha : \widehat{\mathcal{L}}_{\mathcal{G}_\alpha} \to \widehat{K^\times}$. And notice that $\hat{\phi}(\widehat{\mathcal{L}}_{\mathcal{G}_\alpha}) \cap \widehat{U}_{\mathcal{D}_K}$ is trivial. Indeed, if $\boldsymbol{x} \in \mathcal{L}_{\mathcal{G}_\alpha}$ is non-trivial, then there exists v_α such that $\jmath^{v_\alpha}(\boldsymbol{x}) \neq 0$; hence if v is such that $\Phi_\alpha(T_v) \subseteq T_{v_\alpha}$ is open, then $\jmath^v\big(\hat{\phi}(\boldsymbol{x})\big) \neq 0$.

A morphism $\Phi_\alpha : \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \to \mathcal{G}_\alpha$ is called divisorial, if for each complete regular like decomposition graph $\mathcal{G}_{\mathcal{D}_K}$ for $K|k$ one has $\hat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha}) \subset \epsilon \cdot \mathcal{L}_{\mathcal{D}_K}$ for some ℓ-adic unit $\epsilon \in \mathbb{Z}_\ell^\times$. (Notice that it is enough to have the inclusion $\hat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha}) \subset \epsilon \cdot \mathcal{L}_{\mathcal{D}_K}$ for a single "sufficiently rich" complete regular like decomposition graph $\mathcal{D}_K$ for $K|k$.) Moreover, by Pop [14], Fact 4.9, 1), if Φ_α is divisorial, then there exists a unique divisorial lattice $\mathcal{L}_{\mathcal{G}_\alpha}$ for $\mathcal{G}_\alpha$ such that its image $\mathcal{L}_\alpha = \hat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha})$ is contained in $\mathcal{L}_{\mathcal{D}_K}$. And $\mathcal{L}_\alpha \cap \widehat{U}_{\mathcal{D}_K}$ is trivial, because $\hat{\phi}(\widehat{\mathcal{L}}_{\mathcal{G}_\alpha}) \cap \widehat{U}_{\mathcal{D}_K}$ is so.

Definition 5.2. In the above notations, an abstract rational quotient of $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ is every divisorial morphism $\Phi_\alpha : \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \to \mathcal{G}_\alpha$ which satisfies the following conditions:

i) For all vertices $\mathfrak{v}$ of $\mathcal{D}_K$, denoting by $U_{\mathfrak{v}} \subset K^\times$ the $\mathfrak{v}$-units and by $\jmath_{\mathfrak{v}} : \widehat{U}_{\mathfrak{v}} \to \widehat{K\mathfrak{v}^\times}$ the ℓ-adic completion of the $\mathfrak{v}$-reduction homomorphism $U_{\mathfrak{v}} \to K\mathfrak{v}^\times$, one has: If $\jmath_{\mathfrak{v}}$ is non-trivial on $\widehat{\mathcal{L}}_\alpha \cap \widehat{U}_{\mathfrak{v}}$, then $\widehat{\mathcal{L}}_\alpha \subset \widehat{U}_{\mathfrak{v}}$ and $\jmath_{\mathfrak{v}}$ is injective on $\widehat{\mathcal{L}}_\alpha$.

ii) For every finitely generated $\mathbb{Z}_\ell$-module $\Delta \subset \mathcal{L}_{\mathcal{D}_K} \otimes \mathbb{Z}_\ell$ with $\widehat{U}_{\mathcal{D}_K} \subseteq \Delta$, there exist 1-vertices v of $\mathcal{D}_K$ such that $\Delta \subset \widehat{U}_v$ and the ℓ-adic completion $\jmath^v : \widehat{K^\times} \to \mathbb{Z}_\ell$ of $v : K^\times \to \mathbb{Z}$ and $\jmath_v : \widehat{U}_v \to \widehat{Kv^\times}$ satisfy: $\jmath^v(\widehat{\mathcal{L}}_\alpha) \neq 0$ and $\Delta \cap \ker(\jmath_v) = \Delta \cap \widehat{\mathcal{L}}_\alpha$.

We recall that by Pop [14], Proposition 5.6, one has: If $x \in K$ is a general element, i.e., $\kappa_x = k(x)$ is relatively algebraically closed in K, then the canonical projection of Galois groups $\Phi_{\kappa_x} : \Pi_K \to \Pi_{\kappa_x}$ defines an abstract rational quotient $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \to \mathcal{G}_{\kappa_x}$ in the sense above.

We are now in the position to recover the geometric rational quotients of $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ in the case the base field k is an algebraic closure of some finite field.

Proposition 5.3. *Let k is an algebraic closure of a finite field. Then the following hold:*

1) *Every abstract rational quotient $\Phi_\alpha : \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \to \mathcal{G}_\alpha$ of $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ is geometric, i.e., there exists a general element $x \in K$ and an isomorphism of decomposition graphs $\Phi_{\alpha,\kappa_x} : \mathcal{G}_\alpha \to \mathcal{G}_{\kappa_x}$ such that $\Phi_{\kappa_x} = \Phi_{\alpha,\kappa_x} \circ \Phi_\alpha$.*

2) *Let $L|l$ be a function field over an algebraically closed field l, and $\Phi : \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \to \mathcal{H}_{\mathcal{D}_L^{\mathrm{tot}}}$ be an (abstract) isomorphism of decomposition graphs. Then Φ is compatible with geometric rational quotients in the sense that if $\Phi_{\kappa_y} : \mathcal{H}_{\mathcal{D}_L^{\mathrm{tot}}} \to \mathcal{G}_{\kappa_y}$ is a geometric rational quotient of $\mathcal{H}_{\mathcal{D}_L^{\mathrm{tot}}}$, then $\Phi_\alpha := \Phi_{\kappa_y} \circ \Phi$ is a geometric rational quotient of $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$.*

Proof. To 1): Let $\mathcal{D}_K \subseteq \mathcal{D}_K^{\mathrm{tot}}$ be a complete regular like decomposition graph for $K|k$. (Recall that by Pop, Definition/Remark 3.9, 3), there are “many” complete regular like prime divisor graphs for $K|k$, hence “many” complete regular like decomposition graphs.) Since $\Phi_\alpha : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{G}_\alpha$ is divisorial, by Pop [14], Fact 4.9, 1), there exists a unique divisorial lattice $\mathcal{L}_{\mathcal{G}_\alpha}$ for $\mathcal{G}_\alpha$ such that $\mathcal{L}_\alpha = \hat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha})$ is contained in $\mathcal{L}_{\mathcal{D}_K}$. And recall that $\mathcal{L}_\alpha$ is $\mathbb{Z}_{(\ell)}$-saturated in $\mathcal{L}_{\mathcal{D}_K}$, i.e., $\mathcal{L}_{\mathcal{D}_K}/\mathcal{L}_\alpha$ is $\mathbb{Z}_{(\ell)}$-torsion free, because Φ_α is surjective. Further, if $u \in K$ is non-constant, and $\kappa_u \subset K$ is the corresponding relatively algebraically closed subfield, then $\jmath_K(\kappa_u^\times)_{(\ell)}$ is $\mathbb{Z}_{(\ell)}$-saturated in $\mathcal{L}_{\mathcal{D}_K}$, i.e., $\mathcal{L}_{\mathcal{D}_K}/\jmath_K(\kappa_u^\times)_{(\ell)}$ is $\mathbb{Z}_{(\ell)}$-torsion free.[2]

Now since k is an algebraic closure of a finite field, it follows by Lemma 5.1, that for every $\boldsymbol{u} \in \mathcal{L}_{\mathcal{D}_K}$, in particular for $\boldsymbol{u} \in \mathcal{L}_\alpha$, there exists a multiple $\boldsymbol{u}^n$ such that $\boldsymbol{u}^n \in \widehat{U}_{\mathcal{D}_K} \cdot \jmath_K(K^\times)$. Let $n_{\boldsymbol{u}}$ be the minimal positive integer with this property, and set $\boldsymbol{u}^{n_{\boldsymbol{u}}} = \theta \cdot \jmath_K(u)$ for some $u \in K^\times$ and $\theta \in \widehat{U}_{\mathcal{D}_K}$; and notice that $\jmath_K(u)$ and θ with the property above are unique, because $\widehat{U}_{\mathcal{D}_K} \cap \jmath_K(K^\times)$ is trivial, by Lemma 5.1, 1). Further, $n_{\boldsymbol{u}}$ is relatively prime to ℓ, because $\mathcal{L}_{\mathcal{D}_K}/\jmath_K(\kappa_u^\times)_{(\ell)}$ has no non-trivial $\mathbb{Z}_{(\ell)}$-torsion.

We next show that actually $\mathcal{L}_\alpha = \jmath_K(\kappa_u^\times)_{(\ell)}$.

Claim 1. $\mathcal{L}_\alpha \subseteq \widehat{U}_{\mathcal{D}_K} \cdot \jmath_K(\kappa_u^\times)_{(\ell)}$.

By contradiction, suppose that $\mathcal{L}_\alpha$ is not contained in $\widehat{U}_{\mathcal{D}_K} \cdot \jmath_K(\kappa_u^\times)_{(\ell)}$, and let $\boldsymbol{y} \in \mathcal{L}_\alpha$ be such that $\boldsymbol{y} \notin \widehat{U}_{\mathcal{D}_K} \cdot \jmath_K(\kappa_u^\times)_{(\ell)}$. Setting $\boldsymbol{y}^{n_{\boldsymbol{y}}} = \eta \cdot \jmath_K(y)$ for some $y \in K^\times$ as above, since $\boldsymbol{y} \notin \widehat{U}_{\mathcal{D}_K}$, it follows that y is non-constant. Further, since $n_{\boldsymbol{y}}$ is not divisible by ℓ, it follows that $n_{\boldsymbol{y}} \in \mathbb{Z}_{(\ell)}^\times$. Hence $\boldsymbol{y} \notin \widehat{U}_{\mathcal{D}_K} \cdot \jmath_K(\kappa_u^\times)_{(\ell)}$ iff $\boldsymbol{y}^{n_{\boldsymbol{y}}} \notin \widehat{U}_{\mathcal{D}_K} \cdot \jmath_K(\kappa_u^\times)_{(\ell)}$. Hence without loss of generality, we can suppose that $n_{\boldsymbol{y}} = 1$, hence $\boldsymbol{y} = \eta \cdot \jmath_K(y)$ for some $y \in K^\times$ and $\eta \in \widehat{U}_{\mathcal{D}_K}$. Now let Δ_u and Δ_y be the $\mathbb{Z}_\ell$-submodules of $\widehat{K^\times}$ generated by $\widehat{U}_{\mathcal{D}_K}$ and $\jmath_K(u)$, respectively

[2]Recall that for every abelian group A we denote $A_{(\ell)} := A \otimes \mathbb{Z}_{(\ell)}$.

$\widehat{U}_{\mathcal{D}_K}$ and $\jmath_K(y)$. Then since $\jmath_K(y) \notin \widehat{U}_{\mathcal{D}_K} \cdot \jmath_K(\kappa_u^\times)$, it follows that u and y are algebraically independent over k. Thus $\Delta_y \cap \widehat{\kappa_u^\times} = 1$ inside $\widehat{K^\times}$, as well as $\Delta_u \cap \widehat{\kappa_y^\times} = 1$. But then by Pop [14], Proposition 5.5, 3), it follows that for almost all 1-vertices v of $\mathcal{D}_K$ which are non-trivial on κ_u one has: $\Delta_y \subset \widehat{U}_v$, and $\jmath_v$ maps Δ_y injectively into $\widehat{Kv}$. In particular, since $1 \neq \boldsymbol{y} \in \Delta_y$, it follows that $\jmath_v(\boldsymbol{y}) \neq 1$. Therefore, since $\boldsymbol{y} \in \widehat{\mathcal{L}}_\alpha$, it follows by property i) in the Definition 5.2 above, that $\jmath_v$ maps $\widehat{U}_{\mathcal{D}_K} \cdot \widehat{\mathcal{L}}_\alpha$ injectively into $\widehat{Kv}$. Hence $\jmath_v$ is injective on Δ_u too. This is a contradiction, because $1 \neq \jmath_K(u) \in \Delta_u$, but $\jmath_v\big(\jmath_K(u)\big) = 1$, as noticed (and used) in the proof of loc.cit. Thus we conclude that $\mathcal{L}_\alpha \subseteq \widehat{U}_{\mathcal{D}_K} \cdot \jmath_K(\kappa_u^\times)_{(\ell)}$.

Claim 2. $\jmath_K(\kappa_u^\times)_{(\ell)} \subseteq \mathcal{L}_\alpha$.

For $t \in \kappa_u^\times$, let Δ_t be the $\mathbb{Z}_\ell$-submodule of $\widehat{K^\times}$ generated by $\jmath_K(t)$ and $\widehat{U}_{\mathcal{D}_K}$, and notice that $\Delta_t \subset \mathcal{L}_{\mathcal{D}_K} \otimes \mathbb{Z}_\ell$. Then by hypothesis ii) of Definition 5.2 for $\Delta := \Delta_t$, there exists some 1-vertex v of $\mathcal{D}_K$ with $\Delta_t \subset \widehat{U}_v$ and $\jmath^v(\widehat{\mathcal{L}}_\alpha) \neq 0$ and $\Delta_t \cap \ker(\jmath_v) = \Delta_t \cap \widehat{\mathcal{L}}_\alpha$. On the other hand, $\widehat{U}_{\mathcal{D}_K} \subset \ker(\jmath^{v'})$ for all 1-vertices v' of $\mathcal{D}_K$, in particular for v. Thus $\jmath^v$ non-trivial on $\widehat{\mathcal{L}}_\alpha \subseteq \widehat{U}_{\mathcal{D}_K} \cdot \widehat{\kappa_u^\times}$ implies that $\jmath^v$ is non-trivial on $\widehat{\kappa_u^\times}$, and equivalently, v is non-trivial on κ_u. But then since $\kappa_u|k$ is a function in one variable, its residue field at v is $\kappa_u v = k$, thus $\jmath_v$ is trivial on $\widehat{U}_v \cap \widehat{\kappa_u^\times}$. Hence since $\jmath_K(t) \in \Delta_t \subset \widehat{U}_v$ and $\jmath_K(t) \in \jmath_K(\kappa_u) \subset \widehat{\kappa_u^\times}$, it follows that $\jmath_K(t) \in \ker(\jmath_v)$; thus we finally get: $\jmath_K(t) \in \Delta_t \cap \ker(\jmath_v)$. On the other hand, $\Delta_t \cap \ker(\jmath_v) = \Delta_t \cap \widehat{\mathcal{L}}_\alpha$ by the choice of v, and therefore, $\jmath_K(t) \in \Delta_t \cap \widehat{\mathcal{L}}_\alpha \subseteq \widehat{\mathcal{L}}_\alpha$. Hence we finally have $\jmath_K(t) \in \widehat{\mathcal{L}}_\alpha \cap \jmath_K(K^\times) \subset \widehat{\mathcal{L}}_\alpha \cap \mathcal{L}_{\mathcal{D}_K} = \mathcal{L}_\alpha$. The Calim 2 is proved.

In order to conclude that $\mathcal{L}_\alpha = \jmath_K(\kappa_u^\times)_{(\ell)}$, we first notice that by Claim 1 and Claim 2 we have $\jmath_K(\kappa_u^\times)_{(\ell)} \subseteq \mathcal{L}_\alpha \subseteq \widehat{U}_{\mathcal{D}_K} \cdot \jmath_K(\kappa_u^\times)_{(\ell)}$. On the other hand, by the by the discussion before Definition 5.2 above, it follows that $\mathcal{L}_\alpha \cap \widehat{U}_{\mathcal{D}_K}$ is trivial, thus finally, $\mathcal{L}_\alpha = \jmath_K(\kappa_u^\times)_{(\ell)}$.

Taking ℓ-adic duals, it follows that $\ker(\Phi_\alpha) = \ker(\Phi_{\kappa_u})$, and therefore there exists a group isomorphism $\Phi_{\alpha,\kappa_x} : G_\alpha \to \Pi_{\kappa_u}$ such that $\Phi_{\kappa_x} = \Phi_{\alpha,\kappa_x} \circ \Phi_\alpha$. In particular, $\big(\Phi_{\alpha,\kappa_u}(T_{v_\alpha})\big)_{v_\alpha}$ is a system of generators of Π_{κ_u}.

Claim 3. $\big(\Phi_{\alpha,\kappa_u}(T_{v_\alpha})\big)_{v_\alpha}$ consists of all the inertia subgroups of Π_{κ_u} and $\kappa_u|k$ is rational.

Indeed: First, for all prime divisors v of $K|k$ we have $\Phi_{\kappa_u}(T_v) = \Phi_{\alpha,\kappa_x}\big(\Phi_\alpha(T_v)\big)$. On the other hand, for every T_v we have: If $\Phi_\alpha(T_v)$ is non-trivial, it follows bt the definition of Φ_α than there exists a unique T_{v_α} such that $\Phi_\alpha(T_v) \subseteq T_{v_\alpha}$ and T_{v_α} is the unique maximal pro-cyclic subgroup of G_α containing $\Phi_\alpha(T_v)$ (as mentioned at the beginning of this sub-section). Correspondingly, since $\Phi_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$ is a geometric quotient of $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$, the same is true for if Φ_{κ_u}, namely: If $\Phi_{\kappa_u}(T_v)$ is non-trivial, there exists a unique prime divisor v_{κ_u} of $\kappa_u|k$ with $\Phi_{\kappa_u}(T_v) \subset T_{v_{\kappa_u}}$ and $T_{v_{\kappa_u}}$ is the unique maximal pro-cyclic subgroup of Π_{κ_u} containing $\Phi_{\kappa_u}(T_v)$. Since $\Phi_{\kappa_u} = \Phi_{\alpha,\kappa_u} \circ \Phi_\alpha$, from the above we conclude that Φ_{α,κ_u} maps $(T_{v_\alpha})_{v_\alpha}$ isomorphically onto the corresponding $(T_{v_{\kappa_u}})_{v_{\kappa_u}}$. Hence Π_{κ_u} is generated by $(T_{v_{\kappa_u}})_{v_{\kappa_u}}$, and therefore, $\kappa_u|k$ is a rational function field, as claimed.

In order to conclude the proof of assertion 1), we notice that if $(\tau_{v_\alpha})_{v_\alpha}$ is a distinguished system of generators of G_α, then $(\tau_{v_{\kappa_u}})_{v_{\kappa_u}} := \Phi_{\alpha,\kappa_u}\big((\tau_{v_\alpha})_{v_\alpha}\big)$ is a distinguished systems of generators of Π_{κ_u}. Thus finally, $\Phi_{\alpha,\kappa_u} : \mathcal{G}_\alpha \to \mathcal{G}_{\kappa_u}$ is an isomorphism of abstract decomposition groups, see e.g., Pop [14], Definition/Remark 4.5, 5), and this concludes the proof of assertion 1).

To 2): First, by Pop [14], Proposition 5.4, for every isomorphism $\Phi : \mathcal{G}_{\mathcal{D}_K^{\rm tot}} \to \mathcal{H}_{\mathcal{D}_L^{\rm tot}}$ one has: For every (complete regular like) decomposition subgraph $\mathcal{G}_{\mathcal{D}_K}$ of $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ there exists a unique (complete regular like) decomposition subgraph graph $\mathcal{H}_{\mathcal{D}_L}$ of $\mathcal{H}_{\mathcal{D}_L^{\rm tot}}$ such that Φ defines by restriction an isomorphism $\mathcal{G}_{\mathcal{D}_K} \to \mathcal{H}_{\mathcal{D}_L}$. In particular, for every multi-index $\mathfrak{v}$ of $\mathcal{D}_K$ and the corresponding multi-index $\mathfrak{w}$ of $\mathcal{D}_L$ –including the trivial multi-index, i.e., the trivial valuation $\mathfrak{v}_0$ of K, to which the isomorphism $\Phi : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{H}_{\mathcal{D}_L}$ corresponds– the residual morphism $\Phi_{\mathfrak{v}} : \mathcal{G}_{\mathfrak{v}} := \mathcal{G}_{\mathcal{D}_{K\mathfrak{v}}} \to \mathcal{H}_{\mathcal{D}_{L\mathfrak{w}}} =: \mathcal{H}_{\mathfrak{w}}$ is an isomorphism. Hence each $\Phi_{\mathfrak{v}} : \mathcal{G}_{\mathfrak{v}} \to \mathcal{H}_{\mathfrak{w}}$ is divisorial by loc.cit. Proposition 4.7. Thus each corresponding Kummer morphism $\hat{\phi}_{\mathfrak{v}} : \widehat{L\mathfrak{w}^\times} \to \widehat{K\mathfrak{v}^\times}$ is an isomorphism, and by loc.cit. there exists $\epsilon_{\mathfrak{v}} \in \mathbb{Z}_\ell^\times$, which is unique up to multiplication by $\epsilon'_{\mathfrak{v}} \in \mathbb{Z}_{(\ell)}^\times$, such that denoting by $\mathcal{L}_{\mathfrak{v}}$ and $\mathcal{L}_{\mathfrak{w}}$ the canonical divisorial lattices of $\mathcal{G}_{\mathfrak{v}}$, respectively $\mathcal{H}_{\mathfrak{w}}$, one has: $\hat{\phi}_{\mathfrak{v}}(\mathcal{L}_{\mathfrak{w}}) = \epsilon_{\mathfrak{v}} \cdot \mathcal{L}_{\mathfrak{v}}$. Furthermore, the reduction homomorphisms $\jmath_{\mathfrak{v}}$ and $\jmath_{\mathfrak{w}}$ and the Kummer isomorphisms $\hat{\phi}$ and $\hat{\phi}_{\mathfrak{v}}$ give rise to the first commutative diagram below; and if v and w are 1-vertices which correspond to each other, the morphisms $\jmath^v : \widehat{K^\times} \to \mathbb{Z}_\ell$ and $\jmath^w : \widehat{L^\times} \to \mathbb{Z}_\ell$ give rise for some $a_{vw} \in \mathbb{Z}_{(\ell)}$ to the second commutative diagram below; see e.g., Pop [14], Remark 4.3, 3); respectively loc.cit. Remark 4.3, 4) together with loc.cit. Remark 4.5, 5):

$$\begin{array}{ccc} \widehat{U}_{\mathfrak{w}} & \xrightarrow{\hat{\phi}} & \widehat{U}_{\mathfrak{v}} \\ \downarrow{\scriptstyle \jmath_{\mathfrak{w}}} & & \downarrow{\scriptstyle \jmath_{\mathfrak{v}}} \\ \widehat{L\mathfrak{w}^\times} & \xrightarrow{\hat{\phi}_{\mathfrak{v}}} & \widehat{K\mathfrak{v}^\times} \end{array} \qquad \begin{array}{ccc} \widehat{L^\times} & \xrightarrow{\jmath^w} & \mathbb{Z}_\ell \\ \downarrow{\scriptstyle \hat{\phi}} & & \downarrow{\scriptstyle a_{vw}} \\ \widehat{K^\times} & \xrightarrow{\jmath^v} & \mathbb{Z}_\ell \end{array}$$

Since $\jmath_{\mathfrak{v}}(\mathcal{L}_{\mathcal{D}_K} \cap \widehat{U}_{\mathfrak{v}}) \subseteq \mathcal{L}_{\mathfrak{v}}$ and $\jmath_{\mathfrak{w}}(\mathcal{L}_{\mathcal{D}_L} \cap \widehat{U}_{\mathfrak{w}}) \subseteq \mathcal{L}_{\mathfrak{w}}$, it follows that $\hat{\phi}(\mathcal{L}_{\mathcal{D}_L}) = \epsilon \cdot \mathcal{L}_{\mathcal{D}_K}$ for some $\epsilon \in \mathbb{Z}_\ell^\times$ implies $\hat{\phi}_{\mathfrak{v}}(\mathcal{L}_{\mathfrak{w}}) = \epsilon \cdot \mathcal{L}_{\mathfrak{v}}$. Thus we can suppose that $\epsilon_{\mathfrak{v}} = \epsilon$ is independent of $\mathfrak{v}$.

Now let $\Phi_{\kappa_y} : \mathcal{H}_{\mathcal{D}_L^{\rm tot}} \to \mathcal{G}_{\kappa_y}$ be a geometric rational quotient of $\mathcal{H}_{\mathcal{D}_L^{\rm tot}}$. We claim that

$$\Phi_\alpha := \Phi_{\kappa_y} \circ \Phi : \mathcal{G}_{\mathcal{D}_K^{\rm tot}} \to \mathcal{G}_{\kappa_y} =: \mathcal{G}_\alpha$$

is an abstract rational quotient of $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$. Indeed: First, since $\Phi : \Pi_K \to \Pi_L$ is an isomorphism and Φ_{κ_y} is surjective, Φ_α is surjective too. Second, the Kummer morphism $\hat{\phi}_\alpha$ of Φ_α is nothing but $\hat{\phi}_\alpha = \hat{\phi} \circ \hat{\phi}_{\kappa_y}$, and we have $\hat{\phi}(\mathcal{L}_{\mathcal{D}_L}) = \epsilon \cdot \mathcal{L}_{\mathcal{D}_K}$ and $\hat{\phi}_{\kappa_y}(\mathcal{L}_{\kappa_y}) \subset \mathcal{L}_{\mathcal{D}_L}$. And $\mathcal{L}_{\mathcal{G}_\alpha} := \mathcal{L}_{\kappa_y}$ is a divisorial lattice for $\mathcal{G}_\alpha$ which satisfies: $\hat{\phi}_\alpha(\mathcal{L}_{\mathcal{G}_\alpha}) = \hat{\phi}\big(\hat{\phi}_{\kappa_y}(\mathcal{L}_{\kappa_y})\big) \subset \hat{\phi}(\mathcal{L}_{\mathcal{D}_L}) = \epsilon \cdot \mathcal{L}_{\mathcal{D}_K}$. Thus we conclude that Φ_α is divisorial. Finally, the conditions i), ii), of Definition 5.2 follow instantly using the commutative diagrams above and the fact that Φ_{κ_y} is a rational quotient of $\mathcal{H}_{\mathcal{D}_L^{\rm tot}}$, and we omit the straightforward proofs.

Hence by applying assertion 1) we have: Since Φ_α is an abstract rational quotient of $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$, it follows that Φ_α is actually a geometric quotient of $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$, etc. $\square$

6. Proof of Theorem 2.2

First, assertion 1) follows from the above Propositions 3.5 and 4.4. Concerning assertion 2), apply assertion 1) of Theorem 2.2 together with the above Propositions 5.3.

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