Peter Roquette zu seinem 90. Geburtstag gewidmet

DISTINGUISHING EVERY FINITELY GENERATED FIELD OF CHARACTERISTIC $\neq 2$ BY A SINGLE FIELD AXIOM †

THE STRONG ELEMENTARY EQUIVALENCE VS ISOMORPHY PROBLEM IN CHARACTERISTIC $\neq 2$

FLORIAN POP

ABSTRACT. We show that the isomorphy type of every finitely generated field K with $\mathrm{char}(K) \neq 2$ is encoded by a ***single explicit axiom*** $\boldsymbol{\vartheta}_K$ ***in the language of fields***, i.e., for all finitely generated fields L one has: $\boldsymbol{\vartheta}_K$ holds in L if and only if $K \cong L$ as fields. This extends earlier results by JULIA ROBINSON, RUMELY, POONEN, SCANLON, the author, and others.

1. INTRODUCTION

We begin by recalling that a sentence, or an axiom in the language of fields is any formula in the language of fields which has not free variables. One denotes by $\mathfrak{Th}(K)$ the set of all the sentences in the language of fields which hold in a given field K. For instance, by mere definitions, the field axioms are part of $\mathfrak{Th}(K)$ for every field K; further the fact that K is algebraically closed, as well as $\mathrm{char}(K)$ are encoded in $\mathfrak{Th}(K)$. Namely, K is algebraically closed iff K satisfies the scheme of axioms of algebraically closed fields (asserting that every non-zero polynomial $p(T)$ over K has a root in K); respectively one has $\mathrm{char}(K) = p \geqslant 0$ iff K satisfies the char $= p$ scheme of axioms (asserting: $\mathrm{char} = p > 0$ iff $\Sigma_{\mathrm{i}=1}^{p} 1 = 0$, respectively $\mathrm{char} = 0$ iff $\Sigma_{\mathrm{i}=1}^{n} 1 \neq 0$ for all n). On the other hand, if $K := \mathbb{Q}(t)$ is the rational function field in the variable t over $\mathbb{Q}$, then the usual way to say that t is transcendental over $\mathbb{Q}$, namely "$p(t) \neq 0$ *for all non-zero polynomials* $p(T)$ *over* $\mathbb{Q}$" is not a scheme of axioms in the language of fields (because t is not part of the language of fields).

Two fundamental general type results in algebra are the following:

- Algebraically closed fields K, L have $\mathfrak{Th}(K) = \mathfrak{Th}(L)$ iff $\mathrm{char}(K) = \mathrm{char}(L)$.
- Arbitrary fields K, L have $\mathfrak{Th}(K) = \mathfrak{Th}(L)$ iff there are isomorphic ultra-powers ${}^{*}K \cong {}^{*}L$.

Restricting to fields which are at the center of (birational) arithmetic geometry, namely the finitely generated fields K, which are the function fields of integral schemes of finite type, the elementary theory $\mathfrak{Th}(K)$ is both extremely rich and mysterious. The so called *Elementary equivalence vs Isomorphism Problem* (EEIP) is about five decades old, and asks whether $\mathfrak{Th}(K)$ encodes the isomorphy type of K in the class of all the finitely generated

2010 *Mathematics Subject Classification.* Primary 11G30, 14H25; Secondary 03C62, 11G99, 12G10, 12G20, 12L12, 13F30.

Key words and phrases. Elementary equivalence vs isomorphism, first-order definability, e.g. of valuations, finitely generated fields, Milnor K-groups, Galois/étale cohomology, Kato's higher local-global Principles

† **Note**: This manuscript is a revised version of POP [P5]. The main result of this manuscript was also announced by DITTMANN [Di] (using the same technical tools, but expending rather on POP [P4]).

fields; or equivalently, whether there exists a system of axioms in the language of fields which characterizes K among all the finitely generated fields. On the other hand, building on Julia Robinson [Ro1], [Ro2] methods and ideas, Rumely [Ru] showed at the end of the 1970's that for every global field K there exists a sentence $\boldsymbol{\vartheta}_K^{\mathrm{Ru}}$ which characterizes the isomorphy type of K as a global field, i.e., if L is a global field, then $\boldsymbol{\vartheta}_K^{\mathrm{Ru}}$ holds in L iff $K \cong L$ as fields. In other words, the isomorphy type of K as a global field is characterized by a *single explicit axiom* $\boldsymbol{\vartheta}_K^{\mathrm{Ru}}$ *in the language of fields*. This goes far beyond the EEIP in the class of global fields!

Arguably, it is the ***main open question*** in the elementary (or first-order) theory of finitely generated fields whether a fact similar to Rumely's result [Ru] holds for all finitely generated fields K, namely whether there is a field axiom $\boldsymbol{\vartheta}_K$ which characterizes the isomorphy type of K in the class of all finitely generated fields; this question is also called the strong EEIP. We notice that the (strong) EEIP is open in general; see Pop [P2], [P3], for more details and references on the EEPI both over finitely generated fields and function fields over algebraically closed base fields. A first attempt towards tackling the strong EEIP was Scanlon [Sc], and that reduces the strong EEIP for each K to first-order defining "sufficiently many" divisorial valuations of K.[1] Finally, Pop [P4] tackles the strong EEIP for finitely generated fields which are function fields of curves over global fields. In the present note we generalize that result to all finitely generated fields of characteristic $\neq 2$.

Main Theorem. *For every finitely generated field K with* $\mathrm{char}(K) \neq 2$*, there exists a sentence $\boldsymbol{\vartheta}_K$ in the language of fields such that for all finitely generated fields L one has:*

$$\boldsymbol{\vartheta}_K \textit{ holds in } L \textit{ if and only if } L \cong K \textit{ as fields.}$$

The Main Theorem above will be proved in Section 5. One can give three (by some standards similar) proofs. A first-proof follows simply from Scanlon, by invoking Theorem 1.1 below for the definability of geometric prime divisors (thus circumventing the gap in the proof of defining divisorial valuations in Section 3 of loc.cit.). A second proof reduces the Main Theorem above to results by Aschenbrenner–Khélif–Naziazeno–Scanlon [AKNS], by showing that finitely generated integrally closed subdomains in finitely generated fields of characteristic $\neq 2$ are uniformly first-order definable. Among other things, these proofs show that finitely generated fields of characteristic $\neq 2$ are ***bi-interpretable with arithmetic***, see e.g. [Sc], Section 2, and/or [AKNS], Section 2, for a detailed discussion of bi-interpretablility with arithmetic. Third, a more direct proof based on Pop [P2], Poonen [Po1], and consequences of Rumely [Ru] (namely that the number fields are bi-interpretable with arithmetic).

The main step and technical key point in the proof of the Main Theorem is to give formulae val_d, all $d > 0$, in the language of fields, which uniformly first-order define the geometric prime divisors of finitely generated fields K with $\mathrm{char}(K) \neq 2$ and $\dim(K) = d$. That is the content of Theorem 1.1 below, which could be viewed as the main result of this note.

To make these assertions more precise, let us introduce notation and mention a few fundamental facts about finitely generated fields, to be used throughout the manuscript.

For arbitrary fields Ω, let $\kappa_0 \subset \Omega$ denote their prime fields. Recall that the Kronecker dimension of Ω is $\dim(\Omega) = \dim(\kappa_0) + \mathrm{td}(\Omega|\kappa_0)$, where $\mathrm{td}(\Omega|\kappa_0)$ denotes the transcendence degree, and $\dim(\mathbb{F}_p) = 0$, $\dim(\mathbb{Q}) = 1$. We denote by $\kappa := \Omega^{\mathrm{abs}}$ the constant subfield of Ω, i.e., the elements of Ω which are algebraic over the prime field $\kappa_0 \subset \Omega$, and set $\tilde{\Omega} := \Omega[\sqrt{-1}\,]$.

[1] See the discussion below for more about this.

For $\boldsymbol{a} := (a_1, \dots, a_r)$ with $a_i \in \Omega^\times$ we consider the r-fold Pfister form[2] $q_{\boldsymbol{a}}(\boldsymbol{x})$ in the variables $\boldsymbol{x} = (x_1, \dots, x_{2^r})$ and for field extensions $\Omega'|\Omega$ define the image of Ω' under $q_{\boldsymbol{a}}$ as being

$$q_{\boldsymbol{a}}(\Omega') := \{q_{\boldsymbol{a}}(\boldsymbol{x}') \mid \boldsymbol{x}' \in \Omega'^{2^r},\ \boldsymbol{x}' \neq \boldsymbol{0}\}.$$

Next we recall that, using among other things the Milnor Conjecture,[3] by Pop [P2] there are sentences φ_d, and by Poonen [Po1] there is a predicate $\psi^{\text{abs}}(x)$, and formulas $\psi_r(\boldsymbol{t})$ with free variables $\boldsymbol{t} := (t_1, \dots, t_r)$ such that for all finitely generated fields K and $\kappa = K^{\text{abs}} \subset K$, setting $\tilde{K} := K[\sqrt{-1}]$, one has:

- $\dim(K) = d$ iff φ_d holds in K. Actually, $\varphi_d \equiv \big((\varphi^0_d \wedge 2 = 0) \vee (\varphi^0_{d+1} \wedge 2 \neq 0)\big)$, where
$$\varphi^0_r \;\equiv\; \big(\exists\, \boldsymbol{a} = (a_1, \dots, a_r) \text{ s.t. } 0 \notin q_{\boldsymbol{a}}(\tilde{K})\big) \;\&\; \big(\forall\, \boldsymbol{a} = (a_1, \dots, a_{r+1}) \text{ one has } 0 \in q_{\boldsymbol{a}}(\tilde{K})\big).$$
- κ is defined by $\psi^{\text{abs}}(x)$ inside K, i.e., one has $\kappa = \{x \in K \mid \psi^{\text{abs}}(x) \text{ holds in } K\}$.
- $t_1, \dots, t_r \in K$ are algebraically independent over κ iff $\psi_r(t_1, \dots, t_r)$ holds in K.

In particular, for algebraically independent elements $\boldsymbol{t}_r := (t_1, \dots, t_r)$ of K, the relative algebraic closure $k_{\boldsymbol{t}_r}$ of $\kappa(\boldsymbol{t}_r)$ in K is *uniformly first-order definable,* hence so are the maximal global subfields $k_0 \subset K$ of K, as well as the transcendence bases $\mathcal{T} := (t_1, \dots, t_{d_K})$ of $K|\kappa$.

A prime divisor of K is (the valuation ring of) a valuation v whose residue field Kv satisfies

$$\dim(Kv) = \dim(K) - 1.$$

It turns out that prime divisors v of finitely generated fields are discrete valuations, and Kv is a finitely generated fields as well. A prime divisor v of K is called arithmetic, if v is non-trivial on $\kappa = K^{\text{abs}}$ —in particular κ must be a number field, respectively geometric, if v is trivial on κ. Recall that Rumely [Ru] gives formulae val_1 which uniformly first-order define the prime divisors of global fields, and Pop [P4] gives formulae val_2 which uniformly first-order define the *geometric prime divisors* in the case $\dim(K) = 2$. The focus of this note is to give similar formulae val_d for fields $\bullet$ satisfying:

(H) $\quad \bullet$ is finitely generated, $\quad d := \dim(\bullet) > 2, \quad \operatorname{char}(\bullet) \neq 2$

Theorem 1.1. *There is an explicit procedure that, given an integer $d > 1$, produces a first-order formula val_d that in any finitely generated field K of characteristic* $\operatorname{char}(K) \neq 2$ *and Kronecker dimension* $\dim(K) = d$ *defines all the geometric prime divisors of K.*

For the proof see Section 4, Theorem 4.2, and Recipe 4.7 for the concrete form of val_d.

We conclude the Introduction with the following remarks.

First, in the early version [P5], the case of finitely generated fields of characteristic zero was considered/dealt with. The methods and techniques of [P5] are unchanged, except a key technical point of the procedure of giving val_d, namely the old Proposition 3.5, whose new variant Proposition 3.4 below works for all finitely generated fields satisfying Hypothesis (H).

Second, although the formulae val_d are completely explicit, see Recipe 4.7, it is an open question whether these formulae are optimal in any concrete sense; in particular, the formulae val_d do not address the question about the ***complexity of (uniform) definability*** of (some or all) the prime divisors. The complexity of definability of valuations deserves

[2] See e.g. Pfister [Pf1], Ch. 2, for basic facts.

[3] Proved by Vojevodsky, Orlov–Vishik–Vojevodsky, and Rost, see e.g. the survey articles [Pf2], [Kh].

further special attention, because among other things it ties in with previous first-order definability results of valuations (of finitely generated fields and more general fields) by Eisenträger [Ei], Eisenträger–Shlapentokh [E-S], Kim-Roush [K-R], Koenigsmann [Ko1, Ko3], Miller-Shlapentokh [M-Sh], Poonen [Po2], Shlapen- tokh [Sh1], [Sh2], and others. The focus of the aforementioned results and research is yet another open problem in the theory of finitely generated fields and function fields, namely the generalized Hilbert Tenth Problem — which for the time being is open over all number fields, e.g. $\mathbb{Q}$, $\mathbb{C}(t)$, etc.

Third, it is strongly believed that the (strong) EEIP should hold for the function fields $K|k$ over "reasonable" base fields k; in particular, since finitely generated fields of characteristic $\neq 2$ are nothing but function fields K over prime fields with char $\neq 2$, the Main Theorem above asserts that $\mathbb{Q}$ and $\mathbb{F}_p$, $p \neq 2$, are "reasonable." If k is an algebraically closed field, facts proved by Durré [Du], Pierce [Pi], Vidaux [Vi] for $\mathrm{td}(K|k) = 1$, respectively Pop [P2] for $\mathrm{td}(K|k)$ arbitrary, are quite convincing partial results supporting the possibility that algebraically closed fields are "reasonable." Finally, Koenigsmann [Ko2], Poonen-Pop [P-P] give evidence for the fact that the much more general large fields k, as introduced in Pop [P1], e.g. $k = \mathbb{R}, \mathbb{Q}_p$, PAC, etc., should be "reasonable" base fields. These partial/preliminary results over large fields (including the algebraically closed ones) do not involve prime divisors of $K|k$. Two fundamental open questions arise: First, is it possible to recover prime k-divisors of functions fields $K|k$ over large fields k, at least in the case of special classes of large fields, e.g. local fields, or quasi-finite fields? Second, are there alternative approaches (which do not involve prime divisors) for recovering the isomorphy type of K from $\mathfrak{Th}(K)$?

Thanks: I would like to thank the participants at several activities, e.g., AWS 2003, AIM Workshop 2004, INI Cambridge 2005, HIM Bonn 2009, ALANT III in 2014, MFO Oberwolfach in 2016, IHP Paris in 2018, for the debates on the topic and suggestions concerning this problem. Special thanks are due to Bjorn Poonen, Thomas Scanlon, Jakob Stix, and Michael Temkin for discussing technical aspects of the proofs, and to Uwe Jannsen and Moritz Kerz for discussions concerning Kato's higher dimensional Hasse local-global principles. The author would also like to thank the University of Heidelberg and the University of Bonn for the excellent working conditions during his visits in 2015 and 2016 as a *Humbodt Preisträger.*

2. Higher dimensional Hasse local-global principles

A) *Notations and general facts*

For a (possibly trivial) valuation v of K, let $\mathfrak{m}_v \subset \mathcal{O}_v \subset K$ be its valuation ideal and valuation ring ring, $vK := K^\times/U_v$ be its (canonical) value group, and $Kv := \kappa(v) := \mathcal{O}_v/\mathfrak{m}_v$ be its residues field. We denote by V_K the Riemann space of K, i.e., the space of all the (equivalence classes of) valuations of K.

Let X be a scheme of finite type over either $\mathbb{Z}$ or a field k. For $x \in X$, let $X_x := \overline{\{x\}} \subset X$ be the closure of x in X, and recall that $\dim(x) := \dim(X_x)$. Following Kato [Ka], define:

$$X_i := \{x \in X \mid \dim(x) = i\}, \quad X^i := \{x \in X \mid \mathrm{codim}(x) = i\},$$

and recall that if X is projective normal integral, then for all $0 \leqslant i \leqslant \dim(X)$ one has:

$$\dim(X) = \mathrm{codim}(x) + \dim(x), \quad \text{and therefore: } x \in X^i \Leftrightarrow x \in X_{\dim(X)-i}$$

Notations/Remarks 2.1. Let K be a finitely generated field, and $k \subset K$ be a subfield.

1) A model of K is a separated scheme of finite type $\mathcal{X}$ with function field $\kappa(\mathcal{X}) = K$. And a k-model of K is a k-variety, i.e., a separated k-scheme of finite type, with $k(X) = K$.

2) Let a model $\mathcal{X}$ of K, and $v \in V_K$ be given. We say that v has center $x \in \mathcal{X}$ on $\mathcal{X}$, if $\mathcal{O}_x \prec \mathcal{O}_v$, that is, $\mathcal{O}_x \subseteq \mathcal{O}_v$ and $\mathfrak{m}_x = \mathfrak{m}_v \cap \mathcal{O}_x$. By the *valuation criterion* one has: Since $\mathcal{X}$ is separated, every $v \in V_K$ has at most one center on $\mathcal{X}$, respectively: $\mathcal{X}$ is proper iff every valuation $v \in V_K$ has a center on $\mathcal{X}$ (which is then unique).
3) Let a k-model X of K and $v \in V_K$ be given. We say that $x \in X$ is the center of v on X, if $\mathcal{O}_x \prec \mathcal{O}_v$. If so, then $v \in V_{K|k}$. By the *valuation criterion* one has: Since X is separated over k, every v has at most one center on X, respectively that X is a proper k-variety iff every k-valuation $v \in V_{K|k}$ has a center on X (which is then unique).
4) A prime divisor of K is any $v \in V_K$ satisfying the following equivalent conditions:
   i) $\dim(Kv) = \dim(K) - 1$.
   ii) v is discrete, and Kv is finitely generated and has $\dim(Kv) = \dim(K) - 1$.
   iii) v is defined by a prime Weil divisor of a projective normal model $\mathcal{X}$ of K.
5) A prime k-divisor of K is any $v \in V_{K|k}$ satisfying the following equivalent conditions:
   i) $\mathrm{td}(Kv\,|\,k) = \mathrm{td}(K|k) - 1$.
   ii) v is a prime divisor of K which is trivial on k.
   iii) v is defined by a prime Weil divisor of a projective normal model X of $K|k$.
6) Let $\mathcal{D}^1_K \supset \mathcal{D}^1_{\mathcal{X}}$ be the spaces of prime divisors of K, respectively the ones defined by the prime Weil divisors of a quasi-projective normal model $\mathcal{X}$ of K. Further define $\mathcal{D}^1_{K|k} \supset \mathcal{D}^1_X$ correspondingly, where X is a quasi-projective normal k-model of K.
7) In the above notation, let $\mathcal{X}$ and X be projective. Then one has canonical identifications:
$$\mathcal{D}^1_{\mathcal{X}} \;\leftrightarrow\; \mathcal{X}^1 \;=\; \mathcal{X}_{\dim(K)-1}, \qquad \mathcal{D}^1_X \;\leftrightarrow\; X^1 \;=\; X_{\mathrm{td}(K|k)-1}.$$

B) *Local-global principles* (LGP)

Let us first recall the famous Hasse–Brauer–Noether LGP. Let k be a global field, $\mathbb{P}(k)$ be the set of non-trivial places of k, and for $v \in \mathbb{P}(k)$, let k_v be the completion of k with respect to v. Denoting by ${}_n(\ )$ the n-torsion in an Abelian group, e.g. ${}_n(\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n$, the Hasse–Brauer–Noether LGP asserts that one has a canonical exact sequence:
$$0 \to {}_n\mathrm{Br}(k) \to \bigoplus_v {}_n\mathrm{Br}(k_v) \to \mathbb{Z}/n \to 0,$$
where, the first map is the direct sum of all the canonical restriction maps ${}_n\mathrm{Br}(k) \to {}_n\mathrm{Br}(k_v)$; thus implicitly, for every division algebra D over k there exist only finitely many v such that $D \otimes_k k_v$ is not a matrix algebra; and the second map is the sum of the invariants $\sum_v \mathrm{inv}_v$.

It is a fundamental observation by Kato [Ka] that the above local-global principle has higher dimensional variants as follows: First, following Kato loc.cit, for every positive integer n, say $n = mp^r$ with p the characteristic exponent and $(m,p) = 1$, an integer twist i, one sets $\mathbb{Z}/n(0) = \mathbb{Z}/n$, and defines in general $\mathbb{Z}/n(i) := \mu_m^{\otimes i} \oplus W_r\,\Omega^i_{\log}[-i]$, where $W_r\,\Omega_{\log}$ is the logarithmic part of the de Rham–Witt complex on the étale site, see Illusie [Ill] for details. With these notations, for every (finitely generated) field K one has:
$$\mathrm{H}^1\big(K, \mathbb{Z}/n(0)\big) = \mathrm{Hom}_{\mathrm{cont}}(G_K, \mathbb{Z}/n), \qquad \mathrm{H}^2\big(K, \mathbb{Z}/n(1)\big) = {}_n\mathrm{Br}(K),$$
where G_K is the absolute Galois group of K. Thus the cohomology groups $\mathrm{H}^{i+1}\big(K, \mathbb{Z}/n(i)\big)$ have a particular arithmetical significance, and in these notation, the Hasse–Brauer–Noether LGP is a local-global principle for the cohomology group $\mathrm{H}^2\big(K, \mathbb{Z}/n(1)\big)$. Noticing that K is

a global field iff $\dim(K)=1$, KATO had the fundamental idea that for finitely generated fields K with $\dim(K)=d$, there should exist similar local-global principles for $\mathrm{H}^{d+1}\big(K,\mathbb{Z}/n(d)\big)$.

- *The Kato cohomological complex* (KC)

We briefly recall Kato's cohomological complex (similar to complexes defined by the Bloch–Ogus) which is the basis of the higher dimensional Hasse local-global principles, see KATO [Ka], §1, for details. Let L be an arbitrary field, and recall the canonical isomorphism (generalizing the classical Kummer Theory isomorphism) $h^1: L^\times/n\to \mathrm{H}^1\big(L,\mathbb{Z}/n(1)\big)$. As explained in [Ka], §1, the isomorphism h^1 gives rise canonically for all $q\neq 0$ to morphisms, which by the (now proven) Milnor–Bloch–Kato Conjecture are actually isomorphisms:

$$h^q: K_q^{\mathrm{M}}(L)/n\to \mathrm{H}^q\big(L,\mathbb{Z}/n(q)\big),\ \{a_1,\dots,a_q\}/n\mapsto h^1(a_1)\cup\ldots\cup h^1(a_q)=: a_1\cup\ldots\cup a_q.$$

Further, let v be a discrete valuation of L. Then one defines the boundary homomorphism

$$\partial_v:\mathrm{H}^{q+1}\big(L,\mathbb{Z}/n(q+1)\big)\to \mathrm{H}^q\big(Lv,\mathbb{Z}/n(q)\big),$$

defined by $a\xmapsto{\partial_v} v(a)$ if $q=0$, $a\cup a_1\cup\ldots\cup a_q\xmapsto{\partial_v} v(a)\cdot a_1\cup\ldots\cup a_q$ for $a\in L^\times$, $a_1,\dots,a_q\in U_v$ if $q>0$.

Now let X be an excellent integral scheme, with generic point η_X, and recall the notations $X_i, X^i\subset X$; hence $X_0\subset X$ are the closed points, and $X_{\dim(X)}=\{\eta_X\}$. By mere definitions, for every $x_{i+1}\in X_{i+1}$, one has that $X_{x_{i+1},1}\subset X_i$ consists of all the points $x_i\in X_i$ which lie in the closure of $X_{x_{i+1}}:=\overline{\{x_{i+1}\}}$. Since X is excellent, the normalization $\tilde{X}_{x_{i+1}}\to X_{x_{i+1}}$ of $X_{x_{i+1}}$ is a finite morphism. Hence for every $x_i\in X_{x_{i+1},1}$, there a finitely many $\tilde{x}\in \tilde{X}_{x_{i+1}}$ such that $\tilde{x}\mapsto x_i$ under $\tilde{X}_{x_{i+1}}\to X_{x_{i+1}}$, and the following hold: The local rings $\mathcal{O}_{\tilde{x}}$ of all $\tilde{x}\mapsto x_i$ are discrete valuations rings of the residue field $\kappa(x_{i+1})$, say with valuation $v_{\tilde{x}}$, and the residue field extensions $\kappa(\tilde{x})|\kappa(x_i)$ are finite field extensions. Then for every integer $n>1$, which is invertible on X, letting $0\leqslant i<\dim(X)$, one gets a sequence of the form:

(KC) $$\ldots\to\oplus_{x_{i+1}\in X_{i+1}}\mathrm{H}^{i+2}\big(\kappa(x_{i+1}),\mathbb{Z}/n(i+1)\big)\to\oplus_{x_i\in X_i}\mathrm{H}^{i+1}\big(\kappa(x_i),\mathbb{Z}/n(i)\big)\to\ldots$$

where the component $\mathrm{H}^{i+2}\big(\kappa(x_{i+1}),\mathbb{Z}/n(i+1)\big)\to\oplus_{x_i\in X_i}\mathrm{H}^{i+1}\big(\kappa(x_i),\mathbb{Z}/n(i)\big)$ is defined by

$$\textstyle\sum_{\tilde{x}\in\tilde{X}_{x_{i+1}},\,\tilde{x}\mapsto x_i}\mathrm{cor}_{\kappa(\tilde{x})|\kappa(x_i)}\circ\delta_{v_{\tilde{x}}}$$

Theorem 2.2 (KATO [Ka], Proposition 1.7). *Suppose that X is an excellent scheme such that for all p dividing n and $x_i\in X_i$ one has: If $p=\mathrm{char}\big(\kappa(x_i)\big)$, then $[\kappa(x_i):\kappa(x_i)^p]\le p^i$. Then* (KC) *is a complex. In particular, if n is invertible on X, then* (KC) *is a complex.*

That being said, the Kato Conjectures are about aspects of the fact that in *arithmetically significant situations,* the complex (KC) above is exact, excepting maybe for $i=0$, where the homology of (KC) is perfectly well understood. And KATO proved himself several forms of the above local-global Principles in the case X is an arithmetic scheme of dimension $\dim(X)=2$ and having further properties. Among other things, one has:

Theorem 2.3 (KATO [Ka], Corollary, p.145). *Let X be a proper regular integral $\mathbb{Z}$-scheme, $\dim(X)=2$, and $K=\kappa(X)$ having no orderings. Then one has an exact sequence:*

$$0\to\mathrm{H}^3\big(K,\mathbb{Z}/n(2)\big)\to\oplus_{x_1\in X_1}\mathrm{H}^2\big(\kappa(x_1),\mathbb{Z}/n(1)\big)\to\oplus_{x_0\in X_0}\mathrm{H}^1\big(\kappa(x_0),\mathbb{Z}/n\big)\to\mathbb{Z}/n\to 0.$$

Finally, notice that in Theorem 2.3 above, K is finitely generated with $\dim(K)=2$. Unfortunately, for the time being, the above result is not known to hold in the same form in higher dimensions $d:=\dim(K)>2$, although it is conjectured to be so. There are nevertheless partial results concerning the local-global principles involving $\mathrm{H}^{d+1}\big(K,\mathbb{Z}/n(d\,)\big)$. From those results, we pick and choose only what is necessary for our goals, see below.

Notations/Remarks 2.4. Let K be a finitely generated field with constant field κ. We supplement Notations/Remarks 2.1 as follows:

1) $n>1$ is a positive integer not divisible by $\mathrm{char}(K)$.
2) $k_0\subset K$ is subfield with $\dim(k_0)=1$, hence a *global field.*
3) Let $\mathbb{P}(k_0)$ be the set of places of k_0. For $v\in\mathbb{P}(k_0)$, consider/denote the following:
   - Let R_v be the Henselization of the valuation ring $\mathcal{O}_v$.
     In particular, R_v is an excellent Henselian DVR with finite residue field.
   - Let $k_{0v}=\mathrm{Quot}(R_v)$ be the corresponding Henseliazation of k_0 at v.

• *Localizing the global field k_0*

In the above notations, for every $v\in\mathbb{P}(k_0)$, consider the compositum $K_v:=Kk_{0v}$ of K and k_{0v} (in some fixed algebraic closure $\overline{K}$). Then via the restriction functor(s) in cohomology, one gets canonical localization maps $\mathrm{H}^{d+1}\big(K,\mathbb{Z}/n(d)\big)\to\mathrm{H}^{d+1}\big(K_v,\mathbb{Z}/n(d\,)\big)$.

Theorem 2.5 (Jannsen [Ja], Theorem 0.4)**.** *In the above notations, suppose that* $\mathrm{char}(K)$ *does not divide n. Then the localization maps give rise to an embedding*

$$\mathrm{H}^{d+1}\big(K,\mathbb{Z}/n(d\,)\big)\to\oplus_{v\in\mathbb{P}(k_0)}\mathrm{H}^{d+1}\big(K_v,\mathbb{Z}/n(d\,)\big).$$

• *Local-global principles over R_v, $v\in\mathbb{P}(k_0)$*

In the above notations, for every non-archimedean place $v\in\mathbb{P}(k_0)$, let $R_v\subset k_{0v}$ be the (unique) Henselization of the valuation ring $\mathcal{O}_v$ inside k_{0v}, hence recall that R_v is a Henselian discrete valuation ring with residue field $\kappa(v)$ finite. This being said, one has the following:

Theorem 2.6 (Kerz–Saito [K–S], Theorem 8.1)**.** *Suppose that R is either* (i) *a finite field, or* (ii) *a Henselian discrete valuation ring with finite residue field, such that n is invertible in R, and $\mu_n\subset R$. Then for every projective regular flat R-scheme X, the complex* (KC) *for X is exact, with the only exception of the homology group* $\mathrm{H}_0(\mathsf{KC})=\mathbb{Z}/n$ *in the case* (i).

3. Consequences/applications of the local-global principles

In this section we give a few consequences of the higher Hasse local-global principles mentioned above, as well as an arithmetical interpretation of these consequences.

A) *A technical result for later use*

Notations/Remarks 3.1. Let L be a field satisfying Hypothesis (H) from the Introduction. Let $k_0\subset L$ be a global subfield, $n\neq\mathrm{char}(L)$ be a *prime number,* and suppose that $\mu_{2n}\subset k_0$.

1) We let S_0 be the canonical model of k_0, that is: $S_0=\mathrm{Spec}\,\mathcal{O}_\kappa$, if $k_0=\kappa$ is a number field; S_0 is the unique projective smooth κ-curve with $\kappa(S_0)=k_0$, if κ is a finite field.
2) $U_0=\mathrm{Spec}\,R_0\subset S_0$ are open subsets with $n\in R_0^\times$, and set:
$$\Delta_{U_0}:=\{u_0'\in k_0^\times\mid v(u_0'-1)>2\cdot v(n),\,\forall\, v\notin U_0\}.$$
3) For $\boldsymbol{f}:=(f_1,\dots,f_r)$ with $f_i\in L^\times$, and dense open subsets $U_0\subset S_0$, denote:
$$\mathrm{H}_{U_0,\boldsymbol{f}}:=\big\langle u_0'\cup f_1\cup\ldots\cup f_r\mid u_0'\in\Delta_{U_0}\big\rangle\subset\mathrm{H}^{r+1}\big(L,\mathbb{Z}/n(r)\big).$$

Proposition 3.2. *In the above notation, let $\mathcal{Z}$ be a projective smooth U_0-variety with generic fiber $Z := \mathcal{Z} \times_{U_0} k_0$, and function field $L = k_0(Z)$ with $\dim(L) = r$. The following hold:*

1) *The map $\mathrm{H}_{U_0,\boldsymbol{f}} \to \oplus_{z\in Z^1} \mathrm{H}^r\big(\kappa(z), \mathbb{Z}/n(r-1)\big)$ from the Kato complex* (KC) *is injective.*
2) *Let $\boldsymbol{\alpha} = u_0' \cup f_1 \cup \ldots \cup f_r \in \mathrm{H}_{U_0,\boldsymbol{f}}$ and $z \in Z^1$ satisfy $\partial_z(\boldsymbol{\alpha}) \neq 0$, and w be the prime divisor of $L|k_0$ with $\mathcal{O}_w = \mathcal{O}_z$. Then there is an f_i such that $w(f_i) \notin n \cdot wL$.*

Proof. For $\boldsymbol{\alpha} \in \mathrm{H}_{U_0,\boldsymbol{f}}$ non-trivial, proceed as follows:

First, by Jannsen's Theorem 2.5 above, there exists $v \in \mathbb{P}(k_0)$ such that $\boldsymbol{\alpha}$ is non-trivial over $L_v = Lk_{0v}$. In particular, u_0' is not an n^{th} power in k_{0v}, hence $u_0' \in \Delta_{U_0}$. In particular, letting $R := \mathcal{O}_v^h$ be the Henselization of $\mathcal{O}_v$, the base change $\mathcal{Z}_R = \mathcal{Z} \times_{U_0} R$ is a smooth R-variety (because $\mathcal{Z}$ was a smooth U_0-variety). Set $\operatorname{Spec} R = \{\eta_0, \mathfrak{m}\}$.

Second, by the Kerz–Saito Theorem 2.6 above, there are points $z_R \in \mathcal{Z}_R^1$ such that one has: $0 \neq \boldsymbol{\alpha}_{z_R} := \partial_{z_R}(\boldsymbol{\alpha}) \in \mathrm{H}^r\big(\kappa(z_R), \mathbb{Z}/n(r-1)\big)$. Hence setting $k_R := \mathrm{Quot}(R) = k_{0v}$, one has: If $z_R \mapsto \eta_0$ under $\mathcal{Z}_R \to \operatorname{Spec} R$, then z_R lies in the generic fiber $Z_{k_R} = \mathcal{Z}_R \times_R k_R$ of $\mathcal{Z}_R$. Hence letting $z_R \mapsto z$ under $\mathcal{Z}_R \to \mathcal{Z}$, one gets: Since $Z_{k_R} = Z \times_{k_0} k_R$, one has $z \in Z^1$. Second, since $\kappa(z) \hookrightarrow \kappa(z_R)$, it follows that $0 \neq \boldsymbol{\alpha}_z := \partial_z(\boldsymbol{\alpha}) \in \mathrm{H}^r\big(\kappa(z), \mathbb{Z}/n(r-1)\big)$, as claimed. Next suppose that $z_R \mapsto \mathfrak{m}$ under $\mathcal{Z}_R \to \operatorname{Spec} R$. Since $\mathcal{Z}_R$ is a projective smooth R-variety, its special fiber $\mathcal{Z}_{\mathfrak{m}}$ is reduced and has projective smooth integral $\kappa(\mathfrak{m})$-varieties as connected components, $\mathcal{Z}_{z_R} = \overline{\{z_R\}}$ being such one. Since $0 \neq \boldsymbol{\alpha}_z := \partial_z(\boldsymbol{\alpha}) \in \mathrm{H}^r\big(\kappa(z), \mathbb{Z}/n(r-1)\big)$, by Kerz–Saito's Thm 2.6, there is $y \in \mathcal{Z}_z^1$ with $0 \neq \partial_y\big(\partial_{z_R}(\boldsymbol{\alpha})\big) \in \mathrm{H}^{r-1}\big(\kappa(y), \mathbb{Z}/n(r-2)\big)$. On the other hand, $\dim(\mathcal{Z}_{z_R}) = \dim(\mathcal{Z}_R) - 1$, hence $\mathcal{Z}_{z_R}^1 \subset \mathcal{Z}_R^2$. Hence $\operatorname{codim}_{\mathcal{Z}_R}(y) = 2$, and $y \mapsto \mathfrak{m}$ under $\mathcal{Z}_R \to \operatorname{Spec} R$. Let $\mathcal{Z}(y) := \big\{z_R' \in \mathcal{Z}_R^1 \mid y \in \overline{\{z_R'\}}\big\}$. The Kato complex (KC) for the projective smooth R-scheme $\mathcal{Z}_R$ implies: $\sum_{z_R' \in \mathcal{Z}(y)} \partial_y\big(\partial_{z_R'}(\boldsymbol{\alpha})\big) = 0$. Since $z_R \in \mathcal{Z}(y)$ and $\partial_y\big(\partial_{z_R}(\boldsymbol{\alpha})\big) \neq 0$, there must exist points $z_R' \in \mathcal{Z}(y)$ satisfying:

$$z_R' \neq z_R, \quad 0 \neq \partial_y\big(\partial_{z_R'}(\boldsymbol{\alpha})\big) \in \mathrm{H}^{r-1}\big(\kappa(y), \mathbb{Z}/n(r-2)\big).$$

We claim that all $z_R' \in \mathcal{Z}(y)$, $z_R' \neq z_R$ must satisfy: $z_R' \mapsto \eta_0 \in \operatorname{Spec} R$ under $\mathcal{Z}_R \to \operatorname{Spec} R$. By contradiction, suppose that $z_R' \mapsto \mathfrak{m}$, or equivalently, $z_R' \in \mathcal{Z}_{\mathfrak{m}}$. Arguing as above about as we did for z_R, it follows that $\mathcal{Z}_{z_R'} = \overline{\{z_R'\}}$ is a connected component of $\mathcal{Z}_{\mathfrak{m}}$. Since the connected components $\mathcal{Z}_{z_R}$ and $\mathcal{Z}_{z_R'}$ are either identical or disjoint, and $y \in \mathcal{Z}_{z_R} \cap \mathcal{Z}_{z_R'}$, it follows that $z_R' = z_R$, contradiction! $\square$

Notations/Remarks 3.3. In Notations/Remarks 2.4, let $k \subset K$ be a (relatively algebraically closed) subfield with $\mathrm{td}(K|k) = 1$ and $k_0 \subset k$. Then there exists a unique projective normal (or equivalently regular) k-curve C such that $K = k(C)$. The closed points $P \in C$ are in canonical bijection with the prime divisors v of $K|k$ via $\mathcal{O}_P = \mathcal{O}_v$.

Let $f \in K \backslash k$ be given. Our aim in this subsection is to give a criterion — which for $n = 2$ and char $\neq 2$ turns out to be **first-order** — to express the following:

The set $D_f := \{P \in C \mid v_P(f)$ is not divisible by n in $v_P(K)\}$ is non-empty.

In order to do so, we supplement the previous notations and remarks as follows:

1) Let $\boldsymbol{t} := (t_1, \ldots, t_e)$ denote transcendence bases of $k|k_0$ such that t_i are n^{th} powers in K.
2) $\mathbb{A}_{t_i} \subset \mathbb{P}_{t_i}$ are the S_0-affine/projective t_i-lines, and set $\mathbb{A}_{\boldsymbol{t}} := \times_i \mathbb{A}_{t_i} \subset \times_i \mathbb{P}_{t_i} =: \mathbb{P}_{\boldsymbol{t}}$.
3) For $\boldsymbol{a} = (a_1, \ldots, a_e) \in k_0^e$, set $\boldsymbol{u} = (u_i)_i = (t_i - a_i)_i$, and for $U_0, \boldsymbol{a}, f$ as above, consider:
$$\mathrm{H}_{U_0,\boldsymbol{a},f} := \big\langle u_0' \cup u_0 \cup u_1 \cup \ldots \cup u_e \cup f \mid u_0' \in \Delta_{U_0}, u_0 \in k_0^\times \big\rangle \subset \mathrm{H}^{d+1}\big(K, \mathbb{Z}/n(d)\big).$$

Proposition 3.4. *In the above notations, the following are equivalent:*

i) D_f *is non-empty.*
ii) $\exists\, U_{\boldsymbol t}\subset\mathbb{A}_{\boldsymbol t}$ *Zariski open dense such that* $\forall\,\boldsymbol a\in U_{\boldsymbol t}(k_0)$, $\forall\, U_0$ *one has* $\mathrm{H}_{U_0,\boldsymbol a,f}\neq 0$.
iii) $\forall\, U_{\boldsymbol t}\subset\mathbb{A}_{\boldsymbol t}$ *Zariski open dense* $\exists\,\boldsymbol a\in U_{\boldsymbol t}(k_0)$ *such that* $\forall\, U_0$ *one has* $\mathrm{H}_{U_0,\boldsymbol a,f}\neq 0$.

Proof. To i) $\Rightarrow$ ii): The proof of this implication is "easy" and requires just standard facts. Let $X\to\mathbb{P}_{f,\boldsymbol t}$ be the normalization of $\mathbb{P}_{f,\boldsymbol t}:=\mathbb{P}_f\times\mathbb{P}_{\boldsymbol t}$ in the field extension $k_0(f,\boldsymbol t)\hookrightarrow K$. Let $P\in D_f$ be given, hence by mere definitions we have $v_P(f)=m\in\mathbb{Z}$, and m is not divisible by n in v_PK. Then the residue map $\partial_P:\mathrm{H}^{d+1}\big(K,\mathbb{Z}/n(d)\big)\to\mathrm{H}^{d}\big(\kappa(P),\mathbb{Z}/n(d-1)\big)$ restricted to $\mathrm{H}_{U_0,\boldsymbol a,f}$ is $\partial_P(u'_0\cup u_0\cup u_1\cup\ldots\cup u_e\cup f)=(-1)^d m\cdot u'_0\cup u_0\cup u_1\cup\ldots\cup u_e$.

Let $k_0(\boldsymbol t)\hookrightarrow l$ be the separable part of $k_0(\boldsymbol t)\hookrightarrow\kappa(P)$, and $S_P\to S\to\mathbb{P}_{\boldsymbol t}$ be the normalization of $\mathbb{P}_{\boldsymbol t}$ in the finite field extensions $\kappa(P)\hookleftarrow l\hookleftarrow k_0(\boldsymbol t)$. Then $S\to\mathbb{P}_{\boldsymbol t}$ is étale above a dense open subset $U_{\boldsymbol t}\subset\mathbb{A}_{\boldsymbol t}$. Since $\mathbb{A}_{\boldsymbol t}$ is regular, the preimages $s_{\boldsymbol a}\in S$ of $\boldsymbol a=(a_1,\ldots,a_e)\in U_{\boldsymbol t}(k_0)$ are regular points, $\boldsymbol u:=(u_1,\ldots,u_e)=(t_1-a_1,\ldots,t_e-a_e)$ is a system of regular parameters at $s_{\boldsymbol a}$, and $\kappa_{\boldsymbol a}:=\kappa(s_{\boldsymbol a})$ is a finite separable extension of k_0. Hence letting $R,\mathfrak m$ be the local ring $\mathcal{O}_{s_{\boldsymbol a}},\mathfrak m_{a_{\boldsymbol a}}$ of $s_{\boldsymbol a}$, the $\mathfrak m$-adic completion of R is nothing but $\widehat R=\kappa_{\boldsymbol a}[[u_1,\ldots,u_e]]$, which obviously embeds into $\widehat l:=\kappa_{\boldsymbol a}((u_1))\ldots((u_e))$. Since $k_0\hookrightarrow\kappa_{\boldsymbol a}$ is an extension of global fields, the image of $\mathrm{H}_{U_0}=\{u'_0\cup u_0\mid(u'_0,u_0)\in\Delta_{U_0}\}$ under $\mathrm{res}:\mathrm{H}^2\big(k_0,\mathbb{Z}/n(1)\big)\to\mathrm{H}^2\big(\kappa_{\boldsymbol a},\mathbb{Z}/n(1)\big)$ is non-trivial. And if $u'_0\cup u_0$ is non-trivial over $\kappa_{\boldsymbol a}$, and easy induction on e shows that $\boldsymbol\beta:=u'_0\cup u_0\cup u_1\cup\ldots\cup u_e$ is non-trivial over $\widehat l$, thus over $l\subset\widehat l$. Finally, since $\kappa(P)\,|\,l$ is purely inseparable, $\boldsymbol\beta$ is non-trivial over $\kappa(P)$, hence $0\neq\boldsymbol\beta\cup f=u'_0\cup u_0\cup u_1\cup\ldots\cup u_e\cup f\in\mathrm{H}_{U_0,\boldsymbol a,f}$.

To ii) $\Rightarrow$ iii): Clear!

To iii) $\Rightarrow$ i): The (quite involved) proof is by induction on e. For every non-empty subset $I\subset I_e:=\{1,\ldots,e\}$, set $\boldsymbol t_I=(t_i)_{i\in I}$, $k_I:=k_0(\boldsymbol t_I)$, and denote $\mathbb{A}_I:=\times_{i\in I}\mathbb{A}_{t_i}\subset\times_{i\in I}\mathbb{P}_{t_i}=:\mathbb{P}_I$. For $J\subset I\subset I_e$, the canonical projections $\mathbb{P}_I\twoheadrightarrow\mathbb{P}_J$ are open surjective and define $k_J\hookrightarrow k_I$.

Construction$_{\mathcal I}$. Let $\mathcal{I}=(I_\nu)_\nu$ be a chain of subsets of I_e with $|I_\nu|=\nu$, $I_0=\emptyset$. Setting $k_\nu:=k_{I_\nu}$, the canonical projections $\mathbb{P}_{\boldsymbol t}=\mathbb{P}_{I_e}\twoheadrightarrow\ldots\twoheadrightarrow\mathbb{P}_{I_1}$ define $k_0(\boldsymbol t)=k_e\hookleftarrow\ldots\hookleftarrow k_1$. Given Zariski open dense subsets $U_\nu:=U_{I_\nu}\subset\mathbb{P}_{I_\nu}$ with $U_e\to\ldots\to U_1$, and the preimages $\mathbb{P}_{f,U_\nu}\twoheadrightarrow\mathbb{P}_{U_\nu}$ of U_ν under $\mathbb{P}_{f,\boldsymbol t}\twoheadrightarrow\mathbb{P}_{\boldsymbol t}\twoheadrightarrow\mathbb{P}_{I_\nu}$, one has canonical open k_0-immersions:

$$(*)_{\mathcal I}\qquad U_e=\mathbb{P}_{U_e}\hookrightarrow\ldots\hookrightarrow\mathbb{P}_{U_1}\hookrightarrow\mathbb{P}_{I_e}=\mathbb{P}_{\boldsymbol t}\,,\quad \mathbb{P}_{f,U_e}\hookrightarrow\ldots\hookrightarrow\mathbb{P}_{f,U_1}\hookrightarrow\mathbb{P}_{f,\boldsymbol t}=\mathbb{P}_f\times\mathbb{P}_{\boldsymbol t}\,.$$

Let $X_0\to\mathbb{P}_{f,\boldsymbol t}$ be any projective morphism of k_0-varieties defining $k_0(f,\boldsymbol t)\hookrightarrow K$. Using prime to n alterations, see [ILO], Exposé X, Theorem 2.1, there is a projective smooth k_0-variety $\tilde X_0$ and a projective surjective k_0-morphism $\tilde X_0\to X_0$ defining a field extension $K=:K_0:=k_0(X_0)\hookrightarrow k_0(\tilde X_0)=:\tilde K_0$ of degree prime to n. Proceed as follows:

Step 1. Generic part$_{\mathcal I}$. Let $X_1\to\mathbb{P}_{k_1}$ be the generic fiber of the $\mathbb{P}_{I_1}$-morphisms $\tilde X_0\to\mathbb{P}_{\boldsymbol t}$. Then X_1 is a projective regular k_1-variety, and $K_1:=k_1(X_1)=\tilde K_0$. By [ILO], loc.cit., there is a projective smooth k_1-variety $\tilde X_1$ and a projective k_1-morphism $\tilde X_1\to X_1$ defining a field extension $K_1=k_1(X_1)\hookrightarrow k_1(\tilde X_1)=:\tilde K_1$ of degree prime to n. Inductively on ν, for the already constructed projective smooth $k_{\nu-1}$-variety $\tilde X_{\nu-1}$, let $X_\nu\to\mathbb{P}_{k_\nu}$ be the generic fiber of the $\mathbb{P}_{I_\nu}$-morphism $\tilde X_{\nu-1}\to\mathbb{P}_{k_{\nu-1}}$, thus X_ν is a projective regular k_ν-variety. By [ILO], loc.cit., there is a projective smooth k_ν-variety $\tilde X_\nu$ and a projective k_ν-morphism $\tilde X_\nu\to X_\nu$ such that the field extension $K_\nu=k_\nu(X_\nu)\hookrightarrow k_\nu(\tilde X_\nu)=:\tilde K_\nu$ has degree prime to n.

Notice that the generic fibers $\tilde{C}_\nu \to C_\nu \to \mathbb{P}_{f,k_e}$ of the $\mathbb{P}_{k_\nu}$-morphisms $\tilde{X}_\nu \to X_\nu \to \mathbb{P}_{f,k_\nu}$ are finite morphisms of projective regular k_e-curves, hence flat morphisms.

Step 2. $\mathsf{Deformation\ part}_{\mathcal{I}}$. Since being a projective smooth and/or finite flat morphism is an open condition on the base, there are Zariski dense open subsets $U_\nu \subset \mathbb{A}_{I_\nu} \subset \mathbb{P}_{I_\nu}$ such that $(*)_{\mathcal{I}}$ above holds, there are *projective smooth U_ν-varieties* $\tilde{\mathcal{X}}_\nu$, $1 \leqslant \nu \leqslant e$, such that $\tilde{\mathcal{X}}_\nu \to U_\nu$ has as generic fiber the projective smooth k_ν-variety $\tilde{X}_\nu$. And setting $\tilde{\mathcal{X}}_0 := \tilde{X}_0$, and $\mathcal{X}_\nu := \tilde{\mathcal{X}}_{\nu-1} \times_{\mathbb{P}_{I_e}} \mathbb{P}_{U_\nu}$ for $1 \leqslant \nu \leqslant e$, there is a projective morphism of U_ν-varieties $\tilde{\mathcal{X}}_\nu \to \mathcal{X}_\nu$ with generic fiber $\tilde{X}_\nu \to X_\nu$, thus defining the field embedding $\tilde{K}_\nu \hookleftarrow K_\nu$ of degree prime to n degree. Note that $\mathsf{Construction}_{\mathcal{I}}$ realizes the morphisms above in dependence on $\mathcal{I} = (I_\nu)_\nu$, that is, one should rather speak about $\tilde{\mathcal{X}}_{\mathcal{I},\nu} \to \mathcal{X}_{\mathcal{I},\nu} \to U_{\mathcal{I},\nu}$, etc. One the other hand, for all sufficiently small Zariski open dense subsets $U \subset \mathbb{A}_{I_e}$ the following is satisfied:

(†) *For all $\mathcal{I} = (I_\nu)_{1\leqslant\nu\leqslant e}$ one has: $U \subset U_{\mathcal{I},e}$ and $\tilde{\mathcal{X}}_{\mathcal{I},\nu} \to \mathcal{X}_{\mathcal{I},\nu} \to \mathbb{P}_{U_{\mathcal{I},\nu}}$ are flat above U.*

From now on $U_{\boldsymbol{t}} := U \subset \mathbb{A}_{I_e}$ always satisfies condition (†), and we replace $U_{\mathcal{I},\nu}$ by the image $U_\nu := U_{I_\nu}$ of U under $\mathbb{P}_{I_e} \to \mathbb{P}_{I_\nu}$, etc. Hence the objects above satisfy the following:

Hypothesis 3.5. $U_{\boldsymbol{t}} \subset \mathbb{A}_{I_e}$ is a Zariski open subset such that for all $\mathcal{I} = (I_\nu)_{1\leqslant\nu\leqslant e}$, and the resulting open surjective projections $U = U_e \twoheadrightarrow \ldots \twoheadrightarrow U_1$, and the open immersions $U = \mathbb{P}_U \hookrightarrow \ldots \hookrightarrow \mathbb{P}_{U_1} \hookrightarrow \mathbb{P}_{I_e} = \mathbb{P}_{\boldsymbol{t}}$ and $\mathbb{P}_{f,U} \hookrightarrow \ldots \hookrightarrow \mathbb{P}_{f,U_1} \hookrightarrow \mathbb{P}_{f,\boldsymbol{t}} = \mathbb{P}_f \times \mathbb{P}_{\boldsymbol{t}}$, there are/one has:

1) Projective smooth U_ν-varieties $\tilde{\mathcal{X}}_\nu$ generic fiber the projective smooth k_ν-variety $\tilde{X}_\nu$.
2) Projective morphisms of U_ν-varieties $\tilde{\mathcal{X}}_\nu \to \mathcal{X}_\nu$ defining the field embedding $\tilde{K}_\nu \hookleftarrow K_\nu$, where $\mathcal{X}_\nu := \tilde{\mathcal{X}}_{\nu-1} \times_{\mathbb{P}_{I_e}} \mathbb{P}_{U_\nu}$, and $\tilde{\mathcal{X}}_0 := \tilde{X}_0$, thus a canonical morphism $\tilde{\mathcal{X}}_\nu \to \tilde{\mathcal{X}}_{\nu-1}$.
3) Setting $\tilde{\mathcal{X}}_{\nu,U} := \tilde{\mathcal{X}}_\nu \times_{\mathbb{P}_{U_\nu}} U$, one has projective flat U-morphisms of regular U-curves:
$$\tilde{\mathcal{X}}_{e,U} \to \ldots \to \tilde{\mathcal{X}}_{1,U} \to \tilde{\mathcal{X}}_{0,U} \to \mathbb{P}_{f,U} \ \ \textit{defining} \ \ \tilde{K}_e \hookleftarrow \ldots \hookleftarrow \tilde{K}_1 \hookleftarrow \tilde{K}_0 \hookleftarrow k_0(f, \boldsymbol{t}).$$

In particular, for all $\mathcal{I}$ as above, and $\boldsymbol{a} := (a_1, \ldots, a_e) \in U(k_0)$, $\boldsymbol{a}_\nu := (a_i)_{i\in I_\nu} \in U_\nu(k_0)$, the fibers $\tilde{\mathcal{X}}_{\boldsymbol{a}_\nu}$ of $\tilde{\mathcal{X}}_\nu \to U_\nu$ at $\boldsymbol{a}_\nu \in U_\nu$ is a projective smooth k_0-varieties. Hence if $U_0 \subset S_0$ is a sufficiently small Zariski open dense subset (depending on $\boldsymbol{a}$), one has:

$(\ddagger)_{\boldsymbol{a}}$ *For all $\mathcal{I} = (I_\nu)_\nu$, $\tilde{\mathcal{X}}_0$ and all the k_0-varieties $\tilde{\mathcal{X}}_{\boldsymbol{a}_\nu}$ have projective smooth U_0-models.*

Returning to the proof of implication ii) $\Rightarrow$ i), we proceed by proving:

Lemma 3.6. *Under Hypothesis 3.5, let $\boldsymbol{a} \in U_{\boldsymbol{t}}(k_0)$, and U_0 satisfy $(\ddagger)_{\boldsymbol{a}}$. Then for each $0 \neq \boldsymbol{\alpha} \in \mathrm{H}_{U_0,\boldsymbol{a},f}$, there is $P \in C$ such that $\partial_P(\boldsymbol{\alpha})$ is non-trivial, and $v_P(f) \notin n \cdot v_P(K)$.*

Proof of Lemma 3.6. The proof is by induction on e as follows: Since $[\tilde{K}_0 : K]$ is prime to n, the restriction $\mathrm{res}: \mathrm{H}^{d+1}\big(K, \mathbb{Z}/n(d)\big) \to \mathrm{H}^{d+1}\big(\tilde{K}_0, \mathbb{Z}/n(d)\big)$ is injective, hence $\tilde{\boldsymbol{\alpha}} := \mathrm{res}(\boldsymbol{\alpha})$ is nontrivial over $\tilde{K}_0$. Since $\tilde{\mathcal{X}}_0$ has a projective smooth U_0-model, by Proposition 3.2 there is a point $x_0 \in \tilde{\mathcal{X}}_0$ with $\mathrm{codim}_{\tilde{\mathcal{X}}_0}(x_0) = 1$ and $\boldsymbol{\beta} := \partial_{x_0}(\tilde{\boldsymbol{\alpha}})$ is nontrivial in $\mathrm{H}^d\big(\kappa(x_0), \mathbb{Z}/n(d-1)\big)$. Hence the prime divisor $w_0 := w_{x_0}$ of $\tilde{K}_0|k_0$ satisfies:

Case 1. $w_0(u_i) = 0$ for all $i = 1, \ldots, e$. Then setting $u_i \mapsto \overline{u}_i$ under $\mathcal{O}_{w_0} = \mathcal{O}_{x_0} \to \kappa(x_0)$, it follows that $\partial_{x_0}(\tilde{\boldsymbol{\alpha}}) = u_0' \cup u_0 \cup \overline{u}_1 \cup \ldots \cup \overline{u}_e \neq 0$ in $\mathrm{H}^d\big(\kappa(x_0), \mathbb{Z}/n(d-1)\big)$. Hence $\overline{u}_1, \ldots, \overline{u}_e$ must be algebraically independent over k_0, or equivalently, w_0 must be trivial on $k_0(u_1, \ldots, u_e)$. Thus $w_0|_K = v_P$ for some $P \in C$ such that $\partial_P(\boldsymbol{\alpha}) \neq$ over $\kappa(P)$, and $v_P(f) \notin n \cdot v_P K$ (because $w(f) \notin n \cdot w\tilde{K}_0$). Thus finally implying that D_f is non-empty, as claimed.

Case 2. The set $I := \{i \,|\, w_0(u_i) \neq 0\}$ is non-empty. Recalling that t_i is an n^{th} power in K, say $t_i = t^n$, and $u_i = t_i - a_i$, we claim that $w_0(t_i) = 0$. Indeed, by contradiction, let $w_0(t_i) \neq 0$. First, if $w_0(t_i) > 0$, then $\overline{u}_i = a_i$, hence $u'_0 \cup u_0 \cup a_i \in \mathrm{H}^3\big(k_0, \mathbb{Z}/n(2)\big) = 0$ is a sub-symbol of $\partial_{x_0}(\tilde{\boldsymbol{\alpha}})$, implying that $\partial_{x_0}(\tilde{\boldsymbol{\alpha}}) = 0$, contradiction! Second, if $w_0(t_i) < 0$, then $t_i - a_i = t_i(1 - a_i/t_i) = t^n u'_i$ with $u'_i \in 1 + \mathfrak{m}_{w_0}$. Hence $\overline{u}'_i = 1$, and $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$, where the latter symbol is obtained by replacing u_i by u'_i in the symbol $\boldsymbol{\alpha}$. Then $u'_0 \cup u_0 \cup 1 = 0$ is a sub-symbol of $\partial_{x_0}(\tilde{\boldsymbol{\alpha}})$, thus $\partial_{x_0}(\tilde{\boldsymbol{\alpha}}) = 0$, contradiction! Thus conclude that $w_0(t_i) = 0$, hence $w_0(t_i - a_i) \neq 0 \;\Rightarrow\; w_0(t_i - b_i) > 0$. In particular, replacing f by f^{-1} if necessary, w.l.o.g., $w_0(f) \geqslant 0$. Thus finally $u_1, \dots, u_e, f \in \mathcal{O}_{x_0}$, and $u_i \in \mathfrak{m}_{x_0}$ iff $i \in I$.

Consider the images $x_0 \mapsto x_{f,\boldsymbol{a}_I} \mapsto x_{\boldsymbol{a}_I} \mapsto \boldsymbol{a}_I := (a_i)_{i \in I}$ under $\tilde{\mathcal{X}}_0 \to \mathbb{P}_{f,\boldsymbol{t}} \to \mathbb{P}_{I_e} \to \mathbb{P}_I$. Then $\boldsymbol{a}_I \in \mathbb{P}_{U_I}(k_0)$, and $x_{\boldsymbol{a}_I} \in \mathbb{P}_{\boldsymbol{t}}$ is defined by $\boldsymbol{t}_I = \boldsymbol{a}_I$, hence $\mathrm{td}\,\big(\kappa(x_{\boldsymbol{a}_I})|k_0\big) \leqslant e - |I|$. Further, $x_{\boldsymbol{a}_I} \in U$ (because $x_{\boldsymbol{a}_I}$ is a generalization of $\boldsymbol{a}$). Hence $e = \mathrm{td}\,\big(\kappa(x_0)|k_0\big)$, together with $\tilde{\mathcal{X}}_0 \to \mathbb{P}_{f,\boldsymbol{t}} \to \mathbb{P}_{\boldsymbol{t}}$ being flat at $x_0 \mapsto x_{f,\boldsymbol{a}_I} \mapsto x_{\boldsymbol{a}_I} \in U_e$, imply:

$$e - 1 = \mathrm{td}\,\big(\kappa(x_0)|k_0\big) - 1 = \mathrm{td}\,\big(\kappa(x_{f,\boldsymbol{a}_I})|k_0\big) - 1 = \mathrm{td}\,\big(\kappa(x_{\boldsymbol{a}_I})|k_0\big) \leqslant e - |I|, \text{ hence } |I| = 1,$$

and $\mathrm{td}\,\big(\kappa(x_{f,\boldsymbol{a}_I})|k_0\big) - 1 = \mathrm{td}\,\big(\kappa(x_{\boldsymbol{a}_I})|k_0\big)$ implies $f \notin \mathfrak{m}_{x_{f,\boldsymbol{a}_I}} \subset \mathfrak{m}_{x_0}$, thus $f, u_i \in \mathcal{O}^{\times}_{x_0}$, $i \notin I$. Reasoning as in Case 1), the non-triviality of $\partial_{x_0}(\tilde{\boldsymbol{\alpha}})$ implies that the residues $\overline{f}, \overline{u}_i \in \kappa(x_0)$ of f, u_i, $i \notin I$ are algebraically independent over k_0. Equivalently, the residues $\overline{f}, \overline{t}_i$, $i \notin I$ are algebraically independent over k_0, hence $x_{f,\boldsymbol{a}_I} \in \mathbb{P}_{f,\boldsymbol{t}}$ is the generic point of the fiber of $\mathbb{P}_{f,\boldsymbol{t}} \to \mathbb{P}_I$ at $\boldsymbol{a}_I \in \mathbb{P}_I(k_0)$. Since $x_0 \mapsto x_{f,\boldsymbol{a}_I} \mapsto x_{\boldsymbol{a}_I} \mapsto \boldsymbol{a}_I$ under $\tilde{\mathcal{X}}_0 \to \mathbb{P}_{f,\boldsymbol{t}} \to \mathbb{P}_{I_e} \to \mathbb{P}_I$, one has:

$(*)$ *x_0 is a generic point of the fiber $\tilde{\mathcal{X}}_{0,\boldsymbol{a}_I}$ of $\tilde{\mathcal{X}}_0 \to \mathbb{P}_I$ at $\boldsymbol{a}_I \in U_I(k_0)$.*

After renumbering $(t_1, \dots, t_e)$, w.l.o.g., $I = \{e\}$. Considering all the chains $\mathcal{I} = (I_\nu)_\nu$ with $I_1 = \{e\}$, and viewing $\tilde{\mathcal{X}}_\nu \to \mathbb{P}_{U_\nu}$ and $\tilde{\mathcal{X}}_\nu \to \mathcal{X}_\nu \to \mathbb{P}_{f,U_\nu}$ as U_1-morphisms via $U_\nu \to U_1$, let $\tilde{\mathcal{X}}_{\nu,\boldsymbol{a}_1} \to \mathcal{X}_{\nu,\boldsymbol{a}_1} \to \mathbb{P}_{U_\nu,\boldsymbol{a}_1} \to U_{\nu,\boldsymbol{a}_1}$ be the fibers of $\tilde{\mathcal{X}}_\nu \to \mathcal{X}_\nu \to \mathbb{P}_{U_\nu} \to U_\nu$ at $\boldsymbol{a}_1 \in U_1(k_0)$. Let w_0 is the prime divisor of $\tilde{K}_0$ defined by $x_0 \in \tilde{\mathcal{X}}_0$. Then Hypothesis 3.5, 3) implies: The sequences $(w_\nu)_{0 \leqslant \nu \leqslant e}$ of prolongations of w_0 to the tower of extensions $(\tilde{K}_\nu)_\nu$ satisfying $w_{\nu+1}|_{\tilde{K}_\nu} = w_\nu$ are in canonical bijection with the sequences $(x_\nu)_{0 \leqslant \nu \leqslant e}$ of generic points $x_\nu \in \tilde{\mathcal{X}}_{\nu,\boldsymbol{a}_1}$ with $x_{\nu+1} \mapsto x_\nu$ for all $0 \leqslant \nu < e$. Hence denoting $e_{\nu+1} := e(w_{\nu+1}|w_\nu)$, $f_{\nu+1} := f(w_{\nu+1}|w_\nu) = [\kappa(x_{\nu+1}) : \kappa(x_\nu)]$ the ramification index, respectively the residue degree of $w_{\nu+1}|w_\nu$, one has: Since $[\tilde{K}_{\nu+1} : \tilde{K}_\nu]$ is prime to n, by the fundamental equality, for every w_ν there is a prolongation $w_{\nu+1}$ with $e_{\nu+1} f_{\nu+1}$ prime to n. We will call such sequences $(x_\nu)_\nu$ and/or $(w_\nu)_\nu$ prime to n compatible. Notice that letting $\tilde{\mathcal{X}}_{x_\nu} \subset \tilde{\mathcal{X}}_\nu$ be the Zariski closure of x_ν, the morphism $\tilde{\mathcal{X}}_{\nu+1} \to \tilde{\mathcal{X}}_\nu$ gives rise to a morphism $\tilde{\mathcal{X}}_{x_{\nu+1}} \to \tilde{\mathcal{X}}_{x_\nu}$ defining the finite extension $\tilde{L}_\nu := \kappa(x_\nu) \hookrightarrow \kappa(x_{\nu+1}) =: \tilde{L}_{\nu+1}$ of degree $f_{\nu+1}$ prime to n. Further, $\tilde{\mathcal{X}}_{x_1}$ is an irreducible component of the projective smooth k_0-variety $\tilde{\mathcal{X}}_{\boldsymbol{a}_1} = \tilde{\mathcal{X}}_{1,\boldsymbol{a}_1}$, hence $\tilde{\mathcal{X}}_{x_1}$ is itself a projective smooth k_0-variety. And by assumption $(\ddagger)_{\boldsymbol{a}}$, $\tilde{\mathcal{X}}_{x_1}$ has a projective smooth U_0-model. Finally, since $\tilde{L}_0 = \kappa(x_0) \hookrightarrow \kappa(x_1) = \tilde{L}_1$ has degree prime to n, and $\boldsymbol{\beta} = \partial_{x_0}(\tilde{\boldsymbol{\alpha}})$ is non-trivial over $\tilde{L}_0 = \kappa(x_0)$, it follows that $\boldsymbol{\beta}_1 = \mathrm{res}(\boldsymbol{\beta})$ is non-trivial over $\tilde{L}_1$. Hence by Proposition 3.2, there is a point $y \in \tilde{\mathcal{X}}^1_{x_1}$ such that $\partial_y(\boldsymbol{\beta}_1) \neq 0$. Let $\tilde{\mathcal{X}}_1(y) \subset \tilde{\mathcal{X}}^1_1$ be the set of points $x'_1 \in \tilde{\mathcal{X}}_1$ with $\mathrm{codim}_{\tilde{\mathcal{X}}_1}(x'_1) = 1$ and $y \in \overline{\{x'_1\}}$. Then reasoning as in the proof of Proposition 3.2, it follows that there is $x'_1 \in \tilde{\mathcal{X}}_1(y)$ such that, first, $\partial_{x'_1}(\tilde{\boldsymbol{\alpha}}) \neq 0$ over $\kappa(x'_1)$, and second, x_1' is not contained in any fiber $\tilde{\mathcal{X}}_{\boldsymbol{a}_1'}$ of $\tilde{\mathcal{X}}_1 \to U_1$. Equivalently, x_1' lies in the generic fiber X_1 of $\tilde{\mathcal{X}}_1 \to U_1$, and the prime divisor w_1' of $\tilde{K}_1|k_0$ defined by x_1' is trivial on k_1.

Case $e = 1$: One has $\boldsymbol{t} = (t_1)$, hence $k_1 = k_0(\boldsymbol{t})$. Therefore $w_1'|_K = v_P$ for some $P \in C$, and conclude by reasoning as in Case 1 above.

Case $e > 1$: Setting $\mu := \nu - 1$, $J_\mu := I_\nu \backslash I_1$, the inclusions $V_\mu := U_{\nu,\boldsymbol{a}_1} \subset \mathbb{P}_{\nu,\boldsymbol{a}_1} = \mathbb{P}_{J_\mu}$ are open immersions, where $V_0 := \operatorname{Spec} k_0 =: \mathbb{P}_{J_0}$. Given a prime to n compatible sequence $(x_\nu)_{0 \leqslant \nu \leqslant e}$, for every $\nu > 0$ we set: $\tilde{\mathcal{Y}}_\mu := \tilde{\mathcal{X}}_{x_\nu} \subset \tilde{\mathcal{X}}_{\nu,\boldsymbol{a}_1}$, and recall that $\tilde{L}_\mu = k_0(\tilde{\mathcal{Y}}_\mu) = \kappa(x_\nu)$. Since $\tilde{\mathcal{X}}_\nu \to U_\nu$ is projective smooth, its fiber $\tilde{\mathcal{X}}_{\nu,\boldsymbol{a}_1} \to U_{\nu,\boldsymbol{a}_1}$ is projective smooth, hence so is its irreducible component $\tilde{\mathcal{Y}}_\mu \subset \tilde{\mathcal{X}}_{\nu,\boldsymbol{a}_1}$. Further, since $\tilde{\mathcal{X}}_\nu \to \mathcal{X}_\nu = \tilde{\mathcal{X}}_{\nu-1} \times_{\mathbb{P}_{\boldsymbol{t}}} \mathbb{P}_{U_\nu}$ is a projective U_ν-morphism, the transitivity of base change implies that $\mathcal{Y}_\mu := \mathcal{X}_{\nu,\boldsymbol{a}_1} = \tilde{\mathcal{Y}}_{\mu-1} \times_{\mathbb{P}_{\boldsymbol{t}'}} \mathbb{P}_{V_\mu}$, and the resulting V_μ-morphism $\tilde{\mathcal{Y}}_\mu \to \mathcal{Y}_\mu$ is projective defining $\tilde{L}_\mu \hookleftarrow \tilde{L}_{\mu-1}$. Therefore Hypothesis 3.5 holds *mutatis mutandis* in the context below, after replacing e by e', namely:

For the canonical open projections $V := V_{e'} \twoheadrightarrow \ldots \twoheadrightarrow V_1$ and the resulting open immersions $V_{e'} = \mathbb{P}_{V_{e'}} \hookrightarrow \ldots \hookrightarrow \mathbb{P}_{V_1} \hookrightarrow \mathbb{P}_{J_{e'}}$ and $\mathbb{P}_{f,V_{e'}} \hookrightarrow \ldots \hookrightarrow \mathbb{P}_{f,V_1} \hookrightarrow \mathbb{P}_{f,J_{e'}}$, the following hold:

1)′ $\tilde{\mathcal{Y}}_\mu \to V_\mu$ is a projective smooth integral V_μ-variety of dimension $e' + 1 - \mu$ for $1 \leqslant \mu \leqslant e'$.

2)′ $\tilde{\mathcal{Y}}_\mu \to \mathcal{Y}_\mu := \tilde{\mathcal{Y}}_{\mu-1} \times_{\mathbb{P}_{J_{e'}}} \mathbb{P}_{V_\mu}$ is a projective surjective V_μ-morphism for $1 \leqslant \nu \leqslant e'$.

3)′ Setting $\tilde{\mathcal{Y}}_{\mu,V} := \tilde{\mathcal{Y}}_\mu \times_{\mathbb{P}_{J_\mu}} V$ for $0 \leqslant \mu \leqslant e'$, one gets canonically a sequence of projective surjective flat morphisms of V-curves $\tilde{\mathcal{Y}}_{e',V} \to \ldots \to \tilde{\mathcal{Y}}_{1,V} \to \tilde{\mathcal{Y}}_{0,V} \to \mathbb{P}_{f,V}$.

Further, $\boldsymbol{a}' := (a_1, \ldots, a_{e'}) \in V(k_0)$, $\boldsymbol{a}_\mu' := (a_i)_{i \in J_\mu} \in V_\mu(k_0)$, and the fiber $\tilde{\mathcal{Y}}_{\boldsymbol{a}_\mu}$ of $\tilde{\mathcal{Y}}_\mu \to V_\mu$ at $\boldsymbol{a}_\mu' \in V_\mu$ is a projective smooth k_0-variety. Moreover, since each $\tilde{\mathcal{Y}}_{\mu,\boldsymbol{a}_\mu'}$ is an irreducible component of the projective smooth k_0-variety $\tilde{\mathcal{X}}_{\nu,\boldsymbol{a}_\nu}$, the condition $(\ddagger)_{\boldsymbol{a}}$ above implies:

$(\ddagger)_{\boldsymbol{a}'}$ *For all $\mathcal{J} = (J_\mu)_\mu$, $\tilde{\mathcal{Y}}_0$ and all the k_0-varieties $\tilde{\mathcal{Y}}_{\boldsymbol{a}_\mu'}$ have projective smooth U_0-models.*

In particular, since $e' < e$, we can apply the induction hypothesis for $\boldsymbol{\beta} = \partial_{x_0}(\tilde{\boldsymbol{\alpha}})$ as follows: Recall that the canonical projective morphism $\tilde{\mathcal{Y}}_0 := \tilde{\mathcal{X}}_{1,\boldsymbol{a}_1} \to \tilde{\mathcal{X}}_{0,\boldsymbol{a}_1} =: Y_0$ is the fiber of $\tilde{\mathcal{X}}_1 \to \tilde{\mathcal{X}}_{0,U_1}$ at $\boldsymbol{a}_1 \in U_1(k_0)$, and the corresponding extension $L_0 = \kappa(x_0) \hookrightarrow \kappa(x_1) = \tilde{L}_0$ has degree $f(x_1|x_0)$ prime to n. Hence the image $\tilde{\boldsymbol{\beta}} := \operatorname{res}(\boldsymbol{\beta})$ of $\boldsymbol{\beta}$ over $\tilde{L}_0$ is non-trivial. Let C' be the generic fiber of $\tilde{\mathcal{Y}}_{0,V} \to V$, hence of $\tilde{\mathcal{Y}}_0 \to \mathbb{P}_{\boldsymbol{t}'}$, and set $\boldsymbol{u}' := \boldsymbol{t}' - \boldsymbol{a}' = (t_i - a_i)_{1 \leqslant i \leqslant e'}$. Then C' is a projective regular $k_0(\boldsymbol{t}')$-curve, and the images $(f, \boldsymbol{u}') \mapsto (\overline{f}, \overline{\boldsymbol{u}}') \mapsto (f', \boldsymbol{u}')$ under $\mathcal{O}_{x_0} \to \kappa(x_0) \hookrightarrow \tilde{L}_0$ satisfy: $\tilde{\boldsymbol{\beta}} = u_0' \cup u_0 \cup u_1' \cup \ldots \cup u_{e'}' \cup f'$. Hence by the induction hypothesis, there is $P' \in C'$ such that $\partial_{P'}(\tilde{\boldsymbol{\beta}}) \neq 0$ and $v_{P'}(f') \notin n \cdot v_{P'}(\tilde{L}_0)$. In particular, $y := P' \in C' \subset \tilde{\mathcal{Y}}_0$ is a point of codimension one. Recalling that $\tilde{\mathcal{Y}}_0 = \tilde{\mathcal{X}}_{1,\boldsymbol{a}_1} = \tilde{\mathcal{X}}_{\boldsymbol{a}_1}$ is the fiber of $\tilde{\mathcal{X}}_1$ at $\boldsymbol{a}_1 \in U_1(k_0)$, it follows that $y \in \tilde{\mathcal{X}}_1$ is a point of codimension two. By the discussion before Case $e = 1$ above, there exists $x_1' \in \tilde{\mathcal{X}}_1(y)$ with $\partial_{x_1}(\tilde{\boldsymbol{\alpha}}) \neq 0$ and $k_0(t_e) \subset \mathcal{O}_{x_1'}$. On the other hand, $k_0(\boldsymbol{t}') \subset \kappa(y) \subset \kappa(x_1)$, hence finally, $\boldsymbol{t} = (\boldsymbol{t}', t_e)$ consists of w_{x_1}-units. Conclude as in Case 1. □

B) *A criterion for D_f to be non-empty*

Notations/Remarks 3.7. Recalling that v denotes the finite places of k_0, we supplement the previous Notatations/Remarks 3.1 as follows. For every $\delta, \delta_1, \ldots, \delta_e \in k_0^\times$ we define:

1) $U_\delta := \{a \in k_0^\times \mid v(\delta) \neq 0 \text{ or } v(n) > 0 \Rightarrow v(a-1) > 2 \cdot v(n)\ \forall\ v\}$, the n,δ-units.

2) $\Sigma_\delta = \{a \in k_0^\times \mid v(\delta) \neq 0 \Rightarrow v(a) \neq 0\}$, the non-$\delta$-units. We notice the following:
   a) U_δ is a subgroup of $k_0^\times$, and Σ_δ is a U_δ-set, i.e., $U_\delta \cdot \Sigma_\delta = \Sigma_\delta$.
   b) For every finite set $A \subset k_0$ there exist "many" $\delta \in k_0^\times$ such that $A \cap \Sigma_\delta = \emptyset$.
3) $U_\delta^\bullet := U_\delta \times k_0^\times$, and $\boldsymbol{\Sigma_\delta} := \times_i \Sigma_{\delta_i}$ for $\boldsymbol{\delta} := (\delta_1, \ldots, \delta_e)$.
4) For $\boldsymbol{a} = (a_i)_i \in \boldsymbol{\Sigma_\delta}$, set $\boldsymbol{u} := \boldsymbol{t} - \boldsymbol{a}$, and for $\delta_0 \in k_0^\times$ and $\boldsymbol{u}_0 := (u_0', u_0) \in U_{\delta_0}^\bullet$, define
$$\boldsymbol{\alpha}_{\boldsymbol{u}_0, \boldsymbol{a}, f} := u_0' \cup u_0 \cup u_1 \cup \ldots \cup u_e \cup f \in \mathrm{H}^{d+1}\big(K, \mathbb{Z}/n(d)\big)$$
   and consider the subgroup $\mathrm{H}_{\delta_0, \boldsymbol{a}, f} := \big\langle \boldsymbol{\alpha}_{\boldsymbol{u}_0, \boldsymbol{a}, f} \mid \boldsymbol{u}_0 \in U_{\delta_0}^\bullet \big\rangle \subset \mathrm{H}^{d+1}\big(K, \mathbb{Z}/n(d)\big)$.

Key Lemma 3.8. *In the above notations, the following are equivalent:*
 i) *D_f is non-empty.*
 ii) $\exists\, \delta_1\, \forall a_1 \in \Sigma_{\delta_1} \ldots \exists\, \delta_e\, \forall\, a_e \in \Sigma_{\delta_e} \forall\, \delta_0\, \exists\, (u_0', u_0) \in U_{\delta_0}^\bullet$ *such that* $\boldsymbol{\alpha}_{\boldsymbol{u}_0, \boldsymbol{a}, f} \neq 0$.

Proof. The proof follows easily from Proposition 3.4 along the following lines:

To i) $\Rightarrow$ ii): Given i), by Proposition 3.4, $\exists\, U_{\boldsymbol{t}} \subset \mathbb{A}_{\boldsymbol{t}}$ Zariski open dense s.t. $\mathrm{H}_{U_0, \boldsymbol{a}, f} \neq 0$ for all $\boldsymbol{a} \in U_{\boldsymbol{t}}(k_0)$, $U_0 \subset S_0$. Set $\boldsymbol{t}_i = (t_1, \ldots, t_i)$. Then $\varpi_i : \mathbb{A}_{\boldsymbol{t}} \to \mathbb{A}_{\boldsymbol{t}_i}$ defined by $k_0[\boldsymbol{t}_i] \hookrightarrow k_0[\boldsymbol{t}]$ are Zariski open maps. Hence $\Sigma_i := \varpi_i\big(U_{\boldsymbol{t}}(k_0)\big) \subset k_0^i$ is Zariski open dense, and one has:
 a) If $\boldsymbol{a}_i \in \Sigma_i$, and $U_{\boldsymbol{t}, \boldsymbol{a}_i} \subset \mathbb{A}_{\boldsymbol{t}, \boldsymbol{a}_i} \hookrightarrow \mathbb{A}_{\boldsymbol{t}}$ are the fibers of $U_{\boldsymbol{t}} \subset \mathbb{A}_{\boldsymbol{t}}$ at $\boldsymbol{a}_i$ under $\varpi_i : \mathbb{A}_{\boldsymbol{t}} \to \mathbb{A}_{\boldsymbol{t}_i}$, then at the level of k_0-rational points, one has canonical identifications
$$U_{\boldsymbol{t}, \boldsymbol{a}_i}(k_0) \subset \mathbb{A}_{\boldsymbol{t}, \boldsymbol{a}_i}(k_0) = \varpi_i^{-1}(\boldsymbol{a}_i) = \boldsymbol{a}_i \times k_0^{(e-i)} \subset k_0^e = \mathbb{A}_{\boldsymbol{t}}(k_0).$$
 b) In particular, $U_{\boldsymbol{t}, \boldsymbol{a}_i}(k_0) \subset \boldsymbol{a}_i \times k_0^{(e-i)}$ is a Zariski open dense subset for all $\boldsymbol{a}_i \in U_i$.

Proceed by induction on $i = 1, \ldots, e$ as follows:

<u>Step 1</u>. $i = 1$: Since $\Sigma_1 := \varpi_1\big(U_{\boldsymbol{t}}(k_0)\big) \subset k_0$ is Zariski open dense, $A_1 := k_0 \backslash \Sigma_1$ is finite. Hence $\exists\, \delta_1 \in k_0^\times$ such that $\Sigma_{0, \delta_1} \cap A_1 = \emptyset$, thus $\Sigma_{0, \delta_1} \subset \Sigma_1$. Then $\forall\, a_1 \in \boldsymbol{\Sigma}_{\delta_1}$, set $\boldsymbol{a}_1 := (a_1)$.

<u>Step 2</u>. $i \Rightarrow i+1$: Suppose that $\boldsymbol{a}_i = (a_1, \ldots, a_i) \in \varpi_i\big(U_{\boldsymbol{t}}(k_0)\big)$ is inductively constructed. Let $\varpi_{i+1,i} : \mathbb{A}_{\boldsymbol{t}_{i+1}} \to \mathbb{A}_{\boldsymbol{t}_i}$ be the canonical projection. Then $\varpi_i = \varpi_{i+1,i} \circ \varpi_{i+1}$, and all the projections involved are open surjective. Further, by the discussion above, one has that $U_{\boldsymbol{a}_i} := U_{\boldsymbol{t}, \boldsymbol{a}_i}(k_0) \subset \boldsymbol{a}_i \times k_0^{(e-i)}$ is Zariski open dense, and therefore $\varpi_{i+1,i}\big(U_{\boldsymbol{a}_i}\big) \subset \boldsymbol{a}_i \times k_0$ is a dense open subset. Hence there exists $\Sigma_{i+1} \subset k_0$ cofinite such that $\boldsymbol{a}_i \times \Sigma_{i+1} \subset \varpi_{i+1,i}\big(U_{\boldsymbol{a}_i}\big)$, thus $\boldsymbol{a}_i \times \Sigma_{i+1} \subset U_{\boldsymbol{a}_i} \subset \boldsymbol{a}_i \times k_0$ is a Zariski open dense subset. In particular, $A_{i+1} := k_0 \backslash \Sigma_{i+1}$ is finite. Hence $\exists\, \delta_{i+1} \in k_0^\times$ such that $\Sigma_{\delta_{i+1}} \cap A_{i+1} = \emptyset$, and in particular, $\Sigma_{\delta_{i+1}} \subset \Sigma_{i+1}$. Then $\forall\, a_{i+1} \in \Sigma_{\delta_{i+1}}$, setting $\boldsymbol{a}_{i+1} := (\boldsymbol{a}_i, a_{i+1})$, one has: $\boldsymbol{a}_{i+1} \in \varpi_{i+1}\big(U_{\boldsymbol{t}, \boldsymbol{a}_i}(k_0)\big) \subset \varpi_{i+1}\big(U_{\boldsymbol{t}}(k_0)\big)$. This completes the proof of the induction step, thus of the implication i) $\Rightarrow$ ii).

To ii) $\Rightarrow$ i): Let $U_{\boldsymbol{t}} \subset \mathbb{A}_{\boldsymbol{t}}$ be a Zariski dense open subset. Then condition ii) of the Key Lemma 3.8 above, implies that there is $\boldsymbol{a} \in U_{\boldsymbol{t}}(k_0)$ such that $\forall\, U_0 \subset S_0$ one has $\mathrm{H}_{U_0, \boldsymbol{a}, f} \neq 0$. Hence condition iii) from Proposition 3.4 is satisfied, concluding that $D_f \neq \emptyset$. $\square$

4. Uniform definability of the geometric prime divisors of K

In this section we work in the context and notation of the previous sections, but specialize to the case $n = 2 \neq \mathrm{char}$. In particular, for $\boldsymbol{a} := (a_1, \ldots, a_r)$ with $a_i \in K^\times$, by the Milnor Conjecture, $a_1 \cup \ldots \cup a_r \in \mathrm{H}^r\big(K, \mathbb{Z}/n(r-1)\big)$ is trivial iff $0 \in q_{\boldsymbol{a}}(K)$. Therefore:
$$a_1 \cup \ldots \cup a_r = 0 \textit{ is first-order expressible by } \exists\ (x_1, \ldots, x_{2^r}) \neq 0 \textit{ s.t. } q_{\boldsymbol{a}}(x_1, \ldots, x_{2^r}) = 0.$$

Let K satisfy Hypothesis (H), $k_0 \subset K$ be a (relatively algebraically closed) global subfield, $e = \mathrm{td}(K|k_0) - 1 > 0$, $(f, t_1, \dots, t_e)$ be algebraically independent functions over k_0, such that each t_i is an n^{th} power in K. Then K is the function field of a projective normal curve C over $k_0(\boldsymbol{t})$, and f is a non-constant function on C. Finally, recalling the context of the Key Lemma 3.8, let $\boldsymbol{x} := (x_1, \dots, x_{2^{d+1}}) \neq (0, \dots, 0)$ be a system of 2^{d+1} variables, and consider the following uniform first-order formula:

$$\varphi(f, \boldsymbol{t}) \;\equiv\; \exists\, \delta_1 \,\forall a_1 \!\in\! \Sigma_{\delta_1} \dots \exists\, \delta_e \,\forall\, a_e \!\in\! \Sigma_{\delta_e} \forall\, \delta_0 \,\exists\, (u_0', u_0) \!\in\! U^{\bullet}_{\delta_0} \,\forall\, \boldsymbol{x} :\; q_{\boldsymbol{u}_0, \boldsymbol{u}, f}(\boldsymbol{x}) \neq 0.$$

Key Lemma (revisited) 4.1. *In the above notation, the following are equivalent:*

i) D_f *is non-empty.*

ii) $\varphi(f, \boldsymbol{t})$ *holds in* $\tilde{K}$*.*

Proof. As explained above, this is just a reformulation of Key Lemma 3.8. □

Our final aim in this section is to show that the prime divisors of $K|k$ are uniformly first-order definable. Precisely, we will give formulae

$$\mathsf{val}_d\big(\, x; f, \boldsymbol{t}, \boldsymbol{\xi}, \boldsymbol{\delta}, (a_i)_i \in \boldsymbol{\Sigma_\delta}, \delta_0, \boldsymbol{u}_0 \in U^{\bullet}_{\delta_0},\, q_{\boldsymbol{u}_0, \boldsymbol{u}, f}(\boldsymbol{x_\xi})\big)$$

such that evaluating all the variables but x in K, the resulting predicates in the variable x define all the valuation rings $\mathcal{O}_w \subset K$ of prime divisors K which are trivial on $k_0(\boldsymbol{t})$. Here, $q_{(\boldsymbol{u}_0, \boldsymbol{u}, f)}$ is the $(d+1)$-fold Pfister form defined $\boldsymbol{u}_0, \boldsymbol{u}, f$. These formulae involve — among other things — Rumely's [Ru] formulae val_1 which uniformly define the prime divisors of number fields; further, in the case of finitely generated fields of Kronecker dimension two, Pop [P4] gives formulae val_2 which uniformly define the *geometric prime divisors*.

Theorem 4.2. *There exist explicit formulae* val_d *which uniformly define the geometric prime divisors of finitely generated fields* K *with* $\mathrm{char}(K) \neq 2$ *and* $\dim(K) > 1$ *as follows*

$$\mathsf{val}_d\big(\, x; f, \boldsymbol{t}, \boldsymbol{\xi}, \boldsymbol{\delta}, (a_i)_i \in \boldsymbol{\Sigma_\delta}, \delta_0, \boldsymbol{u}_0 \in U^{\bullet}_{\delta_0}, q_{\boldsymbol{u}_0, \boldsymbol{u}, f}(\boldsymbol{x_\xi})\big).$$

The proof of Theorem 4.2 follows from the Recipe 4.7 below.

A) *The uniformly definable subsets* $\Theta_{f,\boldsymbol{t}}$*,* $\overline{\Theta}_{f,\boldsymbol{t}}$ *and semi-local subrings* $\mathfrak{a}_{f,\boldsymbol{t}} \subset R_{f,\boldsymbol{t}}$ *of* K

We recall the formula $\varphi(f, \boldsymbol{t})$ and its negation $\neg\varphi(f, \boldsymbol{t})$ to be used often later

$$\begin{aligned} \varphi(f, \boldsymbol{t}) \;&\equiv\; \exists\, \delta_1 \,\forall a_1 \!\in\! \Sigma_{\delta_1} \dots \exists\, \delta_e \forall\, a_e \!\in\! \Sigma_{\delta_e} \forall\, \delta_0 \,\exists\, (u_0', u_0) \!\in\! U^{\bullet}_{\delta_0} \,\forall\, \boldsymbol{x} :\; q_{\boldsymbol{u}_0, \boldsymbol{u}, f}(\boldsymbol{x}) \neq 0 \\ \neg\varphi(f, \boldsymbol{t}) \;&\equiv\; \forall\, \delta_1 \,\exists\, a_1 \!\in\! \Sigma_{\delta_1} \dots \forall\, \delta_e \,\exists\, a_e \!\in\! \Sigma_{\delta_e} \exists\, \delta_0 \,\forall\, (u_0', u_0) \in U^{\bullet}_{\delta_0} \,\exists\, \boldsymbol{x} :\; q_{\boldsymbol{u}_0, \boldsymbol{u}, f}(\boldsymbol{x}) = 0 \end{aligned}$$

and in order to simply notation and language, we denote/define:

- $\varphi(f, \boldsymbol{t}, \varepsilon) \equiv \big(\varphi(f, \boldsymbol{t})$ holds in $\tilde{K}_\varepsilon\big)$, where $K_\varepsilon = \tilde{K}[\sqrt{\varepsilon}\,]$, $\varepsilon \in K$.
- $\overline{\varphi}\,(f, \boldsymbol{t}, \xi) \equiv \big(\neg\varphi(f, \boldsymbol{t})$ holds in both $\tilde{K}_{\frac{1}{\xi}}$ and $\tilde{K}_{\frac{1}{\xi}-1}\big)$, where $K_\eta = \tilde{K}[\sqrt{\eta}\,]$, $\eta \in K$.

Notations/Remarks 4.3. We supplement Notations/Remarks 3.1, 3.3, as follows.

1) For $\varepsilon \in K$, consider the restriction map $\mathrm{res}_\varepsilon : \mathrm{H}^{d+1}\big(K, \mathbb{Z}/n(d)\big) \to \mathrm{H}^{d+1}\big(K_\varepsilon, \mathbb{Z}/n(d)\big)$.
2) Let $C_\varepsilon \to C$, $P_\varepsilon \mapsto P$, be the normalization of C in the field extension $K_\varepsilon \,|\, K$, and denote by $D_{f,\varepsilon}$ the set of all $P_\varepsilon \in C_\varepsilon$ such that $v_{P_\varepsilon}(f) \notin n \cdot v_{P_\varepsilon} K_\varepsilon$.
3) For every $P \in C$, let $U_P := \mathcal{O}_P^\times$ be the P-units, and let $U_P \subset \mathcal{O}_{v_P}$ be the v_P-units. Then for $\varepsilon \in K^\times$ one has: (i) $\varepsilon \in U_P \cdot K^{\bullet n}$ iff (ii) $v_P(\varepsilon) \in n \cdot v_P K$.

Lemma 4.4. *In the above notations,* $\varepsilon \in \cup_{P\in D_f} U_P\cdot K^{\bullet n} \Leftrightarrow \varphi(f,\boldsymbol{t})$ *holds in* K_ε.
In particular, $\Theta_{f,\boldsymbol{t}} := \cup_{P\in D_f} U_P\cdot K^{\bullet n} \subset K$ *are uniformly first-oder definable in* K *as follows:*

$$\Theta_{f,\boldsymbol{t}} = \{\varepsilon \in K \mid \varphi(f,\boldsymbol{t},\varepsilon)\}$$

Proof. To $\Rightarrow$: Let $\varepsilon \in \cup_{P\in D_f} U_P \cdot K^{\bullet n}$ be given, and $P \in D_f$ be such that $\varepsilon \in U_P \cdot K^{\bullet n}$. Then P is unramified in the extension $K_\varepsilon|K$. Hence if $C_\varepsilon \to C$ is the normalization of C in the field extension $K_\varepsilon \hookleftarrow K$, it follow that $v_{P_\varepsilon}(f) = v_P(f)$ is prime to n. Hence $D_{f,\varepsilon} \neq \emptyset$, and therefore, by Key Lemma 4.1 follows that condition ii) is satisfied over K_ε. Let $k_{0_\varepsilon} = \overline{k_0} \cap K_\varepsilon$ be the field of constants of K_ε. Then $k_{0_\varepsilon}|k_0$ is a finite field extension, and therefore, for every $\delta_\varepsilon \in k_{0_\varepsilon}^\times$ there is $\delta \in k_0^\times$ such that for all v_{0_ε} and $v := v_{0_\varepsilon}|_{k_0}$ one has: $v_{0_\varepsilon}(\delta_\varepsilon) \neq 0$ iff $v(\delta) \neq 0$. In particular, $\Sigma_{\delta_\varepsilon} \cap k_0 = \Sigma_\delta$. Therefore, condition ii) for K_ε implies condition ii) for K.

To $\Rightarrow$: Consider any $\varepsilon \notin \cup_{P\in D_f} U_P \cdot K^{\bullet n}$ that is, $\varepsilon \notin U_P \cdot K^{\bullet n}$ for all $v \in D_f$. Then by Notations/Remarks 3.7, 3), one has $v_P(\varepsilon) \notin n{\cdot}v_P(K)$ for all $P \in D_f$. Hence for all $P \in D_f$, and any prolongation P_ε to K_ε one has $e(P_\varepsilon|P) = n$, thus $v_{P_\varepsilon}(f) = e(P_\varepsilon|P)\, v_P(f) \in n v_{P_\varepsilon}(K_\varepsilon)$. Further, since $v_P(f) \in n\cdot v_P(K)$ for $P \notin D_f$, one has $v_{P_\varepsilon}(f) \in n\cdot v_{P_\varepsilon}(K_\varepsilon)$ for $P_\varepsilon \mapsto P \notin D_f$, thus conclusing that $v_{P_\varepsilon}(f) \in n{\cdot}v_{P_\varepsilon}(K_\varepsilon)$ for all $P_\varepsilon \in C_\varepsilon$. On the other hand, by hypothesis ii), applying Key Lemma 4.1 to K_ε, it follows that $D_{f,\varepsilon}$ is non-empty, contradiction! □

Notations/Remarks 4.5. In the notations from Lemma 4.4 above, we have the following:

1) Let $\eta \in K \setminus \Theta_{f,\boldsymbol{t}}$ be given. Then by Notations/Remarks 4.3, 3), it follows that for all $P \in D_f$ one has: $v_P(\eta) \notin n\cdot v_{P_\eta}(K_\eta)$. In particular, $v_P(\eta) \neq 0$, and therefore one has:
   - If $v_P(\eta) > 0$, then $\eta - 1 \in \mathfrak{m}_P - 1 \subset U_P \subset \Theta_{f,\boldsymbol{t}}$, hence finally $\eta - 1 \in \Theta_{f,\boldsymbol{t}}$.
   - If $v_P(\eta) < 0$, then $v_P(\eta - 1) = v_P(\eta) \notin n\cdot v_P(K)$. Therefore, by the discussion at Notations/Remarks 4.3, 3), it follows that $\eta - 1 \notin U_P \cdot K^{\bullet n}$.
2) Conclude that for $\eta \in K$ the conditions (i), (ii) below are equivalent:

   (i) $\eta, \eta - 1 \notin \Theta_{f,\boldsymbol{t}}$; (ii) $v_P(\eta) < 0$ *for all* $P \in D_f$ *and* $v_P(\eta) \notin n \cdot v_P(K)$ *for all* $P \in D_f$.
3) Hence $\overline{\Theta}_{f,\boldsymbol{t}} := \big\{\xi \in K \mid \frac{1}{\xi}, \frac{1}{\xi} - 1 \notin \Theta_{f,\boldsymbol{t}}\big\} = \{\xi \in K \mid \overline{\varphi}\,(f,\boldsymbol{t},\xi)\}$ are uniformly definable, and

$$(*) \qquad \xi \in \overline{\Theta}_{f,\boldsymbol{t}} \;\; \textit{iff} \;\; \forall\, P \in D_f \;\; \textit{one has:} \;\; v_P(\xi) > 0,\, v_P(\xi) \notin n \cdot v_P(K).$$

Lemma 4.6. *In the above notation, one has* $\mathfrak{a}_{f,\boldsymbol{t}} := \cap_{P\in D_f} \mathfrak{m}_P = \overline{\Theta}_{f,\boldsymbol{t}} - \overline{\Theta}_{f,\boldsymbol{t}}$. *Hence* $\mathfrak{a}_{f,\boldsymbol{t}} \subset K$ *is uniformly definable, thus so is the subring* $R_{f,\boldsymbol{t}} = \cap_{P\in D_f} \mathcal{O}_P = \{\, r \in K \mid r\cdot \mathfrak{a}_{f,\boldsymbol{t}} \subset \mathfrak{a}_{f,\boldsymbol{t}}\}$ *of* K,

$$R_{f,\boldsymbol{t}} = \{r \in K \mid \forall \xi', \xi'' \in K \text{ s.t. } \overline{\varphi}\,(f,\boldsymbol{t},\xi'), \overline{\varphi}\,(f,\boldsymbol{t},\xi'')\; \exists \tilde{\xi}', \tilde{\xi}'' \in K \text{ s.t. } \overline{\varphi}\,(f,\boldsymbol{t},\tilde{\xi}'), \overline{\varphi}\,(f,\boldsymbol{t},\tilde{\xi}'') \wedge r(\xi' - \xi'') = \tilde{\xi}' - \tilde{\xi}''\}$$

Proof. We first prove the equality $\cap_{P\in D_f} \mathfrak{m}_P = \overline{\Theta}_{f,\boldsymbol{t}} - \overline{\Theta}_{f,\boldsymbol{t}}$. For the inclusion "$\subset$" notice that $\overline{\Theta}_{f,\boldsymbol{t}} \subset \mathfrak{m}_P$, $P \in D_f$ by Notations/Remarks 4.3, 3) above. Hence $\overline{\Theta}_{f,\boldsymbol{t}} - \overline{\Theta}_{f,\boldsymbol{t}} \subset \mathfrak{m}_P - \mathfrak{m}_P = \mathfrak{m}_P$, $P \in D_f$, thus finally one has $\overline{\Theta}_{f,\boldsymbol{t}} - \overline{\Theta}_{f,\boldsymbol{t}} \subset \mathfrak{a}_{f,\boldsymbol{t}}$. For the converse inclusion "$\supset$" let $\xi \in \mathfrak{a}_{f,\boldsymbol{t}}$ be arbitrary. Since $\mathfrak{a}_{f,\boldsymbol{t}} = \cap_{P\in D_f} \mathfrak{m}_P$, it follows by Notations/Remarks 4.3, 3), above that $v_P(\xi) > 0$ for all $P \in D_f$. Hence by the weak approximation lemma, it follows that there exists $\xi' \in K$ such that both ξ' and $\xi'' := \xi' - \xi$ satisfy $v_P(\xi'), v_P(\xi'') = 1$. In particular, by Notations/Remarks 4.3, 3), one has $\xi', \xi'' \in \overline{\Theta}_{f,\boldsymbol{t}}$, hence $\xi = \xi' - \xi'' \in \overline{\Theta}_{f,\boldsymbol{t}} - \overline{\Theta}_{f,\boldsymbol{t}}$.

Concerning the assertions about $R_{f,\boldsymbol{t}}$, the first row equalities are well known basic valuation theoretical facts (which follow, e.g. using the weak approximation lemma), whereas the second row equality is simply the definition of $\{r \in K \mid r \cdot \mathfrak{a}_{f,\boldsymbol{t}} \subset \mathfrak{a}_{f,\boldsymbol{t}}\}$ using the explicit definition of $\mathfrak{a}_{f,\boldsymbol{t}} = \overline{\Theta}_{f,\boldsymbol{t}} - \overline{\Theta}_{f,\boldsymbol{t}}$; this also shows/implies the uniform definability of $R_{f,\boldsymbol{t}}$. □

B) *Defining the k-valuation rings of $K|k$*

In the notations and hypotheses the previous sections, recall that $K = k(C)$ for some projective smooth k-curve C. By Riemann-Roch we have: For every closed point $P \in C$ and $m \gg 0$, there exist functions $f \in K$ such that $(f)_\infty = m\,P$. Hence choosing m to be prime to $n = 2$, we have $P \in D_f$. Thus by Lemma 4.6, it follows that $P \in D_f$ and $R_{f,\boldsymbol{t}} = \mathcal{O}_P \cap R^0_{f,\boldsymbol{t}}$, where $R^0_{f,\boldsymbol{t}} = \cap_{P' \in D_f} \mathcal{O}_{P'}$ with $P' \neq P$ the zeros of f which lie in D_f.

For f as above, we set $g := f{+}1$, and notice that $(g)_\infty = mP = (f)_\infty$, etc., and obviously, f and g have no common zeros. We repeat the constructions above with f replaced by g, and get $R_{g,\boldsymbol{t}} = \mathcal{O}_P \cap R^0_{g,\boldsymbol{t}}$, where $R^0_{g,\boldsymbol{t}} = \cap_{Q' \in D_g} \mathcal{O}_{Q'}$ with $Q' \neq P$ the zeros of g from D_g. Since $|\mathrm{div}(f)| \cap |\mathrm{div}(g)| = \{P\}$, by the weak approximation lemma one has:

$$\mathcal{O}_P = R_{f,\boldsymbol{t}} \cdot R_{g,\boldsymbol{t}} := \{r_1 r_2 \mid r_1 \in R_{f,\boldsymbol{t}},\, r_2 \in R_{g,\boldsymbol{t}}\}$$

Therefore, setting $f_1 := f$ and $f_2 := g = f + 1$, we have the following:

$$\begin{aligned}\mathcal{O}_P = R_{f_1,\boldsymbol{t}} \cdot R_{f_2,\boldsymbol{t}} &= \{\, r \in K \mid \exists\, r_i \in R_{f_i,\boldsymbol{t}} \text{ s.t. } r = r_1 r_2 \,\}\\ &= \{\, r \in K \mid \exists\, r_1, r_2 \in K \text{ s.t. } r = r_1 r_2 \text{ and for } i = 1,2 \text{ one has:}\\ &\quad \forall\, \xi'_i, \xi''_i \in K \text{ s.t. } \overline{\varphi}(f_i, \boldsymbol{t}, \xi'_i), \overline{\varphi}(f_i, \boldsymbol{t}, \xi''_i)\ \exists\, \tilde{\xi}'_i, \tilde{\xi}''_i \in K \text{ s.t. } \overline{\varphi}(f_i, \boldsymbol{t}, \tilde{\xi}'_i), \overline{\varphi}(f_i, \boldsymbol{t}, \xi''_i) \text{ and } r_i(\xi'_i - \xi''_i) = \tilde{\xi}'_i - \tilde{\xi}_i \,\}\end{aligned}$$

and finally one recovers $\mathfrak{m}_P$ as being

$$\mathfrak{m}_P = \{r \in K \mid r \in \mathcal{O}_P,\, r^{-1} \notin \mathcal{O}_P\}$$

Hence we have the following *uniform first-oder recipe* to define the prime k_0-divisors of $K|k$.

Recipe 4.7. Recall φ_d, $\psi^{\text{abs}}(x)$, $\psi_r(t_1,\ldots,t_r)$ from the Introduction. If $\dim(K) = 1$, then the prime divisors of K are uniformly first-order definable by the formulae val_1 given by Rumely [Ru]; and if $\dim(K) = 2$, the *geometric* prime divisors of $K|k_0$ are uniformly first-order definable by the formulae val_2 given by Pop [P4].

We next consider the case $\dim(K) > 2$, char $\neq 2$. Letting $k_0 \subset K$ denote global subfields, and setting $e := \dim(K) - 2 = \mathrm{td}(K|k_0) - 1 > 0$, we construct

$$\mathsf{val}_d\big(x; f, \boldsymbol{t}, \boldsymbol{\xi}, \boldsymbol{\delta}, (a_i)_i \in \boldsymbol{\Sigma_\delta}, \delta_0, \boldsymbol{u}_0 \in U^{\bullet}_{\delta_0}, q_{\boldsymbol{u}_0,\boldsymbol{u},f}(\boldsymbol{x_\xi})\big)$$

in a concrete way along the following steps:

1) Consider the systems $\boldsymbol{t} := (t_1,\ldots,t_e)$, of k_0-algebraically independent elements of K with t_i squares in K. These are uniformly definable using the formula $\psi_e(t_1,\ldots,t_e)$ over k_0.
2) Let $\mathcal{P}$ be the set of all the pairs $(f, \boldsymbol{t})$ with $f \in K$ and $(f, \boldsymbol{t})$ algebraically independent such that $\mathrm{H}_{\delta_0,\boldsymbol{u},f}, \mathrm{H}_{\delta_0,\boldsymbol{u},f+1} \neq 0$. Hence $\mathcal{P}$ is uniformly first-order definable as follows:
$$\mathcal{P} := \{(f, \boldsymbol{t}) \mid \varphi(f, \boldsymbol{t}) \wedge \varphi(f{+}1, \boldsymbol{t}) \text{ holds in } K\,\}.$$
3) Therefore, the set $\mathcal{P}_{\mathsf{val}} \subset \mathcal{P}_1$ below are uniformly first-order definable:
$$\mathcal{P}_{\mathsf{val}} := \{(f, \boldsymbol{t}) \in \mathcal{P}_1 \mid \mathcal{O}_{f,\boldsymbol{t}} := R_{f,\boldsymbol{t}} \cdot R_{f+1,\boldsymbol{t}} \text{ is a proper valuation ring of } K\,\}$$
4) Finally note that the above $\mathcal{O}_{f,\boldsymbol{t}}$ are valuation rings of prime k_0-divisors of $K|k_0$, and conversely, for every prime k_0-divisor w of $K|k_0$ there are pairs $(f, \boldsymbol{t}) \in \mathcal{P}_{\mathsf{val}}$ such that the valuation ring $\mathcal{O}_w$ is of the form $\mathcal{O}_w = \mathcal{O}_{f,\boldsymbol{t}}$.

Conclude that the prime divisors of $K|k_0$ are uniformly first-order definable via the set $\mathcal{P}_{\mathsf{val}}$.

5. Proof of the Main Theorem

A) *First proof*: Using Scanlon [Sc]

A first proof follows simply from Scanlon, Theorem 4.1 and Theorem 5.1, applied to the case of characteristic $\neq 2$, using Theorem 1.1 for the definability of valuations (which is essential in both Theorem 4.1 and Theorem 5.1 of loc.cit.). This proof also shows that finitely generated fields of characteristic $\neq 2$ are ***bi-interpretable with the arithmetic.***

B) *Second proof*: Using Aschenbrenner–Khélif–Naziazeno–Scanlon [AKNS]

Recall that one of the main results of [AKNS] asserts that the *finitely generated infinite domains R are bi-interpretable with arithmetic,* see Theorem in the Introduction of loc.cit. In particular, the isomorphy type of any such domain is encoded by a sentence $\boldsymbol{\vartheta}_R$. Thus the Main Theorem from the Introduction follows from the following stronger assertion:

Theorem 5.1. *Let $\mathcal{T} = (t_1, \dots, t_r)$ be independent variables. Then the integral closures $R \subset K$ of $\mathbb{Z}[\mathcal{T}]$ in finite field extensions $\mathbb{Q}(\mathcal{T}) \hookrightarrow K$ are uniformly first-order definable finitely generated domains.*

Proof. Since $\mathbb{Z}[\mathcal{T}]$ is Noetherian, and $\mathbb{Q}(\mathcal{T}) \hookrightarrow K$ are finite separable extensions, the Finiteness Lemma asserts that R is a finite $\mathbb{Z}[\mathcal{T}]$-module, hence finitely generated as ring. The uniform definability of R is though more involved, and uses the uniform definability of generalized geometric prime divisors of $K|k_0$ combined with Rumely [Ru].

Lemma 5.2. *Let A be an integrally closed domain, and $\mathcal{V}$ be a set of valuations of the fraction field $K_A := \mathrm{Quot}(A)$ such that $A = \cap_{v \in \mathcal{V}_A} \mathcal{O}_v$. Let B be the integral closure of A in an algebraic extension $K_B|K_A$, and $\mathcal{W}$ be the prolongation of $\mathcal{V}$ to K_B. Then $B = \cap_{w \in \mathcal{W}} \mathcal{O}_w$.*

Proof. Klar, left to the reader. $\square$

Let R_0 be an integrally closed domain, $L_0 := \mathrm{Quot}(R_0)$, and $\mathcal{V}_0$ be a set of valuations of L_0 such that $R_0 = \cap_{v \in \mathcal{V}} \mathcal{O}_v$. Let $L_1|L_0(t)$ be a finite field extension, $R_1 \subset \tilde{R}_1 \subset L_1$ be the integral closures of $R_0[t] \subset L_0[t]$ in L_1. Then $\kappa(P)$ are finite field extensions of L_0, $P \in \mathrm{Max}(\tilde{R}_1)$ and let $\mathcal{V}^P$ be the prolongation of $\mathcal{V}_0$ to $\kappa(P)$. Finally let $\mathcal{V}_1$ be the set of all the valuations of the form $v_1 := v^P \circ v_P$ with v_P the valuation of $P \in \mathrm{Max}(\tilde{R}_1)$, and $v^P \in \mathcal{V}^P$. Then $v^P = v_1/v_P$, $v_1 L_1 = v^P L^P \times \mathbb{Z}$ lexicographically ordered, and $L_1 v_1 = L^P v^P$. Further, the canonical restriction map $\mathrm{Val}(L_1) \to \mathrm{Val}(L_0)$ gives rise to a well defined surjective maps:

$$\mathcal{V}_1 \to \mathcal{V}^P \to \mathcal{V}, \quad v_1 \mapsto v^P \mapsto v_0 := v^P|_{L_0} = v_1|_{L_0}.$$

Lemma 5.3. *In the above notation, one has $R_1 = \cap_{v_1 \in \mathcal{V}_1} \mathcal{O}_{v_1}$.*

Proof. Lemma 5.2 reduces the problem to the case $L_1 = L_0(t)$, $R_1 = R_0[t]$. For $v_1 = v^P \circ v_P$, $\mathcal{O}_{v_1} \subset \mathcal{O}_{v_P}$, hence $\cap_{v_1 \in \mathcal{V}_1} \mathcal{O}_{v_1} \subset \cap_P \mathcal{O}_{v_P} = L_0[t]$. Thus it is left to prove that $f(t) \in L_0[t]$ satisfies: $v_1(f) \geq 0$ for all $v_1 \in \mathcal{V}_1$ iff $f \in R_0[t]$. This easy exercise is left to the reader. $\square$

Lemma 5.4. *Suppose that all the valuation rings $\mathcal{O}_P$, $P \in \mathrm{Max}(\tilde{R}_1)$ and $\mathcal{O}_{v^P}$, $v^P \in \mathcal{V}^P$ are (uniformly) first-oder definable. Then so are $\mathcal{O}_{v_1}$, $v_1 \in \mathcal{V}_1$ and $R_1 = \cap_{v_1 \in \mathcal{V}_1} \mathcal{O}_{v_1}$.*

Proof. For $v_1 = v^P \circ v_P$, one has $\mathcal{O}_{v_1} = \pi_P^{-1}(\mathcal{O}_{v^P})$, where $\pi_P : \mathcal{O}_P \to \kappa(P) =: L^P$, etc. $\square$

Finally, all of the above can be performed inductively for systems of variables $\mathcal{T} := (t_1, \dots, t_r)$, L_r finite field extension of $L_0(\mathcal{T})$, $R_r \subset \tilde{R}_r \subset L_r$ the integral closures of $R_{r-1}[t_r] \subset L_{r-1}[t_r]$, thus leading to the corresponding sets of all valuations $\mathcal{V}_r$ of L_r the form $v_r = v^P_{r-1} \circ v_P$, where $P \in \mathrm{Max}(\tilde{R}_r)$ and v^P_{r-1} lies in the prolongation $\mathcal{V}^P_{r-1}$ of $\mathcal{V}_{r-1}$ to $\kappa(P)$.

Lemma 5.5. *In the above notation, one has $R_r = \cap_{v_r \in \mathcal{V}_r} \mathcal{O}_{v_r}$. Further, if all the valuation rings $\mathcal{O}_P$, $P \in \mathrm{Max}(\tilde{R}_r)$ and $\mathcal{O}_{v^P}$, $v_P \in \mathcal{V}^P_{r-1}$ are (uniformly) first-order definable, then so are the valuation rings $\mathcal{O}_{v_r}$, $v_r \in \mathcal{V}_r$ and $R_r = \cap_{v_r \in \mathcal{V}_r} \mathcal{O}_{v_r}$.*

Proof. Induction on r reduces everything to $r = 1$. Conclude by using Lemmas 5.3, 5.4. □

Coming back to the proof of Theorem 5.1, let $k_0 = \kappa \subset K$ be the constant subfield of K, and set $R_0 := \mathrm{Spec}\, \mathcal{O}_{k_0}$. Then R is the integral closure of $R_0[\mathcal{T}]$ in K, and Theorem 5.1 above follows from Lemma 5.5 above. □

C) *Third proof*: Using Rumely's result [Ru]

We begin by mentioning that Pop [P4], Theorem 1.2 holds in the following more general form (which might be well known to experts, but we cannot give a precise reference). Namely, let $\mathcal{K}$ be a class of function fields of projective normal geometrically integral curves $K = k(C)$ such that $k \subset K$ and the k-valuations rings $\mathcal{O}, \mathfrak{m}$ of $K|k$ are (uniformly) first-order definable. Then for every non-zero $t \in K$, $e > 0$, the sets

$$\Sigma_{t,e} := \{\mathcal{O}, \mathfrak{m} \mid t \in \mathfrak{m}^e,\ t \notin \mathfrak{m}^{e+1}\}$$

are (uniformly) first-order definable subset of the set of all the valuation rings $\mathcal{O}, \mathfrak{m}$. Hence given $N > 0$, a function $t \in K^\times$ has $\deg(t) := [K : k(t)] = N$ iff the following hold:

i) $\Sigma_{t,N+1} = \emptyset$ and $|\Sigma_{t,e}| \leq N$ for all $0 < e \leqslant N$.

ii) $\dim_k \mathcal{O}/\mathfrak{m} \leq N$ for all $\mathcal{O}, \mathfrak{m} \in \Sigma_{t,e}$, and moreover: $N = \sum_{0<e\leqslant N} \sum_{\mathcal{O},\mathfrak{m}\in\Sigma_{t,e}} e \dim_k \mathcal{O}/\mathfrak{m}$.

In particular, there exists a (uniform) first-order formula $\deg_N(\mathfrak{t})$ such that for every $K = k(C)$ as above, and non-constant $t \in K$ one has:

- $\deg_N(t)$ is true in K iff t has degree N as a function of $K|k$, i.e., $[K : k(t)] = N$.

Now let K be a finitely generated field with $\mathrm{char}(K) = 0$, and $\mathcal{T} = (\boldsymbol{t}_e, t)$ be a transcendence basis of $K|k_0$, where $\boldsymbol{t}_e := (t_1, \ldots, t_e)$. Setting $\boldsymbol{T} := (T_1, \ldots, T_e)$, there exists an absolutely irreducible U-monic polynomial $f_K \in k_0[\boldsymbol{T}, T, U]$, and $u \in K$ such that

$$K = k_0(\boldsymbol{t}_e, t)[u], \quad f_K(\boldsymbol{t}_e, t, u) = 0.$$

In the above notation, the isomorphy type of K is given by the following data:

a) *a transcendence basis $(\boldsymbol{t}_e, t)$ of $K|k_0$ and a non-constant function $u \in K$,*

b) *an absolutely irreducible U-monic polynomial $f_K \in k_0[\boldsymbol{T}_e, T, U]$ such that $f(\boldsymbol{t}_e, t, u) = 0$,*

c) *letting $k = k_{\boldsymbol{t}_e} \subset K$ be the relative algebraic closure of $k_0(\boldsymbol{t}_e)$ in K, and $K = k(C)$ with C a projective smooth geometrically integral k-curve C, one has:*

$$\deg_C(u) := [K : k(u)] = \deg_U(f_K) =: N_{f_K}.$$

Recall that Rumely [Ru] gives a uniform bi-interpretability of number fields with Peano arithmetic. In particular, for number fields k_0 endowed with finite system Σ of n constants, there exists a sentence $\boldsymbol{\vartheta}^{\mathrm{Ru}}_{k_0,\Sigma}$ such that for any other global field l_0 there is an isomorphism

$\imath : k_0 \to l_0$, thus endowing l_0 with the finite system of n constants $\imath(\Sigma)$. Recalling Poonen's sentence ψ_0 and the predicate $\psi^{\rm abs}(x)$, consider the sentence:

$$\boldsymbol{\vartheta}_{k_0,\Sigma} \equiv \psi_0 \wedge \big(\,\boldsymbol{\vartheta}^{\rm Ru}_{k_0,\Sigma} \text{ hols in } \kappa = \kappa(\psi^{\rm abs}(x)\big).$$

Then for all finitely generated fields L one has: If $\boldsymbol{\vartheta}_{k_0,\Sigma}$ holds in L, and l_0 is the field of constants of L, then there is an isomorphism $\imath : k_0 \to l_0$, which endows l_0 with $\imath(\Sigma)$.

A special case of this arises by starting with $K = k_0(\boldsymbol{t}_e, t)[u]$, $f_K(\boldsymbol{t}_e, t, u) = 0$ as above, and letting $\Sigma := \Sigma_{f_K}$ be the system of coefficients $(a_{\boldsymbol{\imath}})_{\boldsymbol{\imath}}$ of f_K. In particular, if L is a finitely generated field with field of constants l_0 such that $\boldsymbol{\vartheta}_{k_0,\Sigma_{f_K}}$ holds in L, then there is an isomorphism $\imath : k_0 \to l_0$ which gives rise to a polynomial $f_L := \imath(f_K)$; and notice that f_L is absolutely irreducible and U-monic, and obviously, $\deg_U(f_L) = N_{f_K}$.

Next let $\mathbf{t}_e = (\mathfrak{t}_i)_{1\leqslant i\leqslant e}$, $\mathfrak{t}$, $\mathfrak{u}$ be variables, and consider "generic" polynomials $f(\mathbf{t}_e, \mathfrak{t}, \mathfrak{u})$ which are monic in $\mathfrak{u}$ and have degree $N_f := \deg_{\mathfrak{u}}(f)$. Further let Σ_f be the system of the coefficients of f. Recalling the algebraic independence formula $\psi_r(\mathfrak{t}_1, \dots, \mathfrak{t}_r)$, in the above context, we denote by $k_{\mathbf{t}_e}$ the relative algebraic closure of $\mathbb{Q}(\mathbf{t}_e)$ in finitely generated fields K in which $\mathbf{t}_e$ are evaluated. In particular, by the discussion above, for every finitely generated field K with constants a number field k_0 and $\mathrm{td}(K) = e + 1$, there is some f_K describing K, and we think of f_K as being obtained by properly specializing the variables $(\mathbf{t}_e, \mathfrak{t}, \mathfrak{u}) \mapsto (\boldsymbol{t}_e, t, u)$ and $f \mapsto f_K$. In particular, $\Sigma_f \mapsto \Sigma_{f_K}$, and $\boldsymbol{\vartheta}_{k_0,\Sigma_{f_K}}$ holds in K. Finally, recalling the sentence φ_d defining $d = \dim(K) = e + 2$, consider the sentence

$$\boldsymbol{\vartheta}_K \;\equiv\; \varphi_d \wedge \big(\exists\, \mathbf{t}_e, \mathfrak{t}, \mathfrak{u} :\, \psi_{e+1}(\mathbf{t}_e, \mathfrak{t}) \wedge f_K(\mathbf{t}_e, \mathfrak{t}, \mathfrak{u}) = 0 \wedge \boldsymbol{\vartheta}_{k_0,\Sigma_{f_K}} \wedge k = k_{\mathbf{t}_e} \wedge \deg_{N_{f_K}}(\mathfrak{u})\,\big)$$

To conclude the proof of the Main Theorem, let L be a finitely generated field with constant field l_0 such that $\boldsymbol{\vartheta}_K$ holds in L. Then one has the following:

a) First, $\dim(L) = d = e + 2 = \dim(K)$.

b) $\boldsymbol{\vartheta}_{k_0,\Sigma_{f_K}}$ holds in l_0, hence one has an isomorphism $\imath : k_0 \to l_0$, thus $\mathrm{td}(K) = e+1 = \mathrm{td}(L)$.

- Let $f_L := \imath(f_K)$ be the image of f_K under $\imath$, and notice that f_L is absolutely irreducible.

c) $\exists\, t'_1, \dots, t'_e, t', u' \in L$ s.t. $\psi_{e+1}(\boldsymbol{t}'_e, t')$ holds, hence $(\boldsymbol{t}'_e, t')$ are algebraically independent. Hence since $\mathrm{td}(L) = e + 1$, it follows that $(\boldsymbol{t}'_e, t')$ is a transcendence basis of L.

d) Setting $l := k_{\boldsymbol{t}'_e}$, one has: $f_L(\boldsymbol{t}'_e, t', u') = 0$ and $\deg_{L|l}(u') = N_{f_K} = N_{f_L}$.

By the discussion above, one has $L = l_0(\boldsymbol{t}'_e, t', u')$, and an isomorphism of fields:

$$\imath_K : K \to L, \quad (\boldsymbol{t}_e, t, u) \mapsto (\boldsymbol{t}'_e, t', u'), \quad \imath_K|_{k_0} = \imath$$

References

[AKNS] Aschenbrenner, M., Khélif, A., Naziazeno, E. and Scanlon, Th., *The logical complexity of finitely generated commutative rings,* Int. Math. Res. Notices (to appear).

[Di] Dittmann, Ph., *Defining Subrings in Finitely Generated Fields of Characteristic Not Two,* See: `arXiv:1810.09333` [math.LO], Oct 22, 2018.

[Du] Duret, J.-L., *Équivalence élémentaire et isomorphisme des corps de courbe sur un corps algebriquement clos,* J. Symbolic Logic **57** (1992), 808–923.

[Ei] Eisenträger, K., *Integrality at a prime for global fields and the perfect closure of global fields of characteristic $p > 2$,* J. Number Theory **114** (2005), 170–181.

[E-S] Eisenträger, K. and Shlapentokh, A., *Hilbert's Tenth Problem over function fields of positive characteristic not containing the algebraic closure of a finite field,* JEMS **19** (2017), 2103–2138.

[Hi] Hironaka, H., *Resolution of singularities of an algebraic variety over a field of characteristic zero,* Annals of Math. **79** (1964), 109–203; 205–326.

[Ill] Illusie, L., *Complèxe de de Rham–Witt et cohomologie cristalline,* Ann. Scie. Ec. Norm. Sup. **12** (1979), 501–661.

[Ja] Jannsen, U., *Hasse principles for higher-dimensional fields,* Annals of Math. **183** (2016), 1–71.

[Kh] Kahn, E., *La conjecture de Milnor (d'apres Voevodsky),* Sém. Bourbaki, Asterisque **245** (1997), 379–418.

[Ka] Kato, K., *A Hasse principle for two dim global fields*, J. reine angew. Math. **366** (1986), 142–180.

[K-S] Kerz. M. and Saito, Sh., *Cohomological Hasse principle and motivic cohomology for arithmetic schemes,* Publ. Math. IHES **115** (2012), 123–183.

[K-R] Kim, H. K. and Roush, F. W., *Diophantine undecidability of* $\mathbb{C}(T_1, T_2)$*,* J. Algebra **150** (1992), 35–44.

[Ko1] Koenigsmann, J., *Defining transcendentals in function fields,* J. Symbolic Logic **67** (2002), 947–956.

[Ko2] Koenigsmann, J., *Defining* $\mathbb{Z}$ *in* $\mathbb{Q}$*,* Annals of Math. **183** (2016), 73–93.

[Ko3] Koenigsmann, J., *Decidability in local and global fields,* Proc. ICM 2018 Rio de Janeiro, **Vol. 2**, 63–78.

[M-S] Merkurjev, A. S. and Suslin, A. A., *K-cohomology of Severi–Brauer variety and norm residue homomorphism,* Math. USSR Izvestia **21** (1983), 307–340.

[M-Sh] Miller, R. and Shlapentokh, A., *On existential definitions of C.E. subsets of rings of functions of characteristic 0,* `arXiv:1706.03302` [math.NT]

[Pf1] Pfister, A., Quadratic Forms with Applications to Algebraic Geometry and Topology, LMS LNM **217**, Cambridge University Press 1995; ISBN 0-521-46755-1.

[Pf2] Pfister, A., *On the Milnor conjectures: history, influence, applications,* Jahresber. DMV **102** (2000), 15–41.

[Pi] Pierce, D., *Function fields and elementary equivalence,* Bull. London Math. Soc. **31** (1999), 431–440.

[Po] Poonen, B., *Uniform first-order definitions in finitely gen. fields,* Duke Math. J. **138** (2007), 1–21.

[P-P] B. Poonen and F. Pop, *First-order characterization of function field invariants over large fields,* in: Model Theory with Applications to Algebra and Analysis, LMS LNM Series **350**, Cambridge Univ. Press 2007; pp. 255–271.

[P1] Pop, F., *Embedding problems over large fields,* Annals of Math. **144** (1996), 1–34.

[P2] Pop, F., *Elementary equivalence versus isomorphism*, Invent. Math. **150** (2002), 385–308.

[P3] Pop, F., *Elementary equivalence of finitely generated fields,* Course Notes Arizona Winter School 2003, see `http://swc.math.arizona.edu/oldaws/03Notes.html`

[P4] Pop, F., *Elementary equivalence versus Isomorphisms II,* Algebra & Number Theory **11** (2017), 2091-2111.

[P5] Pop, F., See `arXiv:1809.00440v1` [math.AG], Sept 3, 2018.

[Ro1] Robinson, Julia, *Definability and decision problems in arithmetic,* J. Symb. Logic **14** (1949), 98–114.

[Ro2] Robinson, Julia, *The undecidability of algebraic fields and rings,* Proc. AMS **10** (1959), 950–957.

[Ru] Rumely, R., *Undecidability and Definability for the Theory of Global Fields*, Transactions AMS **262** No. 1, (1980), 195–217.

[Sc] Scanlon, Th., *Infinite finitely generated fields are biinterpretable with* **N***,* JAMS **21** (2008), 893–908. *Erratum,* J. Amer. Math. Soc. **24** (2011), p. 917.

[Se] Serre, J.-P., *Zeta and L-functions,* in: Arithmetical Algebraic Geometry, Proc. Conf. Purdue 1963, New York 1965, pp. 82–92.

[Sh1] Shlapentokh, A., *First Order Definability and Decidability in Infinite Algebraic Extensions of Rational Numbers,* Israel J. Math. **226** (2018), 579–633.

[Sh2] Shlapentokh, A., *On definitions of polynomials over function fields of positive characteristic,* See `arXiv:1502.02714v1`

[Vi] Vidaux, X., *Équivalence élémentaire de corps elliptiques,* CRAS Série I **330** (2000), 1-4.

Department of Mathematics, University of Pennsylvania
DRL, 209 S 33rd Street, Philadelphia, PA 19104, USA
E-mail address: `pop@math.upenn.edu`
URL: `http://math.penn.edu/~pop`