ON I/OM

IHARA'S QUESTION / ODA–MATSUMOTO CONJECTURE

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Abstract. In this note we prove a pro-ℓ abelian-by-central form of Ihara's question / Oda-Matsumoto conjecture I/OM over arbitrary fields k_0, thus in particular over $\mathbb{Q}$, and show how this implies the full profinite I/OM over k_0. In particular, one gets a non-tautological topological/combinatorial description of the absolute Galois group Gal_{k_0}, thus of $\mathrm{Aut}(\overline{\mathbb{Q}})$.

1. Introduction

The aim of this paper is to give a positive answer to a question by Ihara from the 1980's, which in the 1990's became a conjecture by Oda–Matsumoto, for short I/OM. The I/OM is about giving a topological/combinatorial description of the absolute Galois group of the rational numbers. Actually a stronger result will be proved here, namely the **pro-ℓ abelian-by-central** form of the I/OM. Before announcing the result, let me explain the question/conjecture I/OM in detail.

Let $k_0 \subset \mathbb{C}$ be a fixed subfield, e.g., a number field, and $k \subseteq \mathbb{C}$ be the algebraic closure of k_0 inside $\mathbb{C}$. For every geometrically integral k_0-variety X, let $\overline{X} := X \times_{k_0} k$ be the base change to k, and let $\overline{\pi}(X) := \pi_1(\overline{X}, *)$ denote the algebraic fundamental group of X, where $*$ is a geometric point of X above $\overline{k}$ (which we will not mention anymore in order to simplify notations), thus $*$ defines the absolute Galois group Gal_{k_0} of k_0. Then one has an exact sequence of étale fundamental groups:

$$1 \to \overline{\pi}(X) \to \pi_1(X) \to \mathrm{Gal}_{k_0} \to 1.$$

Let $\mathbf{G}_{\mathrm{out}}$ be the category whose objects are the profinite groups above Gal_{k_0}, i.e., diagrams of continuous morphisms of profinite groups of the form $p_G : G \to \mathrm{Gal}_{k_0}$, and whose morphisms are continuos outer morphisms above Gal_{k_0}, i.e., for given $p_G : G \to \mathrm{Gal}_{k_0}$, $p_H : H \to \mathrm{Gal}_{k_0}$, one has $\mathrm{Hom}_{\mathbf{G}_{\mathrm{out}}}(G, H) := \{f \circ \mathrm{Inn}_{H_0} \mid f : G \to H \text{ with } p_G = p_H \circ f\}$, where Inn_{H_0} are the automorphisms of H defined by conjugation with elements from $\ker(p_H)$. Note that if the center of Gal_{k_0} is trivial, which is usually the case in arithmetical situations, then Inn_{H_0} are precisely the inner automorphisms of H which have trivial image in $\mathrm{Aut}(\mathrm{Gal}_{k_0})$. In particular, in the above geometric situation, we have that $\mathrm{Out}\big(\overline{\pi}(X)\big) = \mathrm{Aut}_{\mathbf{G}_{\mathrm{out}}}\big(\overline{\pi}(X)\big)$, and the ambiguity resulting from choices of base points above k vanishes. Further, the canonical exact sequence above gives rise to a representation $\rho_X : \mathrm{Gal}_{k_0} \to \mathrm{Out}\big(\overline{\pi}(X)\big) =$

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$\mathrm{Aut}_{\mathbf{G}_{\mathrm{out}}}(\overline{\pi}(X))$, and by the functoriality of π_1, the collection of all the representations $(\rho_X)_X$, is compatible with the base changes of k_0-morphisms $f : X \to Y$ of geometrically integral k_0-varieties. In particular, for every small category $\mathcal{V}$ of geometrically integral varieties over k_0, its algebraic fundamental group functor $\overline{\pi}_{\mathcal{V}} : \mathcal{V} \to \mathbf{G}_{\mathrm{out}}$, $X \mapsto \overline{\pi}(X)$, gives rise to a representation

$$\rho_{\mathcal{V}} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}(\overline{\pi}_{\mathcal{V}}),$$

where $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ is the automorphism group of $\overline{\pi}_{\mathcal{V}}$. In down to earth terms, the elements $\Phi \in \mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ are the families $\Phi = (\Phi_X)_{X\in\mathcal{V}}$, $\Phi_X \in \mathrm{Out}(\overline{\pi}(X)) = \mathrm{Aut}_{\mathbf{G}_{\mathrm{out}}}(\overline{\pi}(X))$, which are compatible with $\overline{\pi}(f) : \overline{\pi}(X) \to \overline{\pi}(Y)$ for all morphisms $f : X \to Y$ in $\mathcal{V}$.

Recall that $\overline{\pi}(X)$ is nothing but the profinite completion of the topological fundamental group $\pi_1^{\mathrm{top}}(X(\mathbb{C}), *)$ of the "good" topological space $X(\mathbb{C})$, thus $\overline{\pi}(X)$ is an object of topological/combinatorial nature. Following Grothendieck, one should give a new description of Gal_{k_0} by finding categories $\mathcal{V}$ for which $\rho_{\mathcal{V}}$ maps Gal_{k_0} injectively into $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ and $\mathrm{im}(\rho_{\mathcal{V}}) \subseteq \mathrm{Aut}(\overline{\pi}_{\mathcal{V}})$ has a "nice" description. If so, then via the isomorphism $\rho_{\mathcal{V}}$, we would have a new non-tautological description of Gal_{k_0}. For $k_0 = \mathbb{Q}$, Grothendieck suggested to study $\overline{\pi}_{\mathcal{V}}$, for $\mathcal{V}$ sub-categories of the *Teichmüller modular tower* $\mathcal{T}$ of all the moduli spaces $M_{g,n}$. For instance, if $\mathcal{V}_0 = \{M_{0,4}, M_{0,5}\}$ endowed with "connecting homomorphisms," $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}_0})$ is the famous Grothendieck–Teichmüller group $\widehat{GT}$, which was intensively studied first by Drinfel'd [Dr], Ihara [I1], [I2], [I3], Deligne [De], followed later on by several others, e.g. Hain–Matsumoto [H–M], Harbater–Schneps [H–Sch], Ihara–Matsumoto [I–M], Lochak–Schneps [L–Sch], Nakamura–Schneps [N–Sch], etc.

Concerning the nature of $\rho_{\mathcal{V}}$ in general, little is known, and the situation is quite mysterious. Nevertheless, for the injectivity of $\rho_{\mathcal{V}}$ in the case $k_0 = \mathbb{Q}$, one has: First, Drinfel'd remarked that using Belyi's Theorem [Be] it follows that $\rho_{\mathcal{V}}$ is injective, provided $X := M_{0,4} = \mathbb{P}^1\backslash\{0, 1, \infty\}$ lies in $\mathcal{V}$. Further, Voevodsky showed that the same is true if $X \in \mathcal{V}$ with X some affine open in a curve of genus one, and Matsumoto [Ma] showed that the same holds if $X \in \mathcal{V}$ for some affine hyperbolic curve X, and finally, Hoshi–Mochizuki [H–M] proved that $\rho_{\mathcal{V}}$ is injective as soon as $\mathcal{V}$ contains at least one hyperbolic curve. The question about the surjectivity of the representation $\rho_{\mathcal{V}}$, is less understood, and Ihara asked in the 1980's whether $\rho_{\mathcal{V}}$ is an isomorphism, provided $k_0 = \mathbb{Q}$ and $\mathcal{V}$ is the category of all the (geometrically integral) $\mathbb{Q}$-varieties. Finally, based on some "motivic evidence," Oda–Matsumoto conjectured in the 1990's that the answer to Ihara's Question should be positive. The author showed (in 1999, unpublished), that the answer to I/OM is positive —for even more general fields, thus giving a positive answer to I/OM. Finally, André [An] developed the theory of the tempered fundamental group and introduced the resulting p-adic tempered variant $\widehat{GT}_p$ of the Grothendieck–Teichmuülcer groups $\widehat{GT}$, and using those techniques (re)proved that the I/OM holds over p-adic fields.

We now formulate the pro-ℓ abelian-by-central I/OM, which implies the usual I/OM. Let $\overline{\pi}(X) \to \Pi_X^{\mathrm{c}} \to \Pi_X$ be the pro-ℓ abelian-by-central, respectively pro-ℓ abelian, quotients of $\overline{\pi}$. Since the kernels in this exact sequence are characteristic subgroups, there are canonical projections $\mathrm{Out}(\overline{\pi}(X)) \to \mathrm{Out}(\Pi_X^{\mathrm{c}}) \to \mathrm{Out}(\Pi_X)$. Therefore, for every category $\mathcal{V}$ as above, the canonical morphisms of functors $\overline{\pi}_{\mathcal{V}} \to \Pi_{\mathcal{V}}^{\mathrm{c}} \to \Pi_{\mathcal{V}}$ give rise to group homomorphisms $\mathrm{Aut}(\overline{\pi}_{\mathcal{V}}) \to \mathrm{Aut}(\Pi_{\mathcal{V}}^{\mathrm{c}}) \to \mathrm{Aut}(\Pi_{\mathcal{V}})$. Further, $\mathbb{Z}_\ell^\times$ acts by multiplication on Π_X, and that action lifts to an action on Π^{c}_X (by general group theoretical non-sense). Letting $\mathrm{Aut}^{\mathrm{c}}\big(\Pi_{\mathcal{V}}\big)$ be the image of $\mathrm{Aut}\big(\Pi^{\mathrm{c}}_{\mathcal{V}}\big)$ in $\mathrm{Aut}\big(\Pi_{\mathcal{V}}\big)$ modulo $\mathbb{Z}_\ell^\times$, we get a representation as above:

$$\rho^{\mathrm{c}}_{\mathcal{V}} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\mathrm{c}}\big(\Pi_{\mathcal{V}}\big).$$

Classical pro-ℓ abelian-by-central I/OM:
Prove that $\rho^{\mathrm{c}}_{\mathcal{V}} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\mathrm{c}}\big(\Pi_{\mathcal{V}}\big)$ *is an isomorphism in the case* $k_0 = \mathbb{Q}$ *and* $\mathcal{V} = \mathfrak{Var}_{\mathbb{Q}}$.

We will actually prove a much stronger/more precise assertion than the above classical pro-ℓ abelian-by-central I/OM, which holds over arbitrary base fields k_0 and is as follows:

Let k_0 be an *arbitrary field*, and $U_0 := \mathbb{P}^1\backslash\{0,1,\infty\} \times k_0$ be the k_0-tripod with parameter t_0 (terminology by Hoshi–Mochizuki). For every geometrically integral k_0-variety X, and a basis $\{U_i\}_i$ of Zariski (affine) open neighborhoods of the generic point η_X of X, we consider the category $\mathcal{V}_X = \{U_i\}_i \cup \{U_0\}$ with morphisms the inclusions $U_{i'} \hookrightarrow U_i$ and all the dominant k_0-morphisms $\varphi_i : U_i \to U_0$. Then $\mathrm{Aut}\big(\mathcal{V}_X\big) = 1$ is trivial, thus the automorphism group of $\Pi_{\mathcal{V}_X}$ are the systems $(\Phi_i)_i$ with $\Phi_i \in \mathrm{Aut}^{\mathrm{c}}\big(\Pi_{U_i}\big)$ and $\Phi_0 \in \mathrm{Aut}^{\mathrm{c}}\big(\Pi_{U_0}\big)$ such that for all $U_{i'} \hookrightarrow U_i$ and $\varphi_i : U_i \to U_0$, the resulting diagrams below are commutative:

$$\begin{array}{ccc} \Pi_{U_{i'}} & \xrightarrow{\Phi_{i'}} & \Pi_{U_{i'}} \\ \big\downarrow \mathrm{can} & & \big\downarrow \mathrm{can} \\ \Pi_{U_i} & \xrightarrow{\Phi_i} & \Pi_{U_i} \end{array} \qquad\qquad \begin{array}{ccc} \Pi_{U_i} & \xrightarrow{\Phi_i} & \Pi_{U_i} \\ \big\downarrow \Pi(\varphi_i) & & \big\downarrow \Pi(\varphi_i) \\ \Pi_{U_0} & \xrightarrow{\Phi_0} & \Pi_{U_0} \end{array}$$

Note that every k_0 embedding $\imath_t : k_0(t_0) \hookrightarrow k_0(X)$, $t_0 \mapsto t$, originates from some dominant k_0 morphism $\varphi_i : U_i \to U_0$ for U_i sufficiently small. Therefore, every $(\Phi_i)_i$ gives rise to an automorphism $\Phi \in \mathrm{Aut}^{\mathrm{c}}\big(\Pi_K\big)$ which satisfies: For every embedding $\imath_t : k_0(t_0) \hookrightarrow k_0(X)$, $t_0 \mapsto t \in k_0(X)$, and the corresponding projection $p_t : \Pi_K \to \Pi_{U_0}$ one has:

$$p_t \circ \Phi = \Phi_0 \circ p_t, \ \text{ thus } \ \Phi\big(\ker(p_t)\big) = \ker(p_t).$$

This suggests that a possible way to prove pro-ℓ abelian-by-central I/OM type assertions is to prove their *birational form,* which is the following stronger assertion: First, we say that $\Phi \in \mathrm{Aut}^{\mathrm{c}}\big(\Pi_K\big)$ is compatible with $p_t : \Pi_K \to \Pi_{U_0}$, if $\Phi\big(\ker(p_t)\big) = \ker(p_t)$. Second, to fix notations, let $\mathrm{Aut}^{\mathrm{c}}_{U_0}\big(\Pi_K\big) \subseteq \mathrm{Aut}^{\mathrm{c}}\big(\Pi_K\big)$ be the group of all $\Phi \in \mathrm{Aut}^{\mathrm{c}}\big(\Pi_K\big)$ which are compatible with all the projections p_t, and notice that by the discussion above, one has a canonical embedding $\mathrm{Aut}^{\mathrm{c}}\big(\Pi_{\mathcal{V}_X}\big) \hookrightarrow \mathrm{Aut}^{\mathrm{c}}_{U_0}\big(\Pi_K\big)$. The stronger assertion we have in mind is:

Pro-ℓ abelian-by-central I/OM for $k_0(X)$: *Every* $\Phi \in \mathrm{Aut}^{\mathrm{c}}_{U_0}\big(\Pi_K\big)$ *originates from* Gal_{k_0}.

Let $\mathcal{V}$ be a category of geometrically integral k_0-varieties which for every $X \in \mathcal{V}$ contains some $\mathcal{V}_X$ as defined/introduced. For X, Y in $\mathcal{V}$ and $\mathcal{V}_X = \{U_i\}_i \cup \{U_0\}$ and $\mathcal{V}_Y = \{V_j\}_j \cup \{U_0\}$ contained in $\mathcal{V}$, we say that $\mathcal{V}_X$ dominates $\mathcal{V}_Y$, denoted $\mathcal{V}_Y \prec \mathcal{V}_X$, if for every $V_j \in \mathcal{V}_Y$ there exists $U_i \in \mathcal{V}_X$ and a dominant morphism $U_i \to V_j$ in $\mathcal{V}$. We finally say that $\mathcal{V}$ is connected, if for every X, Y in $\mathcal{V}$ there exist varieties $X_0, \dots, X_{2m}$ in $\mathcal{V}$ such that $X_0 = X$, $X_{2m} = Y$ and for $i \geq 0$ one has: $\dim(X_{2i+1}) > 1$, and $\mathcal{V}_{X_{2i}}, \mathcal{V}_{X_{2i+2}} \prec \mathcal{V}_{X_{2i+1}}$.

Theorem 1.1. *Let k_0 be an arbitrary field. In the above notations the following hold:*

1) *Let X be a geometrically integral k_0-variety with* $\dim(X) > 1$. *Then the canonical homomorphisms* $\rho^{\mathrm{c}}_{\mathcal{V}_X} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\mathrm{c}}\big(\Pi_{\mathcal{V}_X}\big) \hookrightarrow \mathrm{Aut}^{\mathrm{c}}_{U_0}\big(\Pi_K\big)$ *are isomorphisms.*

2) *Let $\mathcal{V}$ be a connected category of geometrically integral k_0-varieties. Then the canonical homomorphisms $\rho^{\mathrm{c}}_{\mathcal{V}} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\mathrm{c}}\big(\Pi_{\mathcal{V}}\big)$ is an isomorphism.*

Therefore, the pro-ℓ abelian-by-central I/OM *holds for the categories $\mathcal{V}_X$ and $\mathcal{V}$ over k_0.*

As a corollary of Theorem 1.1 one has the following strengthening of the classical I/OM: For X and $\mathcal{V}_X$ as above, replacing the geometric pro-ℓ abelian-by-central fundamental group by the full geometric fundamental group $\overline{\pi}_1$, and reasoning as above, is follows that the absolute Galois group G_K equals the projective limit of all the $\overline{\pi}_1(U_i)$, $U_i \in \mathcal{V}_X$. Further, by taking projective limits, every automorphism $\overline{\Phi} = (\overline{\Phi}_i)_i \in \mathrm{Aut}_{\mathbf{G}_{\mathrm{out}}}(\overline{\pi}_{\mathcal{V}})$ gives rise to an automorphism $\overline{\Phi} \in \mathrm{Out}(G_K)$ compatible with all the projections $\overline{p}_t : G_K \to \Pi_{U_0}$, i.e., $\overline{p}_t \circ \overline{\Phi} = \Phi_0 \circ \overline{p}_t$. Finallu, let $\mathrm{Aut}_{U_0}\big(G_K\big)$ be the group of all such (outer) automorphisms of G_K.

Theorem 1.2. *Let k_0 have* $\mathrm{char}(k_0) = 0$. *In the above notations the following hold:*

1) *Let X be geometrically integral k_0-variety with* $\dim(X) > 1$. *Then the canonical homomorphisms $\rho_{\mathcal{V}_X} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}\big(\overline{\pi}_{\mathcal{V}_X}\big) \hookrightarrow \mathrm{Aut}_{U_0}\big(G_K\big)$ are isomorphisms.*
2) *Let $\mathcal{V}$ be a connected category of geometrically integral k_0-varieties. Then the canonical homomorphism $\rho_{\mathcal{V}} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}\big(\overline{\pi}_{\mathcal{V}}\big)$ is an isomorphism.*

Therefore, the full profinite I/OM *holds for the categories $\mathcal{V}_X$ and $\mathcal{V}$ over k_0.*

The main tool in the proof of Theorem 1.1 is the main result of Pop [P3] supplemented by the fact that in the hypothesis of Theorem 1.1 one can recover the *total decomposition graph* $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ of $K = k(X)$, as introduced/defined in Pop [P3], see section 2, as well as its *geometric rational quotients,* see section 3 of this manuscript. Moreover, the recipe to recover this information about $K|k$ in preserved under automorphisms $\Phi \in \mathrm{Aut}^{\mathrm{c}}_{U_0}(\Pi_K)$, see Propositions 2.5 and 3.5 below. Thus by the main result of Pop [P3] it follows that such automorphisms Φ originate from geometry, i.e., there exists $\epsilon \in \mathbb{Z}_\ell^\times$ such that $\epsilon \cdot \Phi$ is defined by some automorphism $\phi : K^{\mathrm{i}}|k \to K^{\mathrm{i}}|k$. One concludes by showing that ϕ is a power of Frobenius on $k_0(X)^{\mathrm{i}}$, thus up to Frobenius twists ϕ is the prolongation of a unique $\tau \in \mathrm{Gal}_{k_0}$ to $K^{\mathrm{i}} = k(X)^{\mathrm{i}}$ under the base change $k_0(X)^{\mathrm{i}} \hookrightarrow k(X)^{\mathrm{i}} = K^{\mathrm{i}}$. The proof of Theorem 1.2 relies heavily on the proof of Theorem 1.1, but employes some little extra ideas too, see section 5.

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2. Recovering the decomposition graphs

A) *Recalling general facts*

Let k be an algebraically closed field with $\mathrm{char}(k) \neq \ell$, and $K|k$ a function field with $d := \mathrm{td}(K|k) > 1$.

Recall that a prime divisor of $K|k$ is a discrete valuation v of K which is trivial on k and has as residue field Kv a function field with $\mathrm{td}(Kv|k) = \mathrm{td}(K|k) - 1$. Further, a valuation $\mathbf{v}$ of $K|k$ is called a prime r-divisor if $\mathbf{v}$ is the valuation theoretical composition $\mathbf{v} = v_r \circ \cdots \circ v_1$, where v_1 is a prime divisor of K, and inductively, v_{i+1} is a prime divisor of the residue function field $K\mathbf{v}_i\,|\,k$, where $\mathbf{v}_i := v_i \circ \cdots \circ v_1$. By definition, the trivial valuation will be considered a generalized prime divisor of rank zero. Finally note that $r \leq \mathrm{td}(K|k)$, and that in the above notations, one has $\mathbf{v}_r(K^\times) \cong \mathbb{Z}^r$ lexicographically.

Next recall that a valuation v of $K|k$, which is not necessarily trivial on k, is called a quasi prime divisor of $K|k$ if, fist, $vK/vk \cong \mathbb{Z}$ as abstract groups, and $Kv\,|\,kv$ is a function field with $\mathrm{td}(Kv\,|\,kv) = \mathrm{td}(K|k) - 1$, and second, no proper coarsening of v satisfies these properties. Further, a valuation $\mathbf{v}$ is called a quasi prime r-divisor of $K|k$ if $\mathbf{v}$ is a valuation theoretical composition $\mathbf{v} = v_r \circ \cdots \circ v_1$, where v_1 is a quasi prime divisor of $K|k$, and inductively, v_{i+1} is a quasi prime divisor of the residue function field $K\mathbf{v}_i\,|\,k\mathbf{v}_i$, where $\mathbf{v}_i := v_i \circ \cdots \circ v_1$.

Note that the quasi prime divisors of $K|k$ are precisely the quasi prime 1-divisors of $K|k$. Further, the prime r-divisors of $K|k$ are precisely the quasi prime r-divisors of $K|k$ which are trivial on k.

- The total (quasi) prime divisor graph (See Pop [P1] for more on quasi prime divisors.)

We define the total prime divisor graph $\mathcal{D}_K^{\mathrm{tot}}$ of $K|k$ to be the half-oriented graph as follows:

a) The vertices of $\mathcal{D}_K^{\mathrm{tot}}$ are the residue fields $K\mathbf{v}$ of all the generalized prime divisors $\mathbf{v}$ of $K|k$ viewed as distinct function fields.

b) For $\mathbf{v} = v_r \circ \cdots \circ v_1$ and $\mathbf{w} = w_s \circ \cdots \circ w_1$, the edges from $K\mathbf{v}$ to $K\mathbf{w}$ are as follows:

i) If $\mathbf{v} = \mathbf{w}$, then the trivial valuation $\mathbf{v}/\mathbf{w} = \mathbf{w}/\mathbf{v}$ of $K\mathbf{v} = K\mathbf{w}$ is the only edge from $K\mathbf{v} = K\mathbf{w}$ to itself; and it is by definition a non-oriented edge.

ii) If $K\mathbf{v} \neq K\mathbf{w}$, then the set of edges from $K\mathbf{v}$ to $K\mathbf{w}$ is non-empty iff $s = r+1$ and $v_i = w_i$ for $1 \leq i \leq r$; and if so, then $w_s = \mathbf{w}/\mathbf{v}$ is the only edge from $K\mathbf{v}$ to $K\mathbf{w}$, and it is by definition an oriented edge.

We define the total quasi prime divisor graph $\mathcal{Q}_K^{\mathrm{tot}}$ of $K|k$ in the same way as the total prime divisor graph was defined, but considering as vertices all the generalized quasi prime divisors instead of the generalized prime divisors of $K|k$. Notice that $\mathcal{D}_K^{\mathrm{tot}} \subset \mathcal{Q}_K^{\mathrm{tot}}$ is a full subgraph. Finally, the total (quasi) divisor graph of $K|k$ has the following functorial properties:

1) *Embeddings.* Let $\imath : L|l \hookrightarrow K|k$ be an embedding of function fields which maps l isomorphically onto k. Then the canonical restriction map of valuations $\mathrm{Val}_K \to \mathrm{Val}_L$, $v \mapsto v|_L$, gives rise to a surjective morphism of the total (quasi) prime divisor graphs which we denote $\varphi_\imath : \mathcal{D}_K^{\mathrm{tot}} \to \mathcal{D}_L^{\mathrm{tot}}$ and $\varphi_\imath : \mathcal{Q}_K^{\mathrm{tot}} \to \mathcal{Q}_L^{\mathrm{tot}}$.

2) *Restrictions.* Given a generalized (quasi) prime divisor $\mathbf{v}$ of $K|k$, let $\mathcal{D}_{\mathbf{v}}^{\mathrm{tot}} \subset \mathcal{Q}_{\mathbf{v}}^{\mathrm{tot}}$ be the set of all the generalized prime, respectively quasi prime, divisors $\mathbf{w}$ of $K|k$ with $\mathbf{v} \leq \mathbf{w}$. Then the map $\mathcal{Q}_{\mathbf{v}}^{\mathrm{tot}} \to \mathrm{Val}_{K\mathbf{v}}$, $\mathbf{w} \mapsto \mathbf{w}/\mathbf{v}$, is an isomorphism of $\mathcal{Q}_{\mathbf{v}}^{\mathrm{tot}}$ onto the total quasi prime divisor graph of $K\mathbf{v}\,|\,k\mathbf{v}$. And if $\mathbf{v}$ is a prime divisor, the above map also defines an isomorphism form $\mathcal{D}_{\mathbf{v}}^{\mathrm{tot}}$ onto the total prime divisor graph of $K\mathbf{v}|k$.

- The total (quasi) decomposition graph (See Pop [P3], section 3, for more details.)

Let $K|k$ be as above. For every valuation v of K we denote by $T_v \subseteq Z_v$ the inertia/decomposition groups of v in Π_K. Recall that for every prime divisor v of $K|k$ one has $T_v \cong \mathbb{Z}_\ell$, and for every prime r-divisor $\mathbf{v}$ one has $T_{\mathbf{v}} \cong \mathbb{Z}_\ell^r$. Further, for generalized prime divisors $\mathbf{v}$ and $\mathbf{w}$ one has: $Z_{\mathbf{v}} \cap Z_{\mathbf{w}} \neq 1$ if and only if $\mathbf{v}, \mathbf{w}$ are not independent as valuations, i.e., $\mathcal{O} := \mathcal{O}_{\mathbf{v}}\,\mathcal{O}_{\mathbf{w}} \neq K$; and if so, then $\mathcal{O}$ is the valuation ring of a generalized prime divisor $\mathbf{u}$ of $K|k$ which turns out to be the unique generalized prime divisor with $T_{\mathbf{u}} = T_{\mathbf{v}} \cap T_{\mathbf{w}}$, and also the unique generalized prime divisor of $K|k$ maximal with the property $Z_{\mathbf{v}}, Z_{\mathbf{w}} \subseteq Z_{\mathbf{u}}$.

In particular, $\mathbf{v} = \mathbf{w}$ iff $T_{\mathbf{v}} = T_{\mathbf{w}}$ iff $Z_{\mathbf{v}} = Z_{\mathbf{w}}$. Further, $\mathbf{v} < \mathbf{w}$ iff $T_{\mathbf{v}} \subset T_{\mathbf{w}}$ strictly iff $Z_{\mathbf{v}} \supset Z_{\mathbf{w}}$ strictly, and $T_{\mathbf{w}}/T_{\mathbf{v}} \cong \mathbb{Z}_\ell^{s-r}$ if $\mathbf{v}$ is a prime r-divisor, and $\mathbf{w}$ is a prime s-divisor.

We conclude that the partial ordering of the set of all the generalized prime divisors $\mathbf{v}$ of $K|k$ *is encoded in the set* of their inertia/decomposition groups $T_{\mathbf{v}} \subseteq Z_{\mathbf{v}}$. In particular, the existence of the trivial, respectively nontrivial, edge from $K\mathbf{v}$ to $K\mathbf{w}$ in $\mathcal{D}_K^{\rm tot}$ is equivalent to $T_{\mathbf{v}} = T_{\mathbf{w}}$, respectively to $T_{\mathbf{v}} \subset T_{\mathbf{w}}$ and $T_{\mathbf{w}}/T_{\mathbf{v}} \cong \mathbb{Z}_\ell$.

Via the Galois correspondence and the functorial properties of the Hilbert decomposition theory for valuations, we attach to the total prime divisor graph $\mathcal{D}_K^{\rm tot}$ of $K|k$ a graph $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ whose vertices and edges are in bijection with those of $\mathcal{D}_K^{\rm tot}$, as follows:

a) The vertices of $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ are the pro-ℓ groups $\Pi_{K\mathbf{v}}$, viewed as distinct pro-ℓ groups.

b) If the edge from $K\mathbf{v}$ to $K\mathbf{w}$ exists, the corresponding edge from $\Pi_{K\mathbf{v}}$ to $\Pi_{K\mathbf{w}}$ is endowed with the pair of groups $T_{\mathbf{w}/\mathbf{v}} \subseteq Z_{\mathbf{w}/\mathbf{v}}$ viewed as subgroups of $\Pi_{K\mathbf{v}}$, thus $\Pi_{K\mathbf{w}} = Z_{\mathbf{w}/\mathbf{v}}/T_{\mathbf{w}/\mathbf{v}}$.

The graph $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ will be called the total decomposition graph of $K|k$.

In a similar way, we attach to $\mathcal{Q}_K^{\rm tot}$ the total quasi decomposition graph of $K|k$, which we denote $\mathcal{G}_{\mathcal{Q}_K^{\rm tot}}$. Clearly, $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ is a full subgraph of $\mathcal{G}_{\mathcal{Q}_K^{\rm tot}}$.

The functorial properties of the graphs of (quasi) prime divisors translate in the following functorial properties of the total (quasi) decomposition graphs:

1) *Embeddings.* Let $\imath : L|l \hookrightarrow K|k$ be an embedding of function fields which maps l isomorphically onto k. Then the canonical projection homomorphism $\Phi_\imath : \Pi_K \to \Pi_L$ is an open homomorphism, and moreover, for every generalized (quasi) prime divisor $\mathbf{v}$ of $K|k$ and its restriction $\mathbf{v}_L$ to L one has: $\Phi_\imath(Z_{\mathbf{v}}) \subseteq Z_{\mathbf{v}_L}$ is an open subgroup, and $\Phi_\imath(T_{\mathbf{v}}) \subseteq T_{\mathbf{v}_L}$ satisfies: $\Phi_\imath(T_{\mathbf{v}}) = 1$ iff $\mathbf{v}_L$ has divisible value group, e.g., $\mathbf{v}_L$ is the trivial valuation. Therefore, $\Phi_\imath$ gives rise to morphisms of total (quasi) decomposition graphs

$$\Phi_\imath : \mathcal{G}_{\mathcal{D}_K^{\rm tot}} \to \mathcal{G}_{\mathcal{D}_L^{\rm tot}}, \quad \Phi_\imath : \mathcal{G}_{\mathcal{Q}_K^{\rm tot}} \to \mathcal{G}_{\mathcal{Q}_L^{\rm tot}}.$$

2) *Restrictions.* Given a generalized (quasi) prime divisor $\mathbf{v}$ of $K|k$, let $pr_{\mathbf{v}} : Z_{\mathbf{v}} \to \Pi_{K\mathbf{v}}$ be the canonical projection. Then for every $\mathbf{w} \geq \mathbf{v}$ we have: $T_{\mathbf{w}} \subseteq Z_{\mathbf{w}}$ are mapped onto $T_{\mathbf{w}/\mathbf{v}} \subseteq Z_{\mathbf{w}/\mathbf{v}}$. Therefore, the total (quasi) decomposition graph of $K\mathbf{v}|k\mathbf{v}$ can be recovered from the one of $K|k$ in a canonical way via $pr_{\mathbf{v}} : Z_{\mathbf{v}} \to \Pi_{K\mathbf{v}}$.

B) *Recovering the divisorial subgroups*

Recall that in the context of Theorem 1.1, we say that $\Phi \in \mathrm{Aut}^{\rm c}(\Pi_K)$ is compatible with the projections if for every k_0-embedding $\imath_t : k_0(t) \hookrightarrow k_0(X)$, $t_0 \mapsto t$, and the corresponding projection $p_t : \Pi_K \to \Pi_{U_0}$ one has: $\Phi \circ p_t = p_t \circ \Phi_0$ for some automorphism $\Phi_0 \in \mathrm{Aut}(\Pi_{U_0})$, where $U_0 := \mathbb{P}^1_k \backslash \{0, 1, \infty\}$ is the tripod.

Proposition 2.1. *In the above context and notations, let $K := k(X)$. The following hold:*

1) *A quasi divisorial subgroup $T_{\mathbf{v}} \subset Z_{\mathbf{v}}$ is divisorial iff for all projections p_t one has: If $p_t(T_{\mathbf{v}})$ is trivial, then $p_t(Z_{\mathbf{v}})$ is either trivial or open in Π_{U_0}.*

2) *If $\Phi \in \mathrm{Aut}^{\rm c}(\Pi_K)$ is compatible with all the projections p_t, then Φ maps divisorial subgroups of Π_K isomorphically onto divisorial subgroups of Π_K.*

Proof. We begin by recalling the following basic facts: Let $N \subset M$ be an extension of function fields over algebraically closed base fields $l \subset k$. For a valuation v of M, let

$w := v|_N$ be its restriction to N. Recall that the Abhyankar inequality for v asserts that $\mathrm{td}(Kv|kv) + \dim_{\mathbb{Q}}(\mathbb{Q}\otimes vM)/vk \leq \mathrm{td}(M|k)$. We say that v has no transcendence defect, if the Abhyankar inequality for v is actually an equality. Recall that quasi prime r-divisors have no transcendence defect. The following hold, see e.g., KUHLMANN [Ku].

a) The valuation v is nontrivial iff the decomposition group Z_v of v in Π_M has infinite index in Π_M.
b) If v has no transcendence defect, then w has no transcendence defect.
c) The canonical projection $pr : \Pi_M \to \Pi_N$ has open image, and one has: $pr(Z_v) \subseteq Z_w$ and $pr(T_v) \subseteq T_w$ are open subgroups too.

To 1): In the context of Theorem 1.1, and notations as above, for every quasi prime divisor $\mathbf{v}$ of $K|k$, and a function subfield $L|k \hookrightarrow K|k$, let $\mathbf{w} := \mathbf{v}|_L$. Then $\mathbf{w}|_k = \mathbf{v}|_k$, thus $\mathbf{v}$ is non-trivial on k iff $\mathbf{w}$ is so. For every k_0-embedding $\imath_t : k_0(t_0) \to k_0(X) \subset K$, $t_0 \mapsto t$, one has $\mathrm{im}(\imath_t) = k_0(t)$, and further the following hold:

First suppose that $\mathbf{v}$ is a prime divisor of $K|k$ and $\mathbf{w}$ be its restriction to $k(t)$. Then by the discussion above we have: If $\mathbf{w}$ is trivial, then $Z_{\mathbf{w}} = \Pi_{k(t)}$, and the image of $Z_{\mathbf{v}}$ in $\Pi_{k(t)}$ is open. Hence $p_t(Z_{\mathbf{v}}) \subseteq \Pi_{U_0}$ is open. And if $\mathbf{w}$ is non-trivial, then $T_{\mathbf{w}} = Z_{\mathbf{w}}$, thus the images of $Z_{\mathbf{v}}$ and $T_{\mathbf{v}}$ in $\Pi_{k(t)}$ are equal –and equal to some open subgroup of $T_{\mathbf{w}} \cong \mathbb{Z}_\ell$. Hence $p_t(T_{\mathbf{v}})$ is trivial iff $p_t(Z_{\mathbf{v}})$ is trivial.

Conversely, let $\mathbf{v}$ be a quasi prime divisor such that for all p_t one has: $p_t(T_{\mathbf{v}}) = 1$ implies $p_t(Z_{\mathbf{v}})$ either open in Π_{U_0} or trivial. By contradiction, suppose that $\mathbf{v}$ is not a prime divisor of $K|k$, or equivalently, $v_k := \mathbf{v}|_k$ is non-trivial. Let $\mathcal{T} = (t_1, \dots, t_d)$ be a transcendence basis of $k_0(X)|k_0$, such that $\mathbf{v}(t_d)$ has a non-trivial image in $\mathbf{v}(K^\times)/\mathbf{v}(k^\times)$, and $t_1, \dots, t_{d-1}$ are $\mathbf{v}$-units and their residues $K\mathbf{v}$ are algebraically independent over $k\mathbf{v}$. Now choose $a \in k_0$ with $v_k(a) < 0$, and define $\imath_t : k(t_0) \to k_0(X) \subseteq K$ by $t_0 \mapsto t := at_1$. Then denoting by $L_0 \subset k(X)$ the relative algebraic closure of $k(t_1, \dots, t_{d-1})$ in K, and $L := L_0 k$, one has: $\mathbf{w} := \mathbf{v}|_L$ is a constant reduction of L_0, hence of L, and $T_{\mathbf{w}} = 1$. Therefore, since $k(t) \subset L$ we have $p_t(T_{\mathbf{v}}) = p(T_{\mathbf{w}}) = 1$. On the other hand, since $t - 1 = t(1 - 1/t)$ and $\mathbf{w}(t) = \mathbf{w}(t_1) + \mathbf{w}(a) < 0$, it follows that $1 - 1/t$ is a $\mathbf{v}$-principal unit. But then $(1 - 1/t)^{1/\ell^e}$ lies in the fixed field of $Z_{\mathbf{v}}$ for all $e > 0$, wheres t^{1/ℓ^e} does not lie in the decomposition field of $\mathbf{v}$ for e sufficiently large. Correspondingly, $(1 - 1/t_0)^{1/\ell^e}$ are fixed by the image of $Z_{\mathbf{v}} \to \Pi_{U_0}$ for all e, whereas t^{1/ℓ^e} is not fixed under that image, provided e sufficiently large. We thus conclude that $p_t(Z_{\mathbf{v}}) \subset \Pi_{U_0}$ is non-trivial and not open, in spite of the fact that $p_t(T_{\mathbf{v}}) = 1$, contradiction!

To 2): By assertion 1) it follows that if $T_v \subset Z_v$ is a divisorial subgroup of Π_K, the image $\Phi(T_v) \subset \Phi(Z_v)$ is a divisorial subgroup of Π_K too. Thus Φ maps divisorial subgroups of Π_K isomorphically onto divisorial subgroups of Π_K. $\square$

C) *Recovering the total decomposition graph*

We begin by recalling some facts from POP [P4], especially Proposition 3.5 of loc.cit.

Definition 2.2.

1) A flag of generalized (quasi) prime divisors for $K|k$ is any sequence $\mathbf{v}_1 \leq \cdots \leq \mathbf{v}_r$ such that each $\mathbf{v}_m$ is a (quasi) prime m-divisor of $K|k$ for $1 \leq m \leq r$.

2) A flag of generalized (quasi) divisorial subgroups of Π_K consists of the sequences of the decomposition/inertia groups $Z_{\mathbf{v}_1} \geq \cdots \geq Z_{\mathbf{v}_r}$, $T_{\mathbf{v}_1} \leq \cdots \leq T_{\mathbf{v}_r}$ defined by a flag of generalized (quasi) prime divisors $\mathbf{v}_1 \leq \cdots \leq \mathbf{v}_r$.

The main result of this subsection is the following:

Proposition 2.3. *In the above context and notations, the following hold:*

1) *The flags of generalized divisorial subgroups of Π_K can be recovered from $\mathcal{G}_{\mathcal{Q}_K^{\rm tot}}$ and Π_K endowed with all the projections p_t, as being those flags of generalized quasi divisorial subgroups $Z_1 \geq \cdots \geq Z_r$, $T_1 \leq \cdots \leq T_r$, which satisfy $T_r \subset \mathbf{Inr.tm}_k(K)$.*

- *Thus $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ can be recovered from $\mathcal{G}_{\mathcal{Q}_K^{\rm tot}}$ and Π_K endowed with all the projections p_t.*

2) *Let $1 \to T_{\mathbf{v}} \to Z_{\mathbf{v}} \xrightarrow{\pi} \Pi_{K\mathbf{v}} \to 1$ be canonical exact sequence of generalized prime divisor $\mathbf{v}$ of $K|k$. Then the generalized divisorial subgroups of $\Pi_{K\mathbf{v}}$ are the images $\pi(T) \subseteq \pi(Z)$ of the generalized divisorial subgroups $T \subseteq Z$ in Π_K which satisfy $Z \subseteq Z_{\mathbf{v}}$ and $T \supseteq T_{\mathbf{v}}$.*

3) *Every $\Phi \in \mathrm{Aut}^{\rm c}_{U_0}(\Pi_K)$ maps the set of all the flags of generalized divisorial subgroups of Π_K bijectively onto itself, and therefore defines an automorphism $\Phi : \mathcal{G}_{\mathcal{D}_K^{\rm tot}} \to \mathcal{G}_{\mathcal{D}_K^{\rm tot}}$.*

Proof. Recall that we denote by $\mathbf{Inr.tm.div}(K) \subset \mathbf{Inr.tm}_k(K)$ the set of all the inertia elements at prime divisors of $K|k$, respectively the inertia all all the k-valuations of $K|k$, i.e., valuations which are trivial on k. The proof Proposition 2.3 above is virtually identical with the one of Pop [P4], Proposition 3.5, but using the sets $\mathbf{Inr.tm.div}(K) \subset \mathbf{Inr.tm}_k(K)$ instead of $\mathbf{Inr.tm.q.div}(K) \subset \mathbf{Inr.tm}(K)$, i.e., the set of all the tame quasi divisorial inertia, respectively the set of all tame inertia elements at all the valuations of K. Therefore, we will not repeat those arguments here, but only do here the following:

a) Indicate below how to recover $\mathbf{Inr.tm.div}(K) \subset \mathbf{Inr.tm}_k(K)$, and show that the recipe to do so is invariant under $\Phi \in \mathrm{Aut}^{\rm c}_{U_0}(\Pi_K)$.

b) Recall the functoriality behavior of $\mathbf{Inr.tm}_k(K)$ necessary to make the proof from Pop [P4], Proposition 3.5, work in our situation.

To the first point (a), recall that Proposition 2.1 above gives a recipe to recover the divisorial subgroups $T_v \subset Z_v$ of Π_K among all the quasi divisorial subgroups $T_{\mathbf{v}} \subset Z_{\mathbf{v}}$. Moreover, that recipe is invariant under $\Phi \in \mathrm{Aut}^{\rm c}_{U_0}(\Pi_K)$, i.e., every such Φ maps divisorial subgroups $T_v \subset Z_v$ of Π_K isomorphically onto divisorial subgroups $T_w \subset Z_w$ of Π_K. Therefore, Φ maps $\mathbf{Inr.tm.div}(K) := \cup_v T_v$ (all prime divisors v) homeomorphically onto itself. On the other hand, by Pop [P2], Introduction, Theorem B, the set $\mathbf{Inr.tm.div}(K)$ is dense in $\mathbf{Inr.tm}_k(K)$. Hence finally, every $\Phi \in \mathrm{Aut}^{\rm c}_{U_0}(\Pi_K)$ maps $\mathbf{Inr.tm.div}(K) \subset \mathbf{Inr.tm}_k(K)$ homeomorphically onto themselves, respectively.

To the second point (b), we recall the following general fact: Let $K|k$ be a function field with $\mathrm{td}(K|k) > 1$ and k algebraically closed of characteristic $\neq \ell$, and $\Pi^{\rm c}_K \to \Pi_K$ be the canonical projection. Let $T_{\mathbf{v}} \subset Z_{\mathbf{v}}$ be a quasi divisorial r-divisor subgroup of $K|k$, and $Z_{\mathbf{v}_1} \geq \cdots \geq Z_{\mathbf{v}_r} = Z_{\mathbf{v}}$, $T_{\mathbf{v}_1} \leq \cdots \leq T_{\mathbf{v}_r} = T_{\mathbf{v}}$ be the corresponding flag of quasi divisorial subgroups. Then inductively on r it follows that $\mathbf{v}$ is a prime r-divisor of $K|k$ if and only if $T_{\mathbf{v}} \subset \mathbf{Inr.tm}_k(K)$. Thus using the fact that $\mathbf{Inr.tm}_k(K) \subset \mathbf{Inr.tm}(K)$ can be recovered in our context by a group theoretical recipe as indicated above, it follows that in our context one can distinguish the flags of generalized prime divisors among the flags of generalized quasi prime divisors. Since the latter can be recovered by [P4], Proposition 3.5, it follows that the former can be recovered in our context. Further, let $\mathbf{v}$ be a generalized prime divisor of $K|k$, and $1 \to T_{\mathbf{v}} \to Z_{\mathbf{v}} \xrightarrow{\pi} \Pi_{K\mathbf{v}} \to 1$ be its canonical exact sequence. Then the generalized (quasi) divisorial subgroups of $\Pi_{K\mathbf{v}}$ are precisely the images $\pi(T), \pi(Z)$ of the generalized (quasi) divisorial subgroups T, Z in Π_K which satisfy $Z \subseteq Z_v$ and $T \supseteq T_v$. Finally, every $\Phi \in \mathrm{Aut}^{\mathrm{c}}(\Pi_K)$ maps the set of flags of generalized quasi divisorial subgroups of Π_K bijectively onto itself, thus defines an automorphism $\Phi : \mathcal{G}_{\mathcal{Q}_K^{\mathrm{tot}}} \to \mathcal{G}_{\mathcal{Q}_K^{\mathrm{tot}}}$. $\square$

3. Recovering the rational quotients

A) *Recovering* 1*-dimensional* k_0*-rational quotients*

In the context of Theorem 1.1, let $p_t : \Pi_K \to \Pi_{U_0}$ be a projection defined by an embedding $\imath_t : k_0(t_0) \to k_0(X) \subset K$, $t_0 \mapsto t$. Reasoning as at the beginning of the proof of Proposition 2.1 above, for v a prime divisor of $K|k$ one has: $p_t(Z_v) \subset \Pi_{U_0}$ is open iff v is trivial on $k(t)$. Let $\kappa_t \subset K$ be the relative algebraic closure of $k(t)$ in K. Taking into account that v is trivial on $k(t)$ iff v is trivial on κ_t, the following are equivalent:

i) $p_t(Z_v) = p_t(T_v)$.

ii) v is non-trivial on κ_t.

In particular, the set $\mathcal{V}_{\kappa_t} = \mathcal{V}_{p_t}$ of all the prime divisors v of $K|k$ which are non-trivial on κ_t, or equivalently that $p_t(Z_v) = p_t(T_v)$, is detected by the projection p_t and can be recovered from $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ endowed with $p_t : \Pi_K \to \Pi_{U_0}$.

We next give a group theoretical recipe to recover the projection $\Phi_{\kappa_t} : \Pi_K \to \Pi_{\kappa_t}$ from $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ endowed with all the divisorial subgroups $T_v \subset Z_v$, $v \in \mathcal{V}_{\kappa_t}$, which means *mutatis mutandis,* from $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ endowed with $p_t : \Pi_K \to \Pi_{U_0}$.

We will do this in a more general setting as follows: Let $u \in K$ be any non-constant function, and let $\kappa_u \subset K$ be the relative algebraic closure of $k(u)$ in K, and let $\mathcal{V}_{\kappa_u}$ be the set of all the prime divisors of $K|k$ which are non-trivial on κ_u. The embedding $\imath_u : \kappa_u|k \hookrightarrow K|k$ gives rise to the group theoretical quotient $\Phi_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$ of Π_K, which we call the (group theoretical) geometric 1-dimensional quotient of Π_K defined by $\kappa_u|k \hookrightarrow K|k$, for short a (group theoretical) 1-dimensional quotient of Π_K, if $\kappa_u|k$ is irrelevant for the context.

We first notice that taking ℓ-adic duals, to give $\Phi_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$ is equivalent to giving the image of the Kummer morphism $\hat{\phi}_{\kappa_u}$ of Φ_{κ_u}, which is the embedding of the ℓ-adic duals:

$$\hat{\phi}_{\kappa_u} : \widehat{\kappa}_u = \mathrm{Hom}_{\mathrm{cont}}(\Pi_{\kappa_u}, \mathbb{Z}_\ell) \hookrightarrow \mathrm{Hom}_{\mathrm{cont}}(\Pi_K, \mathbb{Z}_\ell) = \widehat{K}$$

Recall that for every prime divisor v of $K|k$, we denote by $\jmath^v : \widehat{K} \to \mathrm{Hom}_{\mathrm{cont}}(T_v, \mathbb{Z}_\ell)$ the ℓ-adic dual of the embedding $T_v \hookrightarrow \Pi_K$. And notice that the ℓ-adic completion $\widehat{U}_v$ of the v-units $U_v \subset K^\times$ is precisely $\widehat{U}_v = \ker(\jmath^v)$. We further let $\jmath_v : \widehat{U}_v \to \widehat{Kv}$ be the ℓ-adic dual of $\Pi_K/T_v \to \Pi_{Kv}$ and notice that $\ker(\jmath_v) = \widehat{U_v^1}$ is precisely the ℓ-adic completion of the group of principal v-units $U_v^1 \subset U_v$ of v. For a geometric set of prime divisors D for $K|k$, we define

$$\widehat{K}_{\mathrm{fin}} = \{\boldsymbol{x} \in \widehat{K} \mid \jmath^v(\boldsymbol{x}) = 0 \text{ for all but finitely many } v \in D\}$$

and notice that $\widehat{K}_{\mathrm{fin}}$ does not depend on the geometric set D of $K|k$, thus $\widehat{K}_{\mathrm{fin}}$ is a birational invariant of $K|k$. Indeed, for any two geometric sets of prime divisors D, E, their intersection $D \cap E$ is geometric too, and both $D \backslash D \cap E$ and $E \backslash D \cap E$ are finite, etc. In particular, for every geometric set of prime divisors D for $K|k$ and every $\boldsymbol{x} \in \widehat{K}_{\mathrm{fin}}$ there are only finitely may $v \in D$ such that $\jmath^v(\boldsymbol{x}) \neq 0$. Further, if $\jmath : K^\times \to \widehat{K}$ is the ℓ-adic completion morphism, then $\jmath_K(K^\times) \subset \widehat{K}_{\rm fin}$, hence $\widehat{K}_{\rm fin}$ is ℓ-adically dense in $\widehat{K}$. We finally notice that the same is true correspondingly for the function subfields $\kappa_u|k$, and that under the embedding $\widehat{\kappa}_u \hookrightarrow \widehat{K}$ one has $\widehat{\kappa}_{u,\rm fin} = \widehat{\kappa}_u \cap \widehat{K}_{\rm fin}$. And since $\widehat{K}_{\rm fin} \subset \widehat{K}$ and $\widehat{\kappa}_{u,\rm fin} \subset \widehat{\kappa}_u$ are ℓ-adically dense subgroups and $\widehat{\kappa}_u \hookrightarrow \widehat{K}$ is a topological embedding, in order to detect the image of $\widehat{\kappa}_u \hookrightarrow \widehat{K}$, it is enough to detect the image of $\widehat{\kappa}_{u,\rm fin} \hookrightarrow \widehat{K}_{\rm fin}$ inside $\widehat{K}$.

Proposition 3.1. *In the above notations, the following hold:*

1) $\widehat{\kappa}_{u,\rm fin} = \{\boldsymbol{x} \in \widehat{K}_{\rm fin} \mid \forall\ v \in \mathcal{V}_{\kappa_u}$ *one has: If* $\boldsymbol{x} \in \widehat{U}_v$*, then* $\jmath_v(\boldsymbol{x}) = 1\}$.
2) *If* $p_t : \Pi_K \to \Pi_{U_0}$ *is the projection defined by some* $\imath_t : k_0(t) \to k_0(X) \subset K$*, then the quotient* $\Phi_{\kappa_t} : \Pi_K \to \Pi_{\kappa_t}$ *can be recovered from* $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ *and* Π_K *endowed with* p_t*.*
3) *Moreover, every* $\Phi \in \mathrm{Aut}^{\rm c}_{U_0}(\Pi_K)$ *maps* $\ker(\Phi_{\kappa_t})$ *onto itself for all* $t \in k_0(X)$*.*

Proof. To 1): The direct inclusion "$\subseteq$" proceed as follows: Recall that $v \in \mathcal{V}_{\kappa_u}$ iff v is non-trivial on κ_u. Hence if $v \in \mathcal{V}_{\kappa_u}$, taking into account that $\kappa_u|k$ is a function field in one variable, it follows that the residue field is $\kappa_u v = k$, thus $(U_v \cap \kappa_u)v = k$, and therefore $\jmath_v$ is trivial on the ℓ-adic completion of $U_v \cap \kappa_u$, which is $\widehat{U}_v \cap \widehat{\kappa}_u$.

For the reverse inclusion "$\supseteq$" one has: Let $\boldsymbol{x} \in \widehat{K}_{\rm fin} \backslash \widehat{\kappa}_{u,\rm fin}$ and let Δ be the $\mathbb{Z}_\ell$-submodule generated by $\boldsymbol{x}$, hence $\Delta \subset \widehat{K}_{\rm fin}$. Since one has $\widehat{K}_{\rm fin}/\widehat{\kappa}_{u,\rm fin} \hookrightarrow \widehat{K}/\widehat{\kappa}_u$, and the latter $\mathbb{Z}_\ell$-module is torsion free, it follows that $\widehat{K}_{\rm fin}/\widehat{\kappa}_{u,\rm fin}$ is torsion free too, thus $\Delta \cap \widehat{\kappa}_{u,\rm fin}$ is trivial. But then by Pop [P3], Proposition 40, 3), it follows that for "many" valuations $v \in \mathcal{V}_{\kappa_u}$ one has: $\Delta \subset \widehat{U}_v$ and $\jmath_v$ maps Δ injectively into $\widehat{Kv}$ and therefore, $\jmath_v(\boldsymbol{x}) \neq 1$, etc.

To 2) and 3): Both follow immediately from assertion 1) and the fact that p_t defines $\mathcal{V}_{\kappa_t}$ uniquely, thus in particular, Φ maps the set $\{T_v \subset Z_v \mid v \in \mathcal{V}_{\kappa_t}\}$ bijectively onto itself. $\square$

B) *Recovering the rational quotients*

For readers sake, we first recall a few basic facts from Pop [P3] as systematized in [P4]. Let $K|k$ be a function field with k algebraically closed, and D be a set of prime divisors of $K|k$. We denote by $T_D \subseteq \Pi_K$ the closed subgroup generated by all the T_v, $v \in D$, and set $\Pi_{1,D} := \Pi_K/T_D$, and call it the fundamental group of D. In the case $D = D_{K|k}$ is the set of all the prime divisors of $K|k$, we denote $\Pi_{1,K} := \Pi_{1,D_{K|k}}$, and call it the birational fundamental group of $K|k$. We say that D is geometric, if there exists a normal model $X \to k$ of $K|k$ such that $D = D_X$ is the set of Weil prime divisors of X. (Note that if D is geometric, we can always choose X with $D = D_X$ and X quasi projective.) For $D = D_X$ geometric, there exist canonical projections $\Pi_{1,D} \to \Pi_1(X)$ and $\Pi_{1,D} \to \Pi_{1,K}$. Further notice that the Weil divisor group, respectively divisor class group, $\mathrm{Div}(D) := \mathrm{Div}(X)$, $\mathbf{Cl}(D) := \mathbf{Cl}(X)$, depend on D only and not on the specific X. Recall that by Pop [P3], section 3, especially the discussion starting at Remarks 19, we have the following: Let $\jmath^v : \widehat{K} \to \mathrm{Hom}(T_v, \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell$ be the ℓ-adic dual of $T_v \hookrightarrow \Pi_K$, and setting $\jmath^D := \oplus_v \jmath^v$, consider the resulting canonical exact sequence of ℓ-adically complete $\mathbb{Z}_\ell$-modules:

$$1 \to \widehat{U}_D \longrightarrow \widehat{K} \xrightarrow{\widehat{\jmath}^D} \widehat{\mathrm{Div}}(D) \longrightarrow \widehat{\mathbf{Cl}}(D) \to 0,$$

where $\widehat{\jmath}^D$ is the ℓ-adic completion of $\jmath^D$ and the other morphisms are canonical. It turns out that $\widehat{U}_D$ is the ℓ-adic dual of $\Pi_{1,D}$, and that the isomorphy type of $\widehat{U}_D$ and $\widehat{\mathbf{Cl}}(D)$ can be recovered from Π_K endowed with $(T_v)_{v\in D}$ only. The geometric set D is called complete regular like if $\Pi_{1,K} = \Pi_{1,D}$ and for all geometric sets $\tilde{D} \subseteq D$ one has that $\widehat{\mathbf{Cl}}(\tilde{D}) \cong \widehat{\mathbf{Cl}}(D) \oplus \mathbb{Z}_\ell^r$ with $r := |\tilde{D}\backslash D|$. Further, a subgraph $\mathcal{D}_K \subset \mathcal{D}_K^{\rm tot}$ is called a geometric complete regular like decomposition graph for $K|k$, if it satisfies the following:

i) For every vertex $\mathbf{v}$ of $\mathcal{D}_K$, the set $D_{\mathbf{v}}$ of non-trivial edges of $\mathcal{D}_K$ originating at $K\mathbf{v}$ is complete regular like.

ii) For every vertex $K\mathbf{v}$, the trivial edge at $\mathbf{v}$ belongs to $\mathcal{D}_K$, and every maximal path of oriented edges has length $d = \mathrm{td}(K|k)$.

Recall that by Pop [P3], Proposition 22, there exists a group theoretical recipe by which one can decide whether a decomposition graph $\mathcal{G}_{\mathcal{D}_K} \subset \mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ is complete regular like. Let $\mathcal{G}_{\mathcal{D}_K}$ be complete regular like, and D be the set of its 1-vertices, and $\mathrm{Div}^0(D) \subset \mathrm{Div}'(D)$ be the preimages of the maximal divisible, respectively ℓ-divisible, subgroups $\mathbf{Cl}^0(D) \subseteq \mathbf{Cl}'(D)$ of $\mathbf{Cl}(D)$. Since $\mathbf{Cl}'(D)/\mathbf{Cl}^0(D)$ is a prime to ℓ torsion group, one has $\mathrm{Div}^0(D)_{(\ell)} = \mathrm{Div}'(D)_{(\ell)}$.[1] Further, this subgroup of $\mathrm{Div}(D)_{(\ell)}$ is a birational invariant of $K|k$ which does not depend on the concrete complete regular like decomposition graph $\mathcal{G}_{\mathcal{D}_K}$, and therefore, the preimage $\mathcal{L}_{\mathcal{D}_K} \subset \widehat{K}$ of $\mathrm{Div}'(D)_{(\ell)}$ in $\widehat{K}$ –which depends on $K|k$ only, and not on $\mathcal{D}_K$. We denote $\widehat{U}_K := \widehat{U}_D$ and call $\mathcal{L}_K := \mathcal{L}_{\mathcal{D}_K}$ the canonical $\widehat{U}_K$-divisorial lattice of $\mathcal{G}_{\mathcal{D}_K}$. On the other hand, by [P3] Proposition 23, $\mathcal{L}_{\mathcal{D}_K} \subset \widehat{K}$ can be recovered from $\mathcal{G}_{\mathcal{D}_K} \subset \mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ up to multiplication by ℓ-adic units $\epsilon \in \mathbb{Z}_\ell^\times$, thus its image $\mathrm{Div}'(D)_{(\ell)}$ in $\mathrm{Div}(D)$ can be recovered from $\mathcal{G}_{\mathcal{D}_K} \subset \mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ up to multiplication by ℓ-adic units $\epsilon \in \mathbb{Z}_\ell^\times$ as well. Further, if $\jmath_K : K^\times \to \widehat{K}$ is the ℓ-adic completion homomorphism, by Pop [P3], Proposition 23, one has that $\mathcal{L}_K$ is the unique divisorial $\widehat{U}_K$-lattice which contains $\jmath_K(K^\times)_{(\ell)}$. Finally, the recipes to recover the above invariants of $K|k$ from $\mathcal{G}_{\mathcal{D}_K} \subset \mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ are invariant under isomorphisms of total decomposition graphs $\Phi : \mathcal{G}_{\mathcal{D}_K^{\rm tot}} \to \mathcal{H}_{\mathcal{D}_L^{\rm tot}}$, where $L|l$ is a further function field with l algebraically closed.

Remark 3.2. Let $\jmath_K : K^\times \to \widehat{K}$ be the ℓ-adic completion homomorphism. Then for every divisorial $\widehat{U}_K$-lattice $\mathcal{L}'_K := \epsilon \cdot \mathcal{L}_K$ for $K|k$ there exists a unique $\epsilon' \in \mathbb{Z}_\ell^\times/\mathbb{Z}_{(\ell)}^\times$ such that $\epsilon' \cdot \jmath_K(K^\times) \subset \mathcal{L}'_K$, and if so, then $\epsilon/\epsilon' \in \mathbb{Z}_{(\ell)}^\times$. Further one has the following:

1) For $u \in K^\times$ let $\mathsf{u} \in \mathbb{Z}_{(\ell)} \cdot \jmath_K(u)$ and $\boldsymbol{u} \in \mathbb{Z}_\ell \cdot \mathsf{u}$ be non-trivial (hence in particular, u is non-constant). Then for every prime divisor v of $K|k$ the following hold: $u \in U_v$ iff $\mathsf{u} \in \widehat{U}_v$ iff $\boldsymbol{u} \in \widehat{U}_v$. And if so, then $\jmath_v\big(\jmath_K(u)\big) \neq 1$ iff $\jmath_v(\mathsf{u}) \neq 1$ iff $\jmath_v(\boldsymbol{u}) \neq 1$ in $\widehat{Kv}$. Therefore, the following sets of prime divisors are equal:
   a) $\mathcal{V}_u := \{v \mid u \in U_v \text{ and } \jmath_v\big(\jmath_K(u)\big) \neq 1\}$
   b) $\mathcal{V}_{\mathsf{u}} := \{v \mid \mathsf{u} \in \widehat{U}_v \text{ and } \jmath_v(\mathsf{u}) \neq 1\}$
   c) $\mathcal{V}_{\boldsymbol{u}} := \{v \mid \boldsymbol{u} \in \widehat{U}_v \text{ and } \jmath_v(\boldsymbol{u}) \neq 1\}$

   and these sets are equal to $\mathcal{V}_{\kappa_u} := \{v \mid v \text{ is trivial on } \kappa_u\}$. And by Proposition 3.1, 1), any of the following: $u \in K^\times$ satisfying a), or $\mathsf{u} \in \mathbb{Z}_{(\ell)} \cdot \jmath_K(K^\times)$ satisfying b), or $\boldsymbol{u} \in \mathbb{Z}_\ell \cdot \jmath_K(K^\times)$ satisfying c), enables one to recover the (group theoretical) geometric 1-dimension quotient $\Phi_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$.

2) Let $u \in K$ be a non-constant function. Then $\kappa_u|k$ is a function field in one variable, hence it has a unique complete normal model $X_u \to k$, which is a projective smooth

[1] Recall that for every abelian group A we denote $A_{(\ell)} := A \times \mathbb{Z}_{(\ell)}$.

curve over k. Thus the set of all the prime divisors of $\kappa_u|k$ is in bijection with the closed points of X_u, hence the total prime divisor graph of $\kappa_u|k$ coincides with the unique complete regular like prime divisor graph for $\kappa_u|k$. Correspondingly, the same is true for the total decomposition graph and the unique complete regular like decomposition graph, which we denote by $\mathcal{G}_{\kappa_u}$. We denote by $\widehat{U}_{\kappa_u} \subset \mathcal{L}_{\kappa_u}$ the group of global units, respectively the canonical divisorial $\widehat{U}_{\kappa_u}$-lattice for $\kappa_u|k$.

3) In the above notations, one can recover $\mathcal{G}_{\kappa_u}$ form $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ endowed with the group theoretical 1-dimension quotient $\Phi_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$ be as follows: Let v be a prime divisor of $K|k$. Then $v \in \mathcal{V}_{\kappa_u}$ iff v is non-trivial on κ_u iff $\Phi_{\kappa_u}(T_v) \subset \Pi_{\kappa_u}$ is non-trivial. And if so, and v_α is the restriction of v to κ_u, then $\Phi_{\kappa_u}(T_v) \subseteq T_{v_\alpha}$ is an open subgroup, and moreover, T_{v_α} is the unique maximal pro-cyclic subgroup of Π_{κ_u} which contains $\Phi_{\kappa_u}(T_v)$. Conversely, for every prime divisor v_α of $\kappa_u|k$ there exists some prime divisor $v \in \mathcal{V}_{\kappa_u}$ which restricts to v_α, thus $\Phi_{\kappa_u}(T_v) \subseteq T_{v_\alpha}$ is non-trivial. Since for all prime divisors v_α of $\kappa_u|k$ one has $T_{v_\alpha} = Z_{v_\alpha}$, it follows that the above procedure recovers $\mathcal{G}_{\kappa_u}$ from $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ endowed with the group theoretical 1-dimension quotient $\Phi_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$.

- Moreover, the above procedure not only recovers $\mathcal{G}_{\kappa_u}$, but also recovers the morphism of total decomposition groups $\Phi_{\kappa_u} : \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \to \mathcal{G}_{\kappa_u}$ defined by the group theoretical 1-dimension quotient $\Phi_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$.

4) Next, if $\mathcal{D}_K \subset \mathcal{D}_K^{\mathrm{tot}}$ is complete regular like, $\Phi_{\kappa_u} : \mathcal{G}_{\mathcal{D}_K} \to \mathcal{G}_{\kappa_u}$ is divisorial in the sense of Pop [P3], Definition/Remark 31 and Proposition 40, because Φ_{κ_u} originates from the embedding of function fields $\kappa_u|k \hookrightarrow K|k$. And the Kummer morphism $\hat{\phi}_{\kappa_u} : \widehat{\kappa}_u \to \widehat{K}$ maps $\widehat{U}_{\kappa_u} \subset \mathcal{L}_{\kappa_u}$ injectively into $\widehat{U}_K \subset \mathcal{L}_K$. Thus by loc.cit. 5) and the discussion above, it follows that $\mathcal{L}'_{\kappa_u} := \epsilon \cdot \mathcal{L}_{\kappa_u}$ is the unique divisorial $\widehat{U}_{\kappa_u}$-lattice for $\kappa_u|k$ which is mapped by $\hat{\phi}_{\kappa_u}$ into $\mathcal{L}'_K = \epsilon \cdot \mathcal{L}_K$. Moreover, since $\Pi_K \to \Pi_{\kappa_u}$ is surjective, it follows that $\mathcal{L}'_K / \mathcal{L}'_{\kappa_u} \subset \widehat{K} / \widehat{\kappa}_u$ are torsion free.

5) The fact that a 1-dimensional quotient $\Phi_{\kappa_u} : \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \to \mathcal{G}_{\kappa_u}$ is an abstract rational quotient of $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ in the sense of Pop [P3], section 5, B), and/or Pop [P4], Definition 5.2, is equivalent to the fact that κ_u is a rational function field by Pop [P3], Proposition 41. On the other hand, $\kappa_u|k$ is a rational function field if and only if the inertia groups $(T_{v_\alpha})_{v_\alpha}$ generate Π_{κ_u} if and only if $\widehat{U}_{\kappa_u} = 1$. These equivalent conditions are encoded in $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ endowed with $\Phi_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$ and are equivalent to the fact that $\jmath_K(\kappa_u^\times)_{(\ell)}$ is the canonical divisorial $\widehat{U}_{\kappa_u}$-lattice for $\kappa_u|k$.

6) The elements $x \in K$ such that $\kappa_x = k(x)$, for short **general elements** of K, are quite abundant in K. Indeed, by the discussion from Pop [P3], Fact/Definition 43, for all $u, u', u'' \in K^\times$ with u, u' algebraically independent, u' separable, the following hold:
   a) $au' + u$ is a general element of K for almost all $a \in k$.
   b) $u''/(au' + u)$ is a general element of K for almost all $a \in k$.

Construction 3.3. In the notations from Remark 3.2 bove, we construct inductively two increasing sequences $(\Sigma_n)_n$ and $(\mathcal{L}'_n)_n$ as follows:

Step 1: Recall that Proposition 3.1, 2) above, gives a recipe to recover the 1-dimensional k_0-rational (group theoretical) quotients $\Phi_{\kappa_t} : \Pi_K \to \Pi_{\kappa_t}$. Further, by Remark 3.2 above, especially 5), among all the Φ_{κ_t}, one can single out the ones which are *rational quotients* $\Phi_{\kappa_x} = \Phi_{\kappa_t}$ for some $x \in K$ and $\kappa_t = k(x)$. Note that for these rational quotients we have $\kappa_t = k(x)$, but x does not necessarily lie in $k_0(X)$ itself, but in K! (The latter happens if and only if the relative algebraic closure $\kappa_{0,t}$ of $k_0(t)$ in $k_0(X)$ is the function field of a curve $X_{0,t}$ of genus zero having no k_0-rational points.) We define Σ_1 to be the set of these rational quotients $\Phi_{\kappa_t} = \Phi_{\kappa_x}$ of Π_K with $\kappa_t = k(x)$ for some $x \in K$ as above.

For every $\Phi_{\kappa_x} \in \Sigma_1$, let $\mathcal{L}'_{\kappa_x} = \epsilon \cdot \jmath_K(\kappa_x^\times)_{(\ell)}$ be the unique divisorial lattice for $\mathcal{G}_{\kappa_x}$ which is mapped by $\hat\phi_{\kappa_x}$ into $\mathcal{L}'_K$. We then let $\mathcal{L}'_1 \subset \mathcal{L}'_K$ be the $\mathbb{Z}_{(\ell)}$-submodule generated by all the images $\hat\phi_{\kappa_x}(\mathcal{L}'_{\kappa_x}) \subset \mathcal{L}'_K$, $\Phi_{\kappa_x} \in \Sigma_1$, and notice that by Remark 3.2, 4) and 5), we have:

$$\mathcal{L}'_1 \subseteq \epsilon \cdot \jmath_K(K^\times)_{(\ell)} \subseteq \mathcal{L}'_K.$$

<u>Step $(n+1)$</u>: Now supposing that Σ_n and $\mathcal{L}'_n \subseteq \epsilon \cdot \jmath_K(K^\times)_{(\ell)} \subseteq \mathcal{L}'_K$ are constructed, we proceed as follows: For every non-trivial $\mathsf{u} \in \mathcal{L}'_n$, let $u \in K^\times_{(\ell)}$ be such that $\mathsf{u} \in \mathbb{Z}_{(\ell)} \cdot \jmath_K(u)$, and $\Phi_{\kappa_u} : \Pi_K \to \Pi_{\kappa_u}$ be the 1-dimensional quotient defined by u as indicated at Remark 3.2, 1) above. By Remark 3.2, 5), we can recover the fact $\kappa_u|k$ is a rational function field from $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ endowed with Φ_{κ_u}. Define Σ_{n+1} to be the set of all such rational quotients $\Phi_{\kappa_x} : \Pi_K \to \Pi_{\kappa_x}$. We also notice that by the discussion at Remark 3.2, 4) and 5), it follows that for every $\Phi_{\kappa_x} \in \Sigma_{n+1}$ one has that $\mathcal{L}'_{\kappa_x} = \epsilon \cdot \jmath_K(\kappa_x^\times)_{(\ell)}$ is the unique divisorial lattice for $\kappa_x|k$ such that $\hat\phi_{\kappa_x}(\mathcal{L}'_{\kappa_x}) \subset \mathcal{L}'_K$, and therefore $\hat\phi_{\kappa_x}(\mathcal{L}'_{\kappa_x}) \subset \epsilon \cdot \jmath_K(K^\times)_{(\ell)} \subseteq \mathcal{L}'_K$. Finally, let $\mathcal{L}'_{n+1} \subseteq \mathcal{L}'_K$ be the $\mathbb{Z}_{(\ell)}$-submodule generated by the all the images $\hat\phi_{\kappa_x}(\mathcal{L}'_{\kappa_x})$, $\Phi_{\kappa_x} \in \Sigma_{n+1}$, and notice that $\mathcal{L}'_{n+1} \subseteq \epsilon \cdot \jmath_K(K^\times)_{(\ell)}$. And obviously, $\Sigma_n \subseteq \Sigma_{n+1}$ and $\mathcal{L}'_n \subseteq \mathcal{L}'_{n+1}$ for all $n \geq 1$.

Next let $\Phi \in \mathrm{Aut}^{\rm c}_{U_0}\big(\Pi_K\big)$ be a fixed automorphism. By Proposition 3.1, 3), it follows that Φ gives rise to an isomorphism $\Phi : \mathcal{G}_{\mathcal{D}_K^{\rm tot}} \to \mathcal{G}_{\mathcal{D}_K^{\rm tot}}$, thus maps each complete regular like decomposition graph $\mathcal{G}_{\mathcal{D}_K}$ for $K|k$ onto a complete regular like decomposition graph, say $\mathcal{G}_{\mathcal{E}_K}$, for $K|k$. Thus by Pop [P3], Proposition 30, 2), the Kummer morphism $\hat\phi : \widehat{K} \to \widehat{K}$ of Φ maps $\widehat{U}_K \subset \widehat{K}$ isomorphically onto itself, and there exists $\epsilon_\Phi \in \mathbb{Z}_\ell^\times$ such that $\hat\phi$ maps $\mathcal{L}_K$ onto the multiple $\epsilon_\Phi \cdot \mathcal{L}_K$. Thus replacing Φ by $\epsilon_\Phi^{-1} \cdot \Phi$, we can and <u>will</u> suppose that $\hat\phi$ maps $\mathcal{L}_K$ onto itself, and therefore $\hat\phi$ maps each $\mathcal{L}'_K := \epsilon \cdot \mathcal{L}_K$ isomorphically onto itself as well.

Lemma 3.4. *In the above notations, let $x, y \in K \backslash k$. Then one has:*

1) *The automorphism Φ maps the divisorial subgroups $T_v \subset Z_v$, $v \in \mathcal{V}_x$ isomorphically on the divisorial subgroups $T_w \subset Z_w$, $w \in \mathcal{V}_y$ if and only if $\Phi\big(\ker(\Phi_{\kappa_x})\big) = \ker(\Phi_{\kappa_y})$.*
2) *Suppose that $\Phi\big(\ker(\Phi_{\kappa_x})\big) = \ker(\Phi_{\kappa_y})$. Then the abstract isomorphism of profinite groups $\Phi_{\kappa_x,\kappa_y} : \Pi_{\kappa_x} \to \Pi_{\kappa_y}$ induced by the automorphism $\Phi : \Pi_K \to \Pi_K$ defines an (abstract) isomorphism of decomposition graphs $\Phi_{\kappa_x,\kappa_y} : \mathcal{G}_{\kappa_x} \to \mathcal{G}_{\kappa_y}$ whose Kummer isomorphism satisfies $\hat\phi_{\kappa_x,\kappa_y}(\mathcal{L}'_{\kappa_y}) = \mathcal{L}'_{\kappa_x}$.*

Proof. For every divisorial subgroup $T_v \subset Z_v$ of Π_K and its image $\Phi(T_v) = T_w \subset Z_w = \Phi(T_v)$ under Φ, one has the following, see e.g., Pop [P3], Remark 26:

a) $\hat\phi$ maps $\widehat{U}_w$ isomorphically onto $\widehat{U}_v$.

b) $\hat\phi$ maps $\ker(\jmath_w)$ isomorphically onto $\ker(\jmath_v)$.

In particular, $v \in \mathcal{V}_{\kappa_x}$ iff $\mathsf{x} \in \widehat{U}_v$ and $\jmath_v(\mathsf{x}) \neq 1$ iff $\mathsf{y} \in \widehat{U}_w$ and $\jmath_w(\mathsf{y}) \neq 1$ iff $w \in \mathcal{V}_{\kappa_y}$. Thus by Proposition 3.1, 1) above, using a), b) above, we conclude that $\{T_v \subset Z_v \mid v \in \mathcal{V}_x\}$ is mapped isomorphically onto $\{T_w \subset Z_w \mid w \in \mathcal{V}_y\}$ iff $\hat{\phi}$ maps $\widehat{\kappa}_{y,\mathrm{fin}}$ isomorphically onto $\widehat{\kappa}_{x,\mathrm{fin}}$. On the other hand, by taking ℓ-adic duals, we conclude that $\hat{\phi}$ maps $\widehat{\kappa}_{y,\mathrm{fin}}$ isomorphically onto $\widehat{\kappa}_{x,\mathrm{fin}}$ iff $\Phi\big(\ker(\Phi_{\kappa_x})\big) = \ker(\Phi_{\kappa_y})$. This concludes the proof of assertion 1).

For the proof of assertion 2), notice that by 1) one has: Φ maps $\{T_v \subset Z_v \mid v \in \mathcal{V}_x\}$ isomorphically onto $\{T_w \subset Z_w \mid w \in \mathcal{V}_y\}$. Further, by Remark 3.2, 3), we have: Let $v \in \mathcal{V}_{\kappa_x}$ be given, and v_α be the restriction of v on κ_x. Then $\Phi(T_v) = T_w$ for some prime divisor w of $K|k$ such that $\Phi_{\kappa_y}(T_w) = \Phi_{\kappa_x,\kappa_y}\big(\Phi_{\kappa_x}(T_v)\big)$, thus $\Phi_{\kappa_y}(T_w)$ is non-trivial, because $\Phi_{\kappa_x}(T_v)$ is so, and Φ_{κ_x,κ_y} is an isomorphism. Therefore, the restriction w_β of w to κ_y is non-trivial. Further, since Φ_{κ_x,κ_y} is an isomorphism, and T_{v_α} is the unique maximal pro-cyclic subgroup of Π_{κ_x} containing $\Phi_{\kappa_x}(T_v)$, it follows that $\Phi_{\kappa_x,\kappa_y}(T_{v_\alpha})$ is the unique maximal pro-cyclic subgroup of Π_{κ_y} which contains $\Phi_{\kappa_y}(T_w)$. We conclude that $\Phi_{\kappa_x,\kappa_y}(T_{v_\alpha}) = T_{w_\beta}$. Thus $\Phi_{\kappa_x,\kappa_y} : \mathcal{G}_{\kappa_x} \to \mathcal{G}_{\kappa_y}$ is an (abstract) isomorphism of decomposition groups. Finally, one has $\hat{\phi}_{\kappa_x}(\mathcal{L}'_{\kappa_x}) \subset \mathcal{L}'_K$ and $\hat{\phi}_{\kappa_y}(\mathcal{L}'_{\kappa_y}) \subset \mathcal{L}'_K$. Since $\hat{\phi} \circ \hat{\phi}_{\kappa_y} = \hat{\phi}_{\kappa_x} \circ \hat{\phi}_{\kappa_x,\kappa_y}$, it follows that $\mathcal{L}''_{\kappa_x} := \hat{\phi}_{\kappa_x,\kappa_y}(\mathcal{L}'_{\kappa_y})$ is a divisorial lattice for $\kappa_x|k$ such that

$$\hat{\phi}_{\kappa_x}(\mathcal{L}''_{\kappa_x}) = \hat{\phi}_{\kappa_x}\big(\hat{\phi}_{\kappa_x,\kappa_y}(\mathcal{L}'_{\kappa_y})\big) = \hat{\phi}\big(\hat{\phi}_{\kappa_y}(\mathcal{L}'_{\kappa_y})\big) \subset \hat{\phi}(\mathcal{L}'_K) = \mathcal{L}'_K.$$

Thus $\hat{\phi}_{\kappa_x}(\mathcal{L}''_{\kappa_x}) \subset \mathcal{L}'_K$ and by the uniqueness of $\mathcal{L}'_{\kappa_x}$ with the property that $\hat{\phi}_{\kappa_x}(\mathcal{L}'_{\kappa_x}) \subset \mathcal{L}'_K$, it follows that $\mathcal{L}''_{\kappa_x} = \mathcal{L}'_{\kappa_x}$. Thus we conclude that $\hat{\phi}_{\kappa_x,\kappa_y}(\mathcal{L}'_{\kappa_y}) = \mathcal{L}''_{\kappa_x} = \mathcal{L}'_{\kappa_x}$, as claimed. $\square$

Proposition 3.5. *In the above notations, let $\Sigma := \cup_n \Sigma_n$. Then one has:*

1) *If k_0 is infinite, Σ consists of all the rational quotients $\Phi_{\kappa_x} : \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \to \mathcal{G}_{\kappa_x}$ of $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$.*
2) *Every $\Phi \in \mathrm{Aut}^{\mathrm{c}}_{U_0}(\Pi_K)$ is compatible with the rational quotients of $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$, i.e., for every $\Phi_{\kappa_x} \in \Sigma$ there exists Φ_{κ_y} and an isomorphism $\Phi_{\kappa_x,\kappa_y} : \mathcal{G}_{\kappa_x} \to \mathcal{G}_{\kappa_y}$ such that $\Phi_{\kappa_y} \circ \Phi = \Phi_{\kappa_x,\kappa_y} \circ \Phi_{\kappa_x}$.*

Proof. We first claim that for every $n > 0$ and every $\Phi_{\kappa_x} \in \Sigma_n$ there exists some $\Phi_{\kappa_y} \in \Sigma_n$ such that $\Phi\big(\ker(\Phi_{\kappa_x})\big) = \ker(\Phi_{\kappa_y})$ and $\hat{\phi}(\mathcal{L}'_n) = \mathcal{L}'_n$. We prove that by induction as follows:

$\underline{n=1}$: Recall that Σ_1 consists of all the rational quotients $\Phi_{\kappa_t} : \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \to \mathcal{G}_{\kappa_t}$ of $\mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$ which are defined by projections $p_t : \Pi_K \to \Pi_{U_0}$ as indicated in Proposition 3.1. For every such Φ_{κ_x}, it follows by loc.cit. that $\Phi\big(\ker(\Phi_{\kappa_x})\big) = \ker(\Phi_{\kappa_x})$. Hence by Lemma 3.4, it follows that $\hat{\phi}\big(\hat{\phi}_{\kappa_x}(\mathcal{L}'_{\kappa_x})\big) = \phi_{\kappa_x}(\mathcal{L}'_{\kappa_x})$, and therefore it follows that $\hat{\phi}(\mathcal{L}'_1) = \mathcal{L}'_1$.

$\underline{n \Rightarrow (n+1)}$: By the induction hypothesis, we have $\hat{\phi}(\mathcal{L}'_n) = \mathcal{L}'_n$, and by the construction of $\mathcal{L}'_n$ one has $\mathcal{L}'_n \subset \epsilon \cdot \jmath_K(K^\times)$. Thus $\forall\, \mathsf{x} \in \mathcal{L}'_n\ \exists\, \mathsf{y} \in \mathcal{L}'_n$ such that $\hat{\phi}(\mathsf{x}) = \mathsf{y}$. Now if $x, y \in K^\times$ are such that $\mathsf{x} \in \mathbb{Z}_{(\ell)} \cdot \jmath_K(x)$ and $\mathsf{y} \in \mathbb{Z}_{(\ell)} \cdot \jmath_K(y)$, let Φ_{κ_x} and Φ_{κ_y} be the 1-dimensional quotients defined by κ_x and κ_y as indicated at Remark 3.2, 3). Then by loc.cit., 1), and reasoning as in the proof of Lemma 3.4, 1), it follows that $\{T_v \subset Z_v \mid v \in \mathcal{V}_x\}$ is mapped by Φ isomorphically onto the $\{T_w \subset Z_w \mid w \in \mathcal{V}_y\}$. But then by Lemma 3.4, it follows that there exists an (abstract) isomorphism of decomposition graphs $\Phi_{\kappa_x,\kappa_y} : \mathcal{G}_{\kappa_x} \to \mathcal{G}_{\kappa_y}$ such that $\Phi_{\kappa_x,\kappa_y} \circ \Phi_{\kappa_x} = \Phi_{\kappa_y} \circ \Phi$, and the Kummer morphism $\hat{\phi}_{\kappa_x,\kappa_y} : \widehat{\kappa}_y \to \widehat{\kappa}_x$ of Φ_{κ_x,κ_y} satisfies $\hat{\phi}_{\kappa_x,\kappa_y}(\mathcal{L}'_{\kappa_y}) = \mathcal{L}'_{\kappa_x}$. In particular, Φ_{κ_x} is a rational quotient iff Φ_{κ_y} is so. Thus we conclude that Φ defines a bijection $\Sigma_{n+1} \to \Sigma_{n+1}$ via $\ker(\Phi_{\kappa_x}) \mapsto \ker(\Phi_{\kappa_y})$, provided $\mathsf{y} = \hat{\phi}(\mathsf{x})$ and $\mathsf{x} \in \mathbb{Z}_{(\ell)} \cdot \jmath_K(x)$ and $\mathsf{y} \in \mathbb{Z}_{(\ell)} \cdot \jmath_K(y)$. And notice that if $\ker(\Phi_{\kappa_x}) \mapsto \ker(\Phi_{\kappa_y})$, then

$\hat{\phi}\big(\hat{\phi}_{\kappa_y}(\mathcal{L}'_{\kappa_y})\big) = \hat{\phi}_{\kappa_x}(\mathcal{L}'_{\kappa_x})$. Thus since $\mathcal{L}'_{n+1}$ is generated by all the $\hat{\phi}_{\kappa_x}(\mathcal{L}'_{\kappa_x})$ with $\Phi_{\kappa_x} \in \Sigma_{n+1}$, as well as by all the $\hat{\phi}_{\kappa_y}(\mathcal{L}'_{\kappa_y})$ with $\Phi_{\kappa_y} \in \Sigma_{n+1}$, we get: $\hat{\phi}(\mathcal{L}'_{n+1}) = \mathcal{L}'_{n+1}$, as claimed.

Thus in order to complete the proof of Proposition 3.5, we proceed as follows: First, if k_0 is a finite field, then assertion 2) above is proved already in Pop [P4], Proposition 5.3. Therefore, from now on we will suppose that k_0 is infinite. We show that $\Sigma := \cup_n \Sigma_n$ consists of all the rational quotients. First, since each Σ_n is a set of rational quotients, it follows that all $\Phi_{\kappa_x} \in \Sigma$ are rational quotients of $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$. Thus it is left to prove that if $x \in K$ is a general element, i.e., $\kappa_x = k(x)$, then $\Phi_{\kappa_x} \in \Sigma$. In order to do so, we notice that if $(a_i)_i$ is a system of generators of the k_0-vector space $(k, +)$, then $(a_i)_i$ is a system of generators of the $k_0(X)$-vector space $(K, +)$, because $K = k(X) = k_0(X)k$.

Let $x = a_1t_1 + \cdots + a_nt_n \in K$ for some $a_1, \dots, a_n \in k$ and $t_1, \dots, t_n \in k_0(X)$ be an element of K. We prove by induction on n that if x is a general element of K, then $\Phi_{\kappa_x} \in \Sigma_n$.

First, if $n = 1$, the assertion follows by Proposition 3.1 and the definiton of Σ_1.

Second, let $x = a_1t_1 + \cdots + a_{n+1}t_{n+1} \in K$ be a general element of K, where $a_1, \dots, a_{n+1} \in k$ and $t_1, \dots, t_{n+1} \in k_0(X)$ are all non-zero. If some t_{n+1} is constant, thus $t_{n+1} \in k$, then $x' := x - a_{n+1}t_{a+1}$ satisfies $k(x') = k(x)$, thus x' is a general element of K, $x' = a_1t_1 + \dots a_nt_n$. Thus we are done by the induction hypothesis and the definitions of Σ_n and $\mathcal{L}'_n$. Hence without loss of generality, we can suppose that $t_1, \dots, t_{n+1}$ are non-constant. Next, suppose that $x - a_it_i$ and t_i are algebraically dependent over k for some $1 \le i \le n+1$. Then we have $x - a_it_i \in \kappa_t$, where $t := t_i$, thus $x \in \kappa_t$ as well. Hence we must have $\kappa_t = k(x)$, because x is by hypothesis a general element of K, and by the definition of Σ_1, it follows that $\Phi_{\kappa_x} \in \Sigma_1$. Hence $\Phi_{\kappa_x} \in \Sigma_{n+1}$, because $\Sigma_1 \subseteq \Sigma_{n+1}$. Finally, suppose that $x - a_it_i$ and t_i are algebraically independent over k for all $i = 1, \dots, n+1$. Since x is a general element, it follows that x is separable in K, i.e., x is not a $p = \mathrm{char}(k)$ power in K. Hence there exits some t_i which is separable in K. Thus after a renumbering, we can suppose that t_1 is separable in K, and by Remark 3.2, 6) above, one has: $t := at_1 + t_{n+1}$ and x/t are general elements of K for almost all $a \in k_0$. Since $x = (a_1 - aa_{n+1})t_1 + \sum_{1<\alpha\le n} a_\alpha t_\alpha + a_{n+1}(at_1 + t_{n+1})$ we get:

$$y := x/t - a_{n+1} = (a_1 - aa_{n+1})t_1/t + \sum_{1<\alpha\le n} a_\alpha t_\alpha/t =: b_1t'_1 + \dots b_nt'_n$$

where $b_1 := a_1 - aa_n$ and $b_\alpha := a_\alpha$ for $1 < \alpha \le n$, and $t'_\alpha := t_\alpha/t$ for all α. Since x/t is a general element, so is $y := x/t - a_{n+1}$. Thus by the induction hypothesis, we have $\Phi_{\kappa_y} \in \Sigma_n$. And since $t \in k_0(X)$ is a general element, it follows that $\Phi_{\kappa_t} \in \Sigma_1$ by the definition of Σ_1. Thus since $\Sigma_1 \subseteq \Sigma_n$, one finally has $\Phi_{\kappa_y}, \Phi_{\kappa_t} \in \Sigma_n$. But then by the definition of $\mathcal{L}'_n$ it follows that $\hat{\phi}_{\kappa_t}(\mathcal{L}'_{\kappa_t}) = \eta \cdot \jmath_K(\kappa_t^\times)$ and $\hat{\phi}_{\kappa_y}(\mathcal{L}'_{\kappa_y}) = \eta \cdot \jmath_K(\kappa_y^\times)$ are contained in $\mathcal{L}'_n$. Therefore, $\eta \cdot \jmath_K(x) = \eta \cdot \big(\jmath_K(y + a_{n+1})\, \jmath_K(t)\big)$ lies in $\mathcal{L}'_n$. Since x is a general element of K, or equivalently, Φ_{κ_x} is a rational quotient of $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$, it follows by the definition of Σ_{n+1} that $\Phi_{\kappa_x} \in \Sigma_{n+1}$. This concludes the proof of the fact that all rational projections Φ_{κ_x} of $\mathcal{G}_{\mathcal{D}_K^{\rm tot}}$ lie in $\Sigma = \cup_n \Sigma_n$. $\square$

4. Concluding the proof of Theorem 1.1

A) *Proof of assertion* 1)

First recall that by the discussion from the Introduction, one has a canonical embedding $\mathrm{Aut}^{\rm c}\big(\Pi_{\mathcal{V}_X}\big) \hookrightarrow \mathrm{Aut}^{\rm c}_{U_0}\big(\Pi_K\big)$. Therefore, it is sufficient to prove that the representations $\rho^{\rm c}_{\mathcal{V}_X} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\rm c}\big(\Pi_{\mathcal{V}_X}\big)$ and $\rho^{\rm c}_{\mathcal{V}_X} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\rm c}_{U_0}\big(\Pi_K\big)$ are injective, respectively surjective.

a) $\rho^{\mathrm{c}}_{\mathcal{V}_X} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\mathrm{c}}\big(\Pi_{\mathcal{V}_X}\big)$ is injective.

Indeed, without loss we can suppose that X is quasi-projective, and let $X \hookrightarrow \mathbb{P}^N_{k_0}$ be a k_0-embedding. By base change to $k = \overline{k}_0$, we get an embedding $\overline{X} \hookrightarrow \mathbb{P}^N_k$. Let $\sigma \in \mathrm{Gal}_{k_0}$ be non-trivial. Then there exist general hyperplanes $H \subset \mathbb{P}^N_k$ such that $H \cap \overline{X} \neq H^\sigma \cap \overline{X}$. Since H is general, it follows that H^σ is general, and $H \cap \overline{X}$ and $H^\sigma \cap \overline{X}$ are ample prime Weil divisors of $\overline{X}$. Therefore, there exist functions $u \in K$ whose pole is a non-zero multiple of $H \cap \overline{X}$, thus $\sigma(u)$ has as pole divisor the same non-zero multiple of $H^\sigma \cap \overline{X}$. Thus letting $\overline{U}$ be the complement of the divisor of u in $\overline{X}$, we have: The field extensions $K_n := K[\sqrt[n]{u}\,]$ are unramified over $\overline{U}$ for all $n = \ell^e$, $e \geq 1$; and correspondingly, $K^\sigma_n := K[\sqrt[n]{\sigma(u)}\,]$ are unramified over $\overline{U}^\sigma$ for all $n = \ell^e$, $e \geq 0$. Since K^σ_n is the σ-conjugate of K_n (under any prolongation of σ to K'), it follows that σ acts non-trivially on Π_K.

b) $\rho^{\mathrm{c}}_{\mathcal{V}_X} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\mathrm{c}}_{U_0}\big(\Pi_K\big)$ is surjective.

Let $\Phi \in \mathrm{Aut}^{\mathrm{c}}_{U_0}\big(\Pi_K\big)$ be given. Then Proposition 3.5 above, Φ defines an isomorphism $\mathcal{G}_{\mathcal{D}^{\mathrm{tot}}_K} \to \mathcal{G}_{\mathcal{D}^{\mathrm{tot}}_K}$ which is compatible with all the rational quotients $\Phi_{\kappa_x} : \mathcal{G}_{\mathcal{D}^{\mathrm{tot}}_K} \to \mathcal{G}_{\kappa_x}$. Hence by the Main Theorem from Pop [P3], Introduction, it follows that there exists an isomorphism of fields $\phi : K^{\mathrm{i}} \to K^{\mathrm{i}}$ and some ℓ-adic unit $\epsilon \in \mathbb{Z}^\times_\ell$ such that $\epsilon \cdot \Phi$ is defined by ϕ, i.e., if ϕ' is some prolongation of ϕ to K', then $\epsilon \cdot \Phi(g) = \phi'^{-1} g\, \phi'$ for all $g \in \Pi_K$. We will prove that ϕ maps $k_0(X)^{\mathrm{i}}$ onto itself, and the restriction of ϕ to $k_0(X)^{\mathrm{i}}$ is a power of Frobenius. In order to do so, for $t \in k_0(X)$, consider the projection $p_t : \Pi_K \to \Pi_{U_0}$ defined by $\imath_t : k_0(t_0) \to k_0(X)$, $t_0 \mapsto t$. Let $\Delta := \ker(p_t)$, and $K_t \subset K'$ be the fixed fied of Δ. Then the following hold:

a) $K_t = K[(t'_n, t''_n)_n]$ with ${t'_n}^n = t$ and ${t''_n}^n = t - 1$, where $n = \ell^e$, $e \geq 0$.

b) $\Phi : \Pi_K \to \Pi_K$ is compatible with p_t, thus $\Phi(\Delta) = \Delta$. Equivalently, $\phi(K^{\mathrm{i}}_t) = K^{\mathrm{i}}_t$.

On the other hand, if $u := \phi(t) \in K^{\mathrm{i}}$, then setting $u'_n := \phi'(t'_n)$, $u''_n := \phi'(t''_n)$ one has: ${u'_n}^n = u$, ${u''_n}^n = u - 1$ for all $n = \ell^e$, $e \geq 0$, and $\phi'(K^{\mathrm{i}}_t) = K^{\mathrm{i}}[(u'_n, u''_n)_n]$. Thus we conclude:

$$K^{\mathrm{i}}[(t'_n, t''_n)_n] = K^{\mathrm{i}}_t = \phi'(K^{\mathrm{i}}) = K^{\mathrm{i}}[(u'_n, u''_n)_n].$$

Next recall that for $t \in K$ we definite by $\kappa_t \subset K$ the relative algebraic closure of $k(t)$ in K, and by $\kappa^{\mathrm{i}}_t \subset K^{\mathrm{i}}$ the pure inseparable closure of κ_t.

Claim 1. One has $\kappa^{\mathrm{i}}_t = \kappa^{\mathrm{i}}_u$, thus ϕ maps κ^{i}_t onto itself. Further, $\kappa^{\mathrm{i}}_t[(t'_n, t''_n)_n] = \kappa^{\mathrm{i}}_u[(u'_n, u''_n)_n]$.

Equivalently, we have to prove that $u = \phi(t)$ and t are algebraically dependent over k. By contradiction, suppose that this is not the case, hence u and t are algebraically independent over k. Then there exists a transcendence basis $\mathcal{T}$ of $K^{\mathrm{i}}|k$ of the form $\mathcal{T} = (t_1, \dots, t_d)$ with $t_1 = t$ and $t_2 = u$. Thus letting $l := \overline{k(t)}$ be the algebraic closure of $k(u)$, it follows that $L^{\mathrm{i}} := K^{\mathrm{i}} l$ is the pure inseparable closure of a function field L over l having $(t_2 = t, \dots, t_d)$ as a transcendence basis. Since $t'_n, t''_n \in l$ for all n, one has that $\phi'(K_t) \subset K^{\mathrm{i}}_t \subset lK^{\mathrm{i}} = L^{\mathrm{i}}$. But then $u'_n, u''_n \in L^{\mathrm{i}}$ for all $n = \ell^e$, hence $t_2^{1/n} = u'_n \in L$ for all $n = \ell^e$, contradicting the fact that L is a function field over l with the transcendence basis $(t_2, \dots, t_d)$. This concludes the proof of the first assertion of Claim 1. For the last assertion, by Kummer theory, one has: Since $K^{\mathrm{i}}[t'_n, t''_n] = K^{\mathrm{i}}[u'_n, u''_n]$, there there exist r_{ij} and $\theta, \eta \in K^{\mathrm{i}}$ such that $t = u^{r_{00}}(u-1)^{r_{01}}\theta^n$, $t - 1 = u^{r_{10}}(u-1)^{r_{11}}\eta^n$ and $r_{00}r_{11} - r_{10}r_{01}$ relatively prime to ℓ. On the other hand, since $t, u \in \kappa_t$, it follows that θ^n, $\eta^n \in \kappa^{\mathrm{i}}_t$. Thus since $\kappa^{\mathrm{i}}_t \subset K^{\mathrm{i}}$ is relatively algebraically closed, it follows that $\theta, \eta \in \kappa^{\mathrm{i}}_t$. This concludes the proof of Claim 1.

Now suppose that $t \in k_0(X)$ is a *general element of* K, i.e., $k(t) \subset K$ is a relatively algebraically closed subfield —thus $k(t)^{\mathrm{i}} \subset K^{\mathrm{i}}$ is relatively algebraically closed in K^{i}. [Recall that the general elements $t \in k_0(X)$ of K are quite abundant, as they generate $k_0(X)$.] Since $\phi : K^{\mathrm{i}} \to K^{\mathrm{i}}$ is an isomorphism, if $t \in k_0(X)$ is a general element of K, then setting $u := \phi(t) \in K^{\mathrm{i}}$, one has that $k(u)$ is relatively algebraically closed in $\phi(K)$, and $k(u)^{\mathrm{i}}$ is relatively algebraically closed in K^{i}. In particular, by Claim 1 above and has: If $t \in k_0(X)$ is a general element of K, and $u := \phi(t)$, it follows that

$$k(t)^{\mathrm{i}} = \phi\big(k(t)^{\mathrm{i}}\big) = k(u)^{\mathrm{i}}.$$

Claim 2. The restriction of ϕ to $k_0(X)^{\mathrm{i}}$ is a power of Frobenius.

Indeed, first let $t \in k_0(X)$ be a general element, and $u = \phi(u) \in k_0(X)^{\mathrm{i}}$. Then by the discussion above, one has $k(t)^{\mathrm{i}} = k(u)^{\mathrm{i}}$. Therefore, t is algebraic purely inseparable over $k(u)$, and u is algebraic purely inseparable over $k(t)$. Equivalently, since k is algebraically closed, thus perfect, it there exist $e \geq 0$, $a, b, c, d \in K$ such that $u^{p^{\epsilon}} = (at-b)/(ct-d)$, where $\epsilon = e$ or $\epsilon = -e$. Therefore, $(u-1)^{p^{\epsilon}} = u^{p^{\epsilon}} - 1 = (at-b)/(ct-d) - 1 = \big((a-c)t-(b-d)\big)/(ct-d)$. We notice that $t = 0, 1, \infty$ are the only places of $k(t)^{\mathrm{i}}$ ramified in $k(t)^{\mathrm{i}}[(t'_n, t''_n)_n]$, and $u = 0, 1, \infty$ are the only places of $k(u)^{\mathrm{i}}$ ramified in $k(u)^{\mathrm{i}}[(u'_n, u''_n)_n]$. On the other hand, by Claim 1 we have that $\kappa^{\mathrm{i}}_t[(t'_n, t''_n)_n] = \kappa^{\mathrm{i}}_u[(u'_n, u''_n)_n]$, and since $k(t)^{\mathrm{i}} = k(u)^{\mathrm{i}}$ by the discussion above, it follows as well that $k(t)^{\mathrm{i}}[(t'_n, t''_n)_n] = k(t)^{\mathrm{i}}[(u'_n, u''_n)_n]$. Therefore, $u^{p^{\epsilon}} = (at-b)/(ct-d)$ and $(u-1)^{p^{\epsilon}} = \big((a-c)t-(b-d)\big)/(ct-d)$ have zeros and poles only at $t = 0, 1, \infty$, thus $u^{p^{\epsilon}} \in \{t, \frac{1}{t}, 1-t, \frac{1}{1-t}, \frac{t-1}{t}, \frac{t}{t-1}\}$. In particular, a direct verification shows that $\phi^6(t) = t^{p^{6\epsilon}}$, i.e., the restriction of ϕ to $k(t)^{\mathrm{i}}$ is the m^{th} power of Frobenius, where $m = 6\epsilon$.

Second, let $s, t \in k_0(X)$ be *algebraically independent* general elements. By Claim 1) one has that ϕ maps $\kappa^{\mathrm{i}}_{s+t}$ isomorphically onto itself, and κ_{st} isomorphically onto itself, thus the same is true for the 6^{th} power ϕ^6 of ϕ. On the other hand, by the discussion above one has: $\phi^6(t) = t^{p^m}$ and $\phi^6(s) = s^{p^l}$ for $l = 6\eta$, where $\eta \in \mathbb{Z}$ depends on s. Thus $\phi(s+t) = t^{p^m} + s^{p^l}$ is algebraic over $k(s+t)$, and since s, t are algebraically independent, it follows that $m = l$, thus $\epsilon = \eta$. Let us set $\phi_0 := \mathrm{Frob}^{-\epsilon} \circ \phi$. Then one has that $\phi_0(t) \in \{t, \frac{1}{t}, 1-t, \frac{1}{1-t}, \frac{t-1}{t}, \frac{t}{t-1}\}$ and $\phi_0(s) \in \{s, \frac{1}{s}, 1-s, \frac{1}{1-s}, \frac{s-1}{s}, \frac{s}{s-1}\}$. Recall that by Claim 1) one has that $\phi(s+t)$ is algebraic over $k(s+t)$, and $\phi(st)$ is algebraic over $k(st)$, thus the same is true correspondingly for $\phi_0(s+t)$ and $\phi_0(st)$. Finally, a case by case analysis (unfortunately, there are 14 cases to analyze, but the level is undergrad algebra!) shows that the only possibility for ϕ_0 is to satisfy: $\phi_0(s) = s$ and $\phi_0(t) = t$. Letting $t, s \in k_0(X)$ run through all pairs of general elements which are algebraically independent over k_0, we have: Since $k_0(X)$ is generated by general elements, it follows that ϕ_0 is the identity on $k_0(X)$. Thus we conclude that $\phi = \mathrm{Frob}^{\epsilon}$ on $k_0(X)^{\mathrm{i}}$. This concludes the proof of Claim 2.

Coming back to the proof of assertion 1) of the Theorem 1.1, in the above notations we have: For the given ϕ, let $\phi_0 := \mathrm{Frob}^{-\epsilon} \circ \phi$. Then ϕ_0 is the identity on $k_0(X)^{\mathrm{i}}$ and therefore, ϕ_0 is uniquely determined on $K = k(X)$ by its restriction $\tau := (\phi_0)|_k \in \mathrm{Gal}_{k_0}$. Since $\rho^{\mathrm{c}}_{\mathcal{V}_X} : \mathrm{Gal}_{k_0} \to \mathrm{Aut}^{\mathrm{c}}_{U_0}\big(\Pi_K\big)$ is injective, this concludes the proof of assertion 1).

Language: If $\Phi = \rho^{\mathrm{c}}_{\mathcal{V}_X}(\tau)$, i.e., $\tau \in \mathrm{Gal}_{k_0}$ and $\Phi : \Pi_K \to \Pi_K$ is the automorphism defined by the prolongation of τ to $K = k(X)$, we will simply say that Φ is defined by τ.

B) *Proof of assertion* 2)

Let $\Phi_{\mathcal{V}} := (\Phi_{X'})_{X'} \in \mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}})$ be given. Let X, Y in $\mathcal{V}$ be such that $\mathcal{V}_Y \prec \mathcal{V}_X$ as defined in the Introduction before Theorem 1.2. Then the dominant k_0-map $X \to Y$ gives rise to an embedding of function fields $k_0(Y) \hookrightarrow k_0(X)$, thus to $K := k(X) \hookrightarrow k(Y) =: L$ as well. Further, if $\mathcal{V}_X = \{U_i\}_i \cup \{U_0\}$ and $\mathcal{V}_Y = \{V_j\}_j \cup \{U_0\}$, by the discussion before Theorem 1.1 from the Introduction, by taking limits over $(\Phi_{U_i})_i$ and $(\Phi_{V_j})_j$, we get $\Phi_K \in \mathrm{Aut}^{\mathrm{c}}_{U_0}(\Pi_K)$ and $\Phi_L \in \mathrm{Aut}^{\mathrm{c}}_{U_0}(\Pi_L)$, which are compatible with the canonical projection $p_{KL} : \Pi_K \to \Pi_L$ defined by the dominant k_0-map $X \to Y$. Now suppose that Φ_K is defined by some $\tau \in \mathrm{Gal}_{k_0}$. We claim that Φ_L is then defined by τ too. Indeed, one has commutative diagrams:

$$\begin{array}{ccc} \Pi_K & \xrightarrow{\Phi_K} & \Pi_K \\ \downarrow{\scriptstyle p_{KL}} & & \downarrow{\scriptstyle p_{KL}} \\ \Pi_L & \xrightarrow{\Phi_L} & \Pi_L \end{array} \qquad\qquad \begin{array}{ccc} \Pi_K & \xrightarrow{\mathrm{id}} & \Pi_K \\ \downarrow{\scriptstyle p_{KL}} & & \downarrow{\scriptstyle p_{KL}} \\ \Pi_L & \xrightarrow{\Phi} & \Pi_L \end{array}$$

where the first commutative diagram is the one defined by $\Phi_{\mathcal{V}} := (\Phi_{X'})_{X'} \in \mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}})$ as explained above, whereas the second one is obtained from the first one by composing Φ_K and Φ_L with the automorphisms of Π_K, respectively Π_L, defined by τ^{-1}. On the other hand, since $L \hookrightarrow K$ is a k-embedding of function fields, it follows that the image of $\Pi_K \to \Pi_L$ is open in Π_L, and that the restriction of Φ_L to this open subgroup of Π_L is trivial. We thus conclude that Φ_L is trivial too, as claimed.

To conclude the proof of assertion 2), we proceed as follows: We first fixe some $X \in \mathcal{V}$ with $\dim(X) > 1$, and set $K := k(X)$. For any given $\Phi_{\mathcal{V}} = (\Phi_{X'})_{X'} \in \mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}})$, let $\Phi_{\mathcal{V}_X} \in \mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}_X})$ be the "restriction" of $\Phi_{\mathcal{V}}$ to $\Pi_{\mathcal{V}_X}$. Then by assertion 1) of the Theorem proved above, it follows that $\Phi_{\mathcal{V}_X}$ defines a unique $\Phi_K \in \mathrm{Aut}^{\mathrm{c}}_{U_0}(\Pi_K)$, and that $\Phi_{\mathcal{V}_X}$ and Φ_K are defined by a unique $\tau \in \mathrm{Gal}_{k_0}$. We claim that $\Phi_{\mathcal{V}}$ is actually defined by τ as well. Indeed, let $Y \in \mathcal{V}$ be arbitrary. Since $\mathcal{V}$ is connected, it follows by the definitions that there exist $m > 0$ and X_i in $\mathcal{V}$ such that $X_0 = X$, $X_{2m} = Y$ and for all $0 \le i < m$ one has: $\dim(X_{2i+1}) > 1$, and $\mathcal{V}_{X_{2i}}, \mathcal{V}_{X_{2i+2}} \prec \mathcal{V}_{X_{2i+1}}$. We prove by induction on m that Φ_Y equals the image of τ in $\mathrm{Aut}^{\mathrm{c}}(\Pi_Y)$. Indeed, for every X_j let $L_j := k(X_j)$ be the function field of $\overline{X}_j := X_j \times_{k_0} k$ and $\Psi_j \in \mathrm{Aut}^{\mathrm{c}}(\Pi_{\mathcal{V}_{X_j}}) \hookrightarrow \mathrm{Aut}^{\mathrm{c}}_{U_0}(\Pi_{L_j})$ be defined by the restriction of $\Phi_{\mathcal{V}}$ to $\mathcal{V}_{X_j}$. Then reasoning as above, the dominant k_0-maps $X_{2i+1} \to X_{2i}$, $X_{2i+1} \to X_{2i+2}$ give rise to k-embeddings of function fields $L_{2i}|k \hookrightarrow L_{2i+1}|k$, $L_{2i+2}|k \hookrightarrow L_{2i+1}|k$, thus to open projections $\Pi_{L_{2i+1}} \to \Pi_{L_{2i}}$, $\Pi_{L_{2i+1}} \to \Pi_{L_{2i+2}}$ which are compatible with Ψ_{2i+1} and Ψ_{2i}, Ψ_{2i+1} and Ψ_{2i+2}, respectively.

For $m = 1$ we have: Since $\dim(X_1) > 1$, by the discussion above it follows that if Ψ_1 is defined by some $\tau_1 \in \mathrm{Gal}_{k_0}$, then $\Psi_0 = \Phi_K$ is defined by τ_1 as well. On the other hand, Φ_K is defined by τ, thus $\tau_1 = \tau$. Further, since $X_1 \to X_2 = Y$ is dominant, it follows by the discussion above that if Ψ_2 is defined by some $\tau_1 = \tau$ as well.

Now proceeding by induction, suppose that Φ_{2m-2} is defined by τ. Since by hypothesis we have $\dim(X_{2m-1}) > 1$, it follows by the above discussion that we have: First, Ψ_{2m-1} is defined by some $\tau_{2m-1} \in \mathrm{Gal}_{k_0}$, then so is Ψ_{2m-2}, thus conclude that $\tau_{2m-1} = \tau$. Second, if Ψ_{2m-1} is defined by τ, so is Ψ_{2m}. Thus since Φ_Y is defined by Φ_L via the canonical projection $\Pi_L \to \Pi_Y$, we conclude that Φ_Y is defined by τ.

The proof of Theorem 1.1 is complete.

5. Proof of Theorem 1.2

We first notice that assertion 2) follows from assertion 1) in the same way as assertion 2) of Theorem 1.1 was deduced from assertion 1) of Theorem 1.1. Therefore we will not repeat this standard arguments, but rather give the proof of assertion 1) of Theorem 1.2.

A) *Absolute Galois theory of quasi prime divisors*

Let $\overline{K}|K$ be an algebraic closure of K and $G_K = \mathrm{Aut}_K\big(\overline{K}\big)$ be the absolute Galois group of K. For valuations v of K, and their prolongaions $\overline{v}$ to $\overline{K}$, let $T_{\overline{v}} \subseteq Z_{\overline{v}}$ denote the inertia/decomposition group of $\overline{v}$ in G_K. By general decomposition theory we have: All the prolongations $\overline{v}$ of a fixed v are G_K-conjugated, thus all the inertia/decomposition groups $T_{\overline{v}} \subseteq Z_{\overline{v}}$ are conjugated. Further, if $L|K$ is a sub-extension of $\overline{K}|K$, and $w := \overline{v}|_L$ is the restriction of $\overline{v}$ to L (thus w is one of the prolongations of v to L), then $\overline{w} := \overline{v}$ is one of the prolongations of w to $\overline{L} = \overline{K}$, and $T_{\overline{w}} \subseteq Z_{\overline{w}}$ are the intersection of $T_{\overline{v}} \subseteq Z_{\overline{v}}$ with G_L. Further, if $L|K$ is Galois, then denoting by $T_{w|v} \subseteq Z_{w|v}$ the inertia/decomposition groups of w in $\mathrm{Gal}(L|K)$, it follows that $T_{w|v} \subseteq Z_{w|v}$ are precisely the images of $T_{\overline{v}} \subseteq Z_{\overline{v}}$ under the canonical surjective projection $G_K \to \mathrm{Gal}(L|K)$.

In the above notations, suppose that $L|K$ is a *finite* Galois extension, and let $L' \subset L''$ be the maximal pro-ℓ abelian, respectively abelian-by-central, sub-extensions of $\overline{K}|L$, and $\Pi^{\mathrm{c}}_L \to \Pi_L$ be the corresponding canonical projections. Then one has:

- a) $L'|K$ and $L''|K$ are Galois extensions, and the canonical projections $\Pi^{\mathrm{c}}_L \to \Pi^{\mathrm{c}}_K$ and $p_L : \Pi_L \to \Pi_K$ have open images.
- b) The images of the inertia/decomposition groups $T_w \subseteq Z_w$ of w in Π_L under the projection $p_L : \Pi_L \to \Pi_K$ are open subgroups of $T_v \subseteq Z_v$ respectively.

Next, by general valuation theory, in the above notations one has: v is a (quasi) prime r-divisor of $K|k$ iff w is so for $L|k$. Further, since the total (quasi) decomposition graphs $\mathcal{G}_{\mathcal{D}^{\mathrm{tot}}_K} \subseteq \mathcal{G}_{\mathcal{Q}^{\mathrm{tot}}_K}$ of $K|k$ can be recovered from $\Pi^{\mathrm{c}}_K \to \Pi_K$ endowed with all the projections $p_t : \Pi_K \to \Pi_{U_0}$, we conclude that the total (quasi) decomposition graphs $\mathcal{G}_{\mathcal{D}^{\mathrm{tot}}_L} \subseteq \mathcal{G}_{\mathcal{Q}^{\mathrm{tot}}_L}$ of $L|k$ can be recovered from $\Pi^{\mathrm{c}}_L \to \Pi_L$ endowed with $\Pi^{\mathrm{c}}_K \to \Pi_K$.

Key Lemma 5.1. *In the above notations, for $L|K$ finite Galois, consider the action of $\mathrm{Gal}(L|K)$ on the subsets of Π_L defined by the conjugation. Then for every generalized quasi divisorial subgroup $T_{\mathbf{w}} \subseteq Z_{\mathbf{w}}$ of Π_L one has: $Z_{\mathbf{w}|\mathbf{v}} \subseteq \mathrm{Gal}(L|K)$ is precisely the stabilizer of $T_{\mathbf{w}} \subseteq Z_{\mathbf{w}}$ in $\mathrm{Gal}(L|K)$.*

Proof. For $g \in \mathrm{Gal}(L|K)$ arbitrary, consider the prolongation $\mathbf{w}_g := \mathbf{w} \circ g$ of $\mathbf{v}$ to L. Then $\mathbf{w}_g$ is a generalized quasi prime r-divisor of $L|k$. And since every generalized quasi prime divisor of $L|k$ is uniquely determined by its decomposition group, we have: $Z_{\mathbf{w}} = Z_{\mathbf{w}_g}$ iff $\mathbf{w} = \mathbf{w}_g$ iff $g \in Z_{\mathbf{w}|\mathbf{v}}$. Now suppose that $g \in Z_{\mathbf{w}|\mathbf{v}}$. Then by the functoriality of decomposition theory as briefly explained above, there exists a preimage $g' \in \mathrm{Gal}(L'|K)$ of g which lies in the decomposition group $Z_{\mathbf{w}'|\mathbf{v}}$ of some prolongation $\mathbf{w}'$ of $\mathbf{w}$ to L'. But then $\mathbf{w}' \circ g' = \mathbf{w}'$, hence $g'^{-1} Z_{\mathbf{w}'|\mathbf{v}}\, g' = Z_{\mathbf{w}'|\mathbf{v}}$, thus also $g'^{-1} Z_{\mathbf{w}'|\mathbf{w}}\, g' = Z_{\mathbf{w}'|\mathbf{w}}$, because $Z_{\mathbf{w}'|\mathbf{w}} \subseteq Z_{\mathbf{w}'|\mathbf{v}}$ is a normal subgroup. Thus conclude that g stabilizes $Z_{\mathbf{w}} := Z_{\mathbf{w}'|\mathbf{w}}$. In the same way, if $g \notin Z_{\mathbf{w}|\mathbf{v}}$, then $\mathbf{w}_g \neq \mathbf{w}$, thus $Z_{\mathbf{w}_g} \neq Z_{\mathbf{w}}$. Then reasoning as above, if g' is some preimage of g in $\mathrm{Gal}(L'|K)$ and $\mathbf{w}'_g := \mathbf{w}' \circ g'$, then $\mathbf{w}'_g$ is a prolongation of $\mathbf{w}_g$ to L', thus $Z_{\mathbf{w}'|\mathbf{w}} =: Z_{\mathbf{w}} \neq Z_{\mathbf{w}_g} := Z_{\mathbf{w}'_g|\mathbf{w}_g}$. On the other hand, $Z_{\mathbf{w}'_g|\mathbf{w}} = g'^{-1} Z_{\mathbf{w}'|\mathbf{w}}\, g'$, thus we conclude that $Z_{\mathbf{w}} \neq Z^g_{\mathbf{w}}$, as claimed. □

Remarks 5.2. As a corollary of the Key Lemma 5.1 above we have a description of the inertia/decomposition groups of generalized quasi prime divisors $\mathbf{v}$ in G_K as follows: Let $L_i|K$ be an inductive family of finite Galois sub-extensions of $\overline{K}|K$ such that $\overline{K} = \cup_i L_i$. Then G_K is the projective limit of the projective surjective system of finite groups $G_i := \mathrm{Gal}(L_i|K)$. And if $\overline{v}|v$ are as above, and $v_i := \overline{v}|_{L_i}$ for every i, then $T_{\overline{v}} \subset Z_{\overline{v}}$ is the projective system of all the $T_{v_i|v} \subseteq Z_{v_i|v}$. Therefore we have:

1) Giving a compatible system $(\mathbf{v}_i)_i$ of generalized quasi prime divisors of $(L_i)_i$ above $\mathbf{v}$, i.e., such that $\mathbf{v}_i = \mathbf{v}_j|_{L_i}$ for all $L_i \subseteq L_j$ (and $\mathbf{v} = \mathbf{v}_i|_K$), is equivalent to giving a compatible system $(Z_{\mathbf{v}_i})_i$ of generalized quasi divisorial subgroups in $(\Pi_{L_i})_i$, i.e., such that the canonical projection $p_{L_jL_i} : \Pi_{L_j} \to \Pi_{L_i}$ maps $Z_{\mathbf{v}_j}$ into $Z_{\mathbf{v}_i}$ for all $L_i \subseteq L_j$ (and $p_{L_i} : \Pi_{L_i} \to \Pi_K$ maps $Z_{\mathbf{v}_i}$ into $Z_{\mathbf{v}}$).

(∗) Giving the compatible system $(\mathbf{v}_i)_i$ with $\mathbf{v} = \mathbf{v}_i|_K$ is equivalent to giving a prolongation of $\overline{\mathbf{v}}$ of $\mathbf{v}$ to $\overline{K}$ which is defined by $\overline{\mathbf{v}}|_{L_i} := \mathbf{v}_i$.

2) For $\overline{\mathbf{v}} \leftrightarrow (\mathbf{v}_i)_i$ as above, the decomposition groups $Z_{\mathbf{v}_i|\mathbf{v}} \subseteq G_i$ are precisely the stabilizers of $Z_{\mathbf{v}_i}$ in G_i. And $(Z_{\mathbf{v}_i|\mathbf{v}})_i$ is a surjective projective subsystem of $(G_i)_i$, which has $Z_{\overline{\mathbf{v}}} \subset G_K$ as a projective limit. Thus one can recover $Z_{\overline{\mathbf{v}}} \subset G_K$ from the system of group extensions $1 \to \Pi_{L_i} \to \mathrm{Gal}(L'_i|K) \to \mathrm{Gal}(L_i|K) = G_i \to 1$ endowed with $(Z_{\mathbf{v}_i} \subset \Pi_{L_i})_i$.

3) Concerning the description of the inertia group $T_{\overline{\mathbf{v}}} \subseteq Z_{\overline{\mathbf{v}}}$ one has: $T_{\overline{\mathbf{v}}} \subseteq Z_{\overline{\mathbf{v}}}$ is the unique maximale solvable subgroup of $Z_{\overline{\mathbf{v}}}$, and $T_{\overline{\mathbf{v}}}$ has the following structure: The ramification subgroup $V_{\overline{\mathbf{v}}} \subset T_{\overline{\mathbf{v}}}$ is non-trivial iff $\mathrm{char}(K\mathbf{v}) > 0$. If so, then $V_{\overline{\mathbf{v}}}$ is the unique non-abelian Sylow subgroup of $T_{\overline{\mathbf{v}}}$, and if $V_{\overline{\mathbf{v}}}$ is a pro-p-group then $\mathbf{v}$ is a quasi prime r-divisor of $K|k$ if and only if $T_{\overline{\mathbf{v}}}/V_{\overline{\mathbf{v}}} \cong (\widehat{\mathbb{Z}}/\mathbb{Z}_p)^r$.

4) One has a canonical exact sequence of the form $1 \to T_{\overline{\mathbf{v}}} \to Z_{\overline{\mathbf{v}}} \to G_{K\mathbf{v}} \to 1$, where $K\mathbf{v}|k\mathbf{v}$ is the residue function field of $\mathbf{v}$. In particular, $\mathbf{v}$ is maximal among the quasi prime divisors of $K|k$ iff $K\mathbf{v} = k\mathbf{v}$ iff $G_{K\mathbf{v}}$ is trivial iff $Z_{\overline{\mathbf{v}}} = T_{\overline{\mathbf{v}}}$ iff $Z_{\overline{\mathbf{v}}}$ is prosolvable.

5) We finally notice that the canonical projection $\overline{p}_{L_i} : G_{L_i} \to \Pi_{L_i}$ maps $T_{\overline{\mathbf{v}}_i} \subseteq Z_{\overline{\mathbf{v}}_i}$ onto $T_{\mathbf{v}_i} \subseteq Z_{\mathbf{v}_i}$ for all L_i. In particular, $\overline{p}_K : G_K \to \Pi_K$ maps $T_{\overline{\mathbf{v}}} \subseteq Z_{\overline{\mathbf{v}}}$ onto $T_{\mathbf{v}} \subseteq Z_{\mathbf{v}}$.

Next let $\overline{\Phi} : G_K \to G_K$ be an automorphism of G_K. Since $(G_{L_i})_i$ is the system of all the open normal subgroups of G_K, it follows that $\overline{\Phi}$ maps each G_{L_i} isomorphically onto some G_{M_i}, where $(M_i)_i$ is just a degree preserving permutation of the $(L_i)_i$.

6) Since the kernels of the canonical projections $\overline{p}^{\mathrm{c}}_K : G_K \to \Pi^{\mathrm{c}}_K$ and $\overline{p}_K : G_K \to \Pi_K$ are characteristic in G_K, it follows that $\overline{\Phi}$ gives rise to isomorphisms $\Phi^{\mathrm{c}}_K : \Pi^{\mathrm{c}}_K \to \Pi^{\mathrm{c}}_K$ and $\Phi_L : \Pi_K \to \Pi_K$ such that Φ is the abelianization of Φ^{c}.

7) Moreover, if $L|K$ is one of the $L_i|K$, and $M|K$ is the corresponding $M_i|K$, the same is true correspondingly for each of the canonical projections $\overline{p}^{\mathrm{c}}_L : G_L \to \Pi^{\mathrm{c}}_L$, $\overline{p}_L : G_L \to \Pi_L$, and $\overline{p}^{\mathrm{c}}_M : G_M \to \Pi^{\mathrm{c}}_M$, $\overline{p}_M : G_M \to \Pi_M$. Moreover, $\overline{\Phi} : G_K \to G_K$ gives rise to isomorphisms $\Phi^{\mathrm{c}}_L : \Pi^{\mathrm{c}}_L \to \Pi^{\mathrm{c}}_M$, $\Phi_L : \Pi_L \to \Pi_M$ which satisfy: $\Phi^{\mathrm{c}}_L \circ p^{\mathrm{c}}_M = \overline{\Phi} \circ p^{\mathrm{c}}_M$, respectively $\Phi_L \circ p_L = \overline{\Phi} \circ p_M$.

8) If $p^{\mathrm{c}}_L : \Pi^{\mathrm{c}}_L \to \Pi^{\mathrm{c}}_K$ and $p_L : \Pi_L \to \Pi_K$ are the canonical projections, then $\overline{p}^{\mathrm{c}}_K = p^{\mathrm{c}}_L \circ p^{\mathrm{c}}_L$ and $\overline{p}_K = p_L \circ \overline{p}_L$, and correspondingly for $M|K$. Finally Φ^{c}_L and Φ_L are compatible

with $\overline{p}^{\mathrm{c}}_L$ and $\overline{p}^{\mathrm{c}}_M$, respectively, $\overline{p}_L$ and $\overline{p}_M$, i.e., one has commutative diagrams:

$$(*) \qquad \begin{array}{ccc} \Pi^{\mathrm{c}}_L & \xrightarrow{\Phi^{\mathrm{c}}_L} & \Pi^{\mathrm{c}}_M \\ \downarrow p^{\mathrm{c}}_L & & \downarrow p^{\mathrm{c}}_M \\ \Pi^{\mathrm{c}}_K & \xrightarrow{\Phi^{\mathrm{c}}} & \Pi^{\mathrm{c}}_K \end{array} \qquad\qquad \begin{array}{ccc} \Pi_L & \xrightarrow{\Phi_L} & \Pi_M \\ \downarrow p_L & & \downarrow p_M \\ \Pi_K & \xrightarrow{\Phi} & \Pi_K \end{array}$$

9) Since Φ_{L_i} is the abelianization of $\Phi^{\mathrm{c}}_{L_i}$, by the characterization of the quasi r-divisorial subgroups, see Pop [P4], especially Proposition 3.5, one has: Let $T_{\mathbf{v}_i} \subseteq Z_{\mathbf{v}_i}$ be a quasi r-divisorial subgroup for $L_i|k$. Then $\Phi_{L_i}(T_{\mathbf{v}_i}) \subseteq \Phi_{L_i}(Z_{\mathbf{v}_i})$ is a quasi r-divisorial subgroup of Π_{M_i}, say equal to $T_{\mathbf{w}_i} \subseteq Z_{\mathbf{w}_i}$ for some quasi prime divisor $\mathbf{w}_i$ of $M_i|k$. Let $\mathbf{v}$, $\mathbf{w}$ be the restrictions of $\mathbf{v}_i$, $\mathbf{w}_i$ to K, respectively. Then $p_{L_i}(T_{\mathbf{v}_i}) \subseteq p_{L_i}(Z_{\mathbf{v}_i})$ are open subgroups in $T_{\mathbf{v}} \subseteq Z_{\mathbf{v}}$, respectively, and $p_{L_i}(T_{\mathbf{w}_i}) \subseteq p_{M_i}(Z_{\mathbf{w}_i})$ are open subgroups in $T_{\mathbf{w}} \subseteq Z_{\mathbf{w}}$. And $\Phi_K : \Pi_K \to \Pi_K$ maps $T_{\mathbf{v}} \subseteq Z_{\mathbf{v}}$ isomorphically onto $T_{\mathbf{w}} \subseteq Z_{\mathbf{w}}$.

10) Performing the above steps for every finite Galois extension $L_i|K$ and the corresponding $M_i|K$, it follows that the isomorphism $\Phi_{L_i|K} : \mathrm{Gal}(L_i|K) \to \mathrm{Gal}(M_i|K)$ induced by $\overline{\Phi}$ maps the stabilizer of $Z_{\mathbf{v}_i}$ in $\mathrm{Gal}(L_i|K)$ isomorphically onto the stabilizer of $Z_{\mathbf{w}_i}$ in $\mathrm{Gal}(M_i|K)$. Thus taking limits we get: If the system $(\mathbf{v}_i)_i$ is compatible, say defining a prolongation $\overline{\mathbf{v}}$ of $\mathbf{v}$ to $\overline{K}$, then the system $(\mathbf{w}_i)_i$ is compatible as well, and defines a prolongation $\overline{\mathbf{w}}$ of $\mathbf{w}$ to $\overline{K}$, and $\overline{\Phi}$ maps $T_{\overline{\mathbf{v}}} \subseteq Z_{\overline{\mathbf{v}}}$ isomorphically onto $T_{\overline{\mathbf{w}}} \subseteq Z_{\overline{\mathbf{w}}}$.

B) *Proof of assertion* 1) *of Theorem 1.2*

The proof uses in an essential way Theorem 1.1. Reasoning as in the proof of asertion 1) of Theorem 1.1, we instantly see that assertion 1) of Theorem 1.2 is equivalent to the fact that for every $\overline{\Phi} \in \mathrm{Aut}_{U_0}(G_K)$ there exists an automorphism $\overline{\phi}$ of $\overline{K}$ which is trivial on $k_0(X)$ and defines $\overline{\Phi}$ by $g \mapsto \overline{\phi}^{-1} g\, \overline{\phi}$ for all $g \in G_K$. Note that if $\overline{\phi}$ exsits, then $\overline{\phi}$ is unique, because $\mathrm{char}(k) = 0$, thus $k_0(X) \hookrightarrow K$ is an Galois extension.

- Let $\overline{\Phi} \in \mathrm{Aut}_{U_0}(G_K)$ be given.

First recall that since the kernels of the homomorphisms $G_K \to \Pi^{\mathrm{c}}_K \to \Pi_K$ are characteristic, one has canonical homomorphisms $\mathrm{Aut}(G_K) \to \mathrm{Aut}(\Pi^{\mathrm{c}}_K) \to \mathrm{Aut}(\Pi_K)$, and furthermore, the image of $\mathrm{Aut}(G_K) \to \mathrm{Aut}(\Pi_K)$ equals the image of $\mathrm{Out}(G_K) \to \mathrm{Aut}(\Pi_K)$, and these images are contained in $\mathrm{Aut}^{\mathrm{c}}(\Pi_K)$. And obviously, directly from the definition it follows that since $\overline{\Phi} \in \mathrm{Aut}_{U_0}(G_K)$, its image $\Phi \in \mathrm{Aut}^{\mathrm{c}}(\Pi_K)$ lies actually in $\mathrm{Aut}^{\mathrm{c}}_{U_0}(\Pi_K)$. Hence by Theorem 1.1, 1), it follows that there exists (a unique) $\tau \in \mathrm{Gal}_{k_0}$ which defines Φ, i.e., there exists $\epsilon \in \mathbb{Z}_\ell^\times$ such that if ϕ is a prolongation of τ to $K = k(X)$ and ϕ' is a prolongation of ϕ to K', then $\epsilon \cdot \Phi(g) = \phi'^{-1} g\, \phi'$ for all $g \in \Pi_K$. Thus letting $\overline{\Phi}_\tau$ be any prolongation of ϕ' to $\overline{K}$, it follows that $\overline{\Phi}_\tau \in \mathrm{Aut}_{U_0}(G_K)$ equals $\rho_{\mathcal{V}_X}(\tau)$ up to inner G_K-conjugation, and $\overline{\Phi} \circ \overline{\Phi}_\tau^{-1}$ has a trivial image in $\mathrm{Aut}^{\mathrm{c}}_{U_0}(\Pi_K)$. Equivalently, by the definition of ϕ, it follows that $\overline{\Phi} \circ \overline{\Phi}_\tau^{-1}$ is the multiplication by ϵ^{-1} on Π_K. Thus replacing $\overline{\Phi}$ by $\overline{\Phi} \circ \overline{\Phi}_\tau^{-1} \in \mathrm{Aut}_{U_0}(G_K)$, assertion 1) is equivalent to the following:

Main Claim. *Let $\overline{\Phi} \in \mathrm{Aut}(G_K)$ be such that its image $\Phi \in \mathrm{Aut}(\Pi_K)$ is $\Phi = \epsilon^{-1} \cdot \mathrm{id}$ for some $\epsilon \in \mathbb{Z}_\ell^\times$. Then $\overline{\Phi}$ is the conjugation by some $\overline{\phi} \in G_K$ on G_K, and in particular $\epsilon = 1$.*

In order to prove the Main Claim, we first notice that by Remarks 5.2, 5), above, in the notations from there it follows that for every generalized (quasi) prime r-divisor $\mathbf{v}$ of $K|k$ and some prolongation $\overline{\mathbf{v}}$ to $\overline{K}$ one has: $p_K(T_{\overline{\mathbf{v}}}) = T_{\mathbf{v}}$ and $p_K(Z_{\overline{\mathbf{v}}}) = Z_{\mathbf{v}}$. Thus denoting by $\mathbf{w}$ and $\overline{\mathbf{w}}$ the generalized (quasi) prime r-divisor $K|k$ and its prolongation to $\overline{K}$ with $\overline{\Phi}(T_{\overline{\mathbf{v}}}) = T_{\overline{\mathbf{w}}}$ and $\overline{\Phi}(Z_{\overline{\mathbf{v}}}) = Z_{\overline{\mathbf{w}}}$ one has: $p_K(Z_{\overline{\mathbf{w}}}) = \Phi(Z_{\mathbf{v}}) = \epsilon^{-1} \cdot Z_{\mathbf{v}} = Z_{\mathbf{v}}$. We therefore conclude that $\mathbf{v} = \mathbf{w}$, thus $\overline{\mathbf{w}}$ is itself a prolongation of $\mathbf{w} = \mathbf{v}$ to $\overline{K}$. In other words, for every generalized (quasi) prime divisor $\mathbf{v}$ of $K|k$, the automorphism $\overline{\Phi}$ maps the conjugacy class of inertia/decomposition groups $T_{\overline{\mathbf{v}}}^{\sigma} \subseteq Z_{\overline{\mathbf{v}}}^{\sigma}$, $\sigma \in G_K$, onto itself.

Now let $X \to k$ be a projective smooth model of $K|k$ (which exists by our hypothesis that the characteristic is zero). Let $U \subset X$ be a Zariski open subset. Then the maximal Galois sub-extension $\mathcal{K}_U|K$ of $\overline{K}|K$ which is unramified over U has Galois group isomorphic to $\pi_1(U)$, thus topologically finitely generated. Hence for every bound N, there exist only finitely many open normal subgroups $\Delta \subset \pi_1(U)$ such that $|\pi_1(U)/\Delta| \leq N$. Hence the intersection $\Delta_N := \cap_{\Delta} \Delta$ of all such Δ is a characteristic subgroup of $\pi_1(U)$.

Using this we get: Let $L_i|K$ be a finite Galois extension, and let $U_{L_i} \subset X$ be the open subset over which $L_i|K$ is unramified. If $M_i|K$ is the Galois extension corresponding to $L_i|K$ under $\overline{\Phi}$, then $[L_i : K] = [M_i : K]$. Further, by the discussion above, for $\mathbf{v}$ a prime divisor of $K|k$ defined by some Weil prime divisor of X one has: $\mathbf{v}$ is unramified in $L_i|K$ iff the prime divisor $\mathbf{w}$ corresponding to $\mathbf{v}$ under $\overline{\Phi}$ is unramified in $M_i|K$. On the other hand, by the discussion above, one has $\mathbf{v} = \mathbf{w}$, hence we conclude that $\mathbf{v}$ is unramified in $M_i|K$. Therefore, $\mathrm{Gal}(M_i|K)$ is a quotient of $\pi_1(U_{L_i})$ and has order $N_{L_i} := [L_i : K]$. Thus $G_K \to \mathrm{Gal}(M_i|K)$ factors through $G_K \to \pi_1(U_{L_i}) \to \pi_1(U_{L_i})/\Delta_{L_i}$. Clearly, this is the case for all the powers $\overline{\Phi}^n$ of $\overline{\Phi}$, thus finally $\ker\big(G_K \to \pi_1(U_{L_i})/\Delta_{L_i}\big)$ is an open subgroup of G_K which is mapped by $\overline{\Phi}$ onto itself and is contained in G_{L_i}.

We conclude that there there exists an inductive system $(G_\mu)_\mu$ of open normal subgroups of G_K having $\cap_\mu G_\mu = 1$ such that $\overline{\Phi}(G_\mu) = G_\mu$ for all μ. For every G_μ, let $K_\mu|K$ be the finite Galois sub-extension of $\overline{K}|K$ with $G_{K\mu} = G_\mu$, hence $\overline{K} = \cup_\mu K_\mu$. Further, $\overline{\Phi}$ gives rise to a compatible system of automorphisms $\Phi_\mu : \mathrm{Gal}(K_\mu|K) \to \mathrm{Gal}(K_\mu|K)$, and $\overline{\Phi}$ is the projective limit of the system $(\Phi_\mu)_\mu$.

To fix and simplify notations, let $L := K_\mu$ be fixed. Then $\overline{\Phi}$ gives rise to a group automorphism $\Psi : \Pi_L \to \Pi_L$ which together with the canonical group projection $pr : \Pi_L \to \Pi_K$ give rise to an isomorphism $\Psi : \mathcal{G}_{\mathcal{D}_L^{\mathrm{tot}}} \to \mathcal{G}_{\mathcal{D}_L^{\mathrm{tot}}}$, respectively a quotient $pr : \mathcal{G}_{\mathcal{D}_L^{\mathrm{tot}}} \to \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}}$, of total decomposition graphs, and the Kummer isomorphism $\psi : \widehat{L} \to \widehat{L}$ attached to Ψ, and the inclusion $\imath : \widehat{K} \to \widehat{L}$ attached to pr, fit into the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{G}_{\mathcal{D}_L^{\mathrm{tot}}} & \xrightarrow{\Psi} & \mathcal{G}_{\mathcal{D}_L^{\mathrm{tot}}} \\ \big\downarrow{\scriptstyle pr} & & \big\downarrow{\scriptstyle pr} \\ \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} & \xrightarrow{\epsilon^{-1}} & \mathcal{G}_{\mathcal{D}_K^{\mathrm{tot}}} \end{array} \qquad\qquad \begin{array}{ccc} \widehat{K} & \xrightarrow{\epsilon} & \widehat{K} \\ \big\downarrow{\scriptstyle \imath} & & \big\downarrow{\scriptstyle \imath} \\ \widehat{L} & \xrightarrow{\psi} & \widehat{L} \end{array}$$

in which ϵ^{-1} and ϵ are the multiplication morphisms defined by ϵ^{-1} respectively ϵ, where $\epsilon \in \mathbb{Z}_\ell^\times$ is the one introduced above in the Main Claim.

Claim. Ψ *is compatible with rational quotients.*

Let $Y \to k$ be a projective normal model of $L|k$ on which $\mathrm{Gal}(L|K)$ acts. Without loss of generality, we can suppose that Y is complete regular like (in the sense of the discussion in Remark 3.2, 4), and that the quotient $X := \mathrm{Gal}(K_\mu|K)\backslash Y$ of Y by $\mathrm{Gal}(K_\mu|K)$ is a complete regular model for $K|k$.

Let $\tilde{u} \in L$ be such that its $\mathrm{Gal}(L|K)$-conjugates $(\tilde{u}_\theta)_\theta$ are k-linearly independent. Let $t \in K$ be non-constant such that the pole divisor $(\tilde{u})_\infty$ of $\tilde{u}$ is contained in the pole divisor $(t)_\infty$ of t, and t is not in the k-subspace generated by $(\tilde{u}_\theta)_\theta$. Then for almost all $c \in k$, the $\mathrm{Gal}(L|K)$-conjugates $u_\theta = \theta(u)$ of $u := \tilde{u}/t + c \in L$ have the properties:

a) u_θ are general element of L, i.e., $\kappa_{u_\theta} := k(u_\theta)$ are relatively algebraically closed in L.

b) $(u_\theta)_\theta$ are k-linearly independent.

c) The pole divisor of u_θ is $(u_\theta)_\infty = (t)_0$ thus it lies in the image of $\mathrm{Div}(X) \to \mathrm{Div}(Y)$.

Next recall that denoting by $\mathbb{P}^1_{u_\theta}$ the projective u_θ-line over k, the embedding $k(u_\theta) \hookrightarrow L$ is defined by a k-rational map $\varphi_\theta : Y \dashrightarrow \mathbb{P}^1_{u_\theta}$, say with domain some open subset $U_\theta \subset Y$. Notice that if $\theta' = \tau\theta$ in $\mathrm{Gal}(L|K)$, then $u_{\theta'} = \tau(u_\theta)$, and $\varphi_{\theta'} = \varphi_\theta^\tau$. Thus in particular, $U_{\theta'} = U_\theta^\tau$. Therefore, setting $V := \cap_\theta U_\theta$, it follows that $\mathrm{Gal}(L|K)$ acts on $V \subset Y$, and all the rational maps φ_θ are defined on V. Finally, since $Y\backslash V$ is $\mathrm{Gal}(K_\mu|K)$-invariant, after performing a properly chosen sequence of $\mathrm{Gal}(K_\mu|K)$-invariant blowups and normalizing again, one can suppose that $V = Y$, i.e., the rational maps φ_θ are actually morphisms $\varphi_\theta : Y \to \mathbb{P}^1_{u_\theta}$. Now since the geometric generic fiber of each φ_θ is integral –which is equivalent to the fact that $k(u_\theta)$ is relatively algebraically closed in L, it follows that the fibers of the k-morphisms $\varphi_\theta : Y \to \mathbb{P}^1_{u_\theta}$ are geometrically integral on an open subset $U \subset \mathbb{P}^1_{u_\theta}$. This means that for all $a \in U(k)$ the fiber $X_{\theta a} \subset Y$ of φ_θ at $u_\theta = a$ is a geometrically integral Weil prime divisors of Y for each $\theta \in \mathrm{Gal}(L|K)$. In other words, the Y-divisor of the function $u_\theta - a$ is of the form $(u_\theta - a) = v_{\theta a} - (t)_0$ with $(t)_0$ in the image of $\mathrm{Div}(X) \to \mathrm{Div}(Y)$ and $v_{\theta a}^\tau = v_{\theta' a}$ if $\theta' = \tau\theta$ in $\mathrm{Gal}(L|K)$.

Now let $\jmath_K : K^\times \to \widehat{K}$ and $\jmath_L : L^\times \to \widehat{L}$ be the ℓ-adic completion homomorphisms, and recall that the canonical divisorial $\widehat{U}_L$ lattice $\mathcal{L}_L \subset \widehat{K}$ is the unique divisorial $\widehat{U}_L$ lattice in $\widehat{L}$ which contains $\jmath_L(L^\times)$. Further, for every divisorial $\widehat{U}_L$-lattice $\mathcal{L}'_L$, there exists $\eta \in \mathbb{Z}_\ell^\times$ unique up to multiplication with rational ℓ-adic units such that $\mathcal{L}'_K$ has a non-trivial intersection with $\eta \cdot \jmath_K(K^\times)$, and if so, then $\eta \cdot \jmath_K(K^\times) \subseteq \mathcal{L}'_K$. And clearly, $\eta \cdot \jmath_K(K^\times) \subseteq \mathcal{L}'_K$ if and only if $\eta \cdot \jmath_L(L^\times) \subseteq \mathcal{L}'_L$.

Next recall that via $p : \Pi_L \to \Pi_K$ one gets a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \to & \widehat{U}_K & \to & \widehat{K} & \xrightarrow{\mathrm{div}} & \widehat{\mathrm{Div}}(X)_{(\ell)} & \to & \widehat{\mathbf{Cl}}(X) \\
 & & \downarrow\imath & & \downarrow\imath & & \downarrow\mathrm{div}_p & & \downarrow\mathrm{can}_p \\
1 & \to & \widehat{U}_L & \to & \widehat{L} & \xrightarrow{\mathrm{div}} & \widehat{\mathrm{Div}}(Y)_{(\ell)} & \to & \widehat{\mathbf{Cl}}(Y)
\end{array}$$

and that via Ψ one gets a commutative diagram with exact rows of the form:

$$\begin{array}{ccccccccc}
1 & \to & \widehat{U}_L & \to & \widehat{L} & \xrightarrow{\mathrm{div}} & \widehat{\mathrm{Div}}(Y) & \to & \widehat{\mathbf{Cl}}(Y) \\
 & & \downarrow\psi & & \downarrow\psi & & \downarrow\mathrm{div}_\Psi & & \downarrow\mathrm{can}_\Psi \\
1 & \to & \widehat{U}_L & \to & \widehat{L} & \xrightarrow{\mathrm{div}} & \widehat{\mathrm{Div}}(Y) & \to & \widehat{\mathbf{Cl}}(Y)
\end{array}$$

where div_Ψ is the canonical abstract divisor map defined by Ψ and can_Ψ is the canonical isomorphism making the diagram commutative. We proceed as follows:

Step 1. Since $\operatorname{div}\big(\jmath_L(u_\theta - a)\big) = v_{\theta a} - (t)_0$, and $(t)_0$ lies in the image of $\operatorname{Div}(X) \to \operatorname{Div}(Y)$, from the commutativity of the above diagrams, it follows that the divisor of $\psi\big(\jmath_L(u_\theta - a)\big)$ is of the form $\eta \cdot w - \epsilon \cdot (t)_0$ with w a prime divisor of $L|k$ which is $\operatorname{Gal}(L|K)$-conjugate to $v_{\theta a}$, and $\eta \in \mathbb{Z}_\ell^\times$. Since the conjugates of $v_{\theta a}$ are precisely all the $v_{\tau a}$ with $\tau \in \operatorname{Gal}(K_\mu|K)$, it follows that $w = v_{\tau a}$ for some τ. Since $v_{\tau a} - (t)_0 = (u_\tau - a)$ is a principal divisor, its image in $\widehat{\mathbf{Cl}}(X)$ is trivial, hence the image of

$$\eta \cdot w - \epsilon \cdot (t)_0 = (\eta - \epsilon) \cdot v_{\tau a} + \epsilon \cdot \big(v_{\tau a} - (t)_0\big) = (\eta - \epsilon) \cdot v_{\tau a} + \epsilon \cdot (u_\tau - a)_\infty$$

in $\widehat{\mathbf{Cl}}(X_\mu)$ equals $(\eta - \epsilon)[v_{\tau a}]$, where $[v_{\tau a}]$ is the image of $v_{\theta a}$ in $\widehat{\mathbf{Cl}}(X_\mu)$. On the other hand, $v_{\theta a} - (t)_0 = (u_\theta - a)$ is principal and is the divisor of $\jmath_L(u_\theta - a)$, hence it has a trivial image in $\widehat{\mathbf{Cl}}(X)$. Thus by the commutativity of the diagram above, it follows that $(\eta - \epsilon)[v_{\tau a}] = 0$. Since $[v_{\tau a}] \neq 0$, we conclude that $\eta = \epsilon$. Therefore, we get:

$$\psi\big(\jmath_L(u_\theta - a)\big) = \epsilon \cdot \big(\boldsymbol{u}\, \jmath_L(u_\tau - a)\big)$$

for some $\boldsymbol{u} \in \widehat{U}_{K_\mu}$. Notice that for a fixed θ, the elements τ as well as $\boldsymbol{u}$ could *anteriori* depend on $a \in U(k)$. We claim that actually for every θ there exists some τ such that $\psi\big(\jmath_L(u_\theta - a)\big) = \epsilon \cdot \jmath_L(u_\tau - a)$ for all $a \in U(k)$ and $\psi\big(\jmath_L(\kappa_{u_\theta}^\times)\big) = \epsilon \cdot \jmath_L(\kappa_{u_\tau}^\times)$. Indeed, since the set of all the $a \in U(k)$ is infinite, there exists $\tau \in \operatorname{Gal}(L|K)$ such that the set

$$\Sigma_\tau := \{a \in U(k) \mid \psi\big(\jmath_L(u_\theta - a)\big) = \epsilon \cdot \big(\boldsymbol{u}\, \jmath_L(u_\tau - a)\big) \text{ for some } \boldsymbol{u} \in \widehat{U}_L$$

is infinite. For a fixed $b \in k$ let $\mathsf{x} = \jmath_L(u_\theta - b)$ and $\boldsymbol{x} := \psi(\mathsf{x})$. To simplify notations, for $a \in U(k)$ and $\theta, \tau \in \operatorname{Gal}(K_\mu|K)$, let $\kappa_{\theta a}$ be the residue field of $v_{\theta a}$, and let $\jmath_{\theta a} : \widehat{U}_{v_{\theta a}} \to \widehat{\kappa}_{\theta a}$ be the reduction homomorphism as introduced at the beginning of section 3. Further define $\kappa_{\tau a}$ and $\jmath_{\tau a} : \widehat{U}_{v_{\tau a}} \to \widehat{\kappa}_{\tau a}$ correspondingly. Then for all $a \in U(k)$ with $a \neq b$ one has: x is a $v_{\theta a}$-unit with $v_{\theta a}$-residue equal to $a - b \in k^\times$. Thus $\jmath_{\theta a}(\mathsf{x}) = 1$. Now suppose that $a \in \Sigma$. Then, if $v_{\tau a}$ is the prime divisor of $K_\mu|k$ corresponding to $v_{\theta a}$ under Ψ, one has commutative diagrams of the form

$$\begin{array}{ccc} \widehat{U}_{v_{\theta a}} & \xrightarrow{\psi} & \widehat{U}_{v_{\tau a}} \\ \big\downarrow \jmath_{\theta a} & & \big\downarrow \jmath_{\tau a} \\ \widehat{\kappa}_{\theta a} & \xrightarrow{\hat{\phi}_{\theta a}} & \widehat{\kappa}_{\tau a} \end{array}$$

where $\hat{\phi}_{\theta a}$ is defined by the residual isomorphism $\Phi_{\theta a} : \Pi_{Kv_{\theta a}} \to \Pi_{Kv_{\tau a}}$. Hence $\jmath_{\theta a}(\mathsf{x}) = 1$ implies $\jmath_{\tau a}\big(\psi(\mathsf{x})\big) = \hat{\phi}_{\theta a}\big(\jmath_{\theta a}(\mathsf{x})\big) = 1$ for all $a \in \Sigma$. Next recall that by the discussion at the beginning of section 3, especially the proof of Proposition 3.1, for every $\boldsymbol{y} \in \widehat{L}_{\text{fin}} \backslash \widehat{\kappa}_{u_\tau}$ and almost all $a \in k$ one has: If $v_{\tau a}$ is the (unique) zero of $u_\tau - a$, then $\jmath_{\tau a}(\boldsymbol{y}) \neq 1$. In particular, since $\mathsf{x} = \jmath_L(u_\theta - a) \in \mathcal{L}_L \subset \widehat{L}_{\text{fin}}$ and $\boldsymbol{x} := \psi(\mathsf{x}) \in \widehat{L}_{\text{fin}}$ satisfy $\jmath_{\tau a}(\boldsymbol{x}) = \jmath_{\tau a}\big(\psi(\mathsf{x})\big) = 1$ for all $a \in \Sigma$, it follows that $\boldsymbol{x} \in \epsilon \cdot \mathcal{L}_L \cap \widehat{\kappa}_{u_\tau} = \epsilon \cdot \jmath_L(\kappa_{u_\tau}^\times)_{(\ell)}$. Since the set of all the $\mathsf{x} = \jmath_L(u_\theta - b)$ with $b \in k$ generate $\jmath_L(\kappa_{u_\theta}^\times)$, we conclude that $\psi\big(\jmath_L(\kappa_{u_\theta}^\times)\big) \subseteq \epsilon \cdot \jmath_L(\kappa_{u_\tau}^\times)$. By symmetry, the opposite inclusion holds too, thus finally $\psi\big(\jmath_L(\kappa_{u_\theta}^\times)\big) = \epsilon \cdot \jmath_L(\kappa_{u_\tau}^\times)$. Moreover, for all $a \in U(k)$ one has $\psi\big(\jmath_L(u_\theta - a)\big) = \epsilon \cdot \jmath_L(u_\tau - a)$.

Step 2. Let $z \in L$ be an arbitrary non-constant element. Let $y \in K$ be a general element in L such that the pole divisor of y contains the pole divisor of z. Then $y + d$ is a general element of L for all $d \in k$, and for almost all $d \in k$ one has: $z/(y + d)$ is a general element of L with pole divisor equal to the zero divisor $(y+d)_0$ of $y+d$. Finally, if $u = \tilde{u}/t + c$ is as considered above, then for almost all $c, d \in k$ one has: $x := (\tilde{u}/t + c)z/(y+d)$ is a general element of L, its pole divisor is $(t)_0 + (y+d)_0$, thus it lies in the image of $\mathrm{Div}(X) \to \mathrm{Div}(X_\mu)$. Further, the system of $\mathrm{Gal}(L|K)$-conjugates $(x_\rho)_\rho$ of x is linearly independent over k. Thus by the discussion from above, every x_ρ is a general element of L and for every ρ there exists σ such that $\psi\big(\jmath_L(\kappa_{x_\rho}^\times)_{(\ell)}\big) = \epsilon \cdot \jmath_L(\kappa_{x_\sigma}^\times)_{(\ell)}$. Since ψ is the identity on $\jmath_L(\kappa_t^\times)_{(\ell)} \subset \mathcal{L}'_K$, and $z = x(t+d)/u \in \kappa_x\, \kappa_t\, \kappa_u$, we finally conclude that there exists $\tau, \sigma \in \mathrm{Gal}(K_\mu|K)$ such that:

$$\psi\big(\jmath_L(z)\big) \in \epsilon \cdot \big(\jmath_L(\kappa_{x_\tau})_{(\ell)}\, \jmath_L(\kappa_t^\times)_{(\ell)}\, \jmath_L(\kappa_{u_\sigma}{}^\times)_{(\ell)}\big) \subset \epsilon \cdot \jmath_L(L^\times)_{(\ell)}.$$

Since z was arbitrary, we get $\psi\big(\jmath_L(L^\times)_{(\ell)}\big) \subseteq \epsilon \cdot \jmath_L(L^\times)_{(\ell)}$, thus ψ maps $\jmath_L(L^\times)_{(\ell)}$ isomorphically onto $\epsilon \cdot \jmath_L(L^\times)_{(\ell)}$. In particular, Ψ is compatible with rational projections, and the Claim above is proved.

Coming back to the proof of the Main Claim, recall that by the main result of Pop [P3], Introduction, it follows that there exists a unique field automorphism $\phi_L : L \to L$ and a unique $\eta \in \mathbb{Z}_\ell^\times$ such that $\eta \cdot \Psi$ is defined by ϕ_L, and in particular, $\psi = \eta^{-1} \cdot \hat{\phi}_L$ on $\widehat{L}$. Since ψ is the multiplication by ϵ on $\widehat{K} \subseteq \widehat{L}$, it follows that $\eta^{-1} \cdot \hat{\phi}_L$ is the multiplication by ϵ on $\widehat{K}$ too, hence $\hat{\phi}_L$ is the multiplication by $\epsilon\,\eta$ on $\widehat{K}$. Since $\hat{\phi}_L$ is the ℓ-adic completion of a field isomorphism, it follows that $\epsilon\,\eta = 1$. Thus finally $\epsilon^{-1} \cdot \Psi$ is defined by the field isomorphism ϕ_L of L, and ϕ_L is unique with this property.

Now let $K_\mu \subseteq K_\nu$ for some $G_\nu \subseteq G_\mu$ as in the Main Claim. Then reasoning as above, one concludes that $\epsilon^{-1} \cdot \Phi_\nu$ is defined by a unique field K-automorphism ϕ_ν of K_ν, and $\phi_\mu = \phi_\nu|_{K_\mu}$. Thus finally the compatible system of K-automorphisms $(\phi_\mu)_\mu$ give rise to a K-automorphism $\bar{\phi}$ of $\overline{K}$ defined by $\bar{\phi}|_{K_\mu} := \phi_\mu$ and each ϕ_μ is defined by $\epsilon^{-1} \cdot \Phi_\mu$.

Finally, in order to conclude the proof of the Main Claim, let $\overline{\Phi}_\phi$ be the automorphism of G_K defined by the $\bar{\phi}$-conjugation. Then $\overline{\Phi} \circ \overline{\Phi}_\phi^{-1}$ is an automorphism of G_K which induces on every Π_{K_μ} the multiplication by ϵ^{-1} for the fixed given $\epsilon \in \mathbb{Z}_\ell^\times$ independent of K_μ. We claim that $\epsilon = 1$. Indeed, let $\tilde{K} := K(\ell)$ be the maximal pro-ℓ sub-extension of $\overline{K}|K$. Then by a standard argument it follows that the automorphism of $G_K(\ell) := \mathrm{Gal}(\tilde{K}|K)$ defined by $\overline{\Phi} \circ \overline{\Phi}_\phi^{-1}$ maps every $g \in G_K(\ell)$ to its power $g^{\epsilon^{-1}}$. From this easily follows that $\epsilon = 1$. This concludes the proof the the Main Claim, thus of assertion 1) of Theorem 1.2.

References:

[An] André, Y., *On a geometric description of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q})$ *and a p-adic avatar of* $\widehat{GT}$, Duke Math. J. **119** (2003), 1–39.

[Be] Belyi, G. V., *On Galois extensions of a maximal cyclotomic field,* Mathematics USSR Izvestija, Vol. **14** (1980), no. 2, 247–256. (Original in Russian: Izvestiya Akademii Nauk SSSR, vol. **14** (1979), no. 2, 269–276.)

[Bo] Bogomolov, F. A., *On two conjectures in birational algebraic geometry,* in: Algebraic Geometry and Analytic Geometry, ICM-90 Satellite Conference Proceedings, eds A. Fujiki et all, Springer Verlag Tokyo 1991.

[BOU] Bourbaki, Algèbre commutative, Hermann Paris 1964.

[D1] Deligne, P., *Le groupe fondamental de la droite projective moins trois points,* in: Galois groups over **Q**, Math. Sci. Res. Inst. Publ. **16**, 79–297, Springer 1989.

[Dr] Drinfeld, V. G., *On quasi-triangular quasi-Hopf algebras and on a group that is closely connected with* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (Russian), Algebra i Analiz **2**, no. 4 (1990), 149–181; translation in: Leningrad Math. J. **2**, no. 4 (1991), 829–860.

[GGA] Geometric Galois Actions I, LMS LNS Vol **242**, eds L. Schneps – P. Lochak, Cambridge Univ. Press 1998.

[G1] Grothendieck, A., *Letter to Faltings, June 1983*, See [GGA].

[G2] Grothendieck, A., *Esquisse d'un programme, 1984*. See [GGA].

[H–M] Hain, R. and Matsumoto, M., *Tannakian fundamental groups associated to Galois groups,* in: Galois groups and Fundamental groups, ed. Schneps, MSRI Pub. Series **41**, 2003, pp.183–216.

[H–Sch] Harbater, D. and Schneps, L., *Fundamental groups of moduli and the Grothendieck-Teichmüller group,* Trans. AMS, Vol. **352** (2000), 3117-3148.

[H–M] Hoshi, Y. and Mochizuki, Sh., *On the combinatorial anabelian geometry of nodally nondegenerate outer representations,* Hiroshima Math. J. **41**, Number 3 (2011), 275–342.

[I1] Ihara, Y., *On Galois represent. arising from towers of covers of* $\mathbb{P}^1\backslash\{0,1,\infty\}$, Invent. Math. **86** (1986), 427–459.

[I2] Ihara, Y., *Braids, Galois groups, and some arithmetic functions,* Proceedings of the ICM'90, Vol. I, II, Math. Soc. Japan, Tokyo, 1991, pp.99–120.

[I3] Ihara, Y., *On beta and gamma functions associated with the Grothendieck-Teichmller group* II, J. reine angew. Math. **527** (2000), 1–11.

[I–M] Ihara, Y. and Matsumoto, M., *On Galois actions on profinite completions of braid groups,* in: Recent developments in the inverse Galois problem (Seattle, WA, 1993), 173–200, Contemp. Math. **186** AMS Providence, RI, 1995.

[I–N] Ihara, Y. and Nakamura, H., *Some illustrative examples for anabelian geometry in high dimensions,* in: Geometric Galois actions I, 127–138, LMS LNS **242**, Cambridge Univ. Press, Cambridge, 1997.

[Kh] Koch, H., *Die Galoissche Theorie der p-Erweiterungen,* Math. Monogr. **10**, Berlin 1970.

[L–Sch] Lochak, P., and Schneps, L., *A cohomological interpretation of the Grothendieck-Teichmüller group. Appendix by C. Scheiderer,* Invent. Math. **127** (1997), 571–600.

[Ma] Matsumoto, M., *Galois representations on profinite braid groups on curves,* J. reine angew. Math. **474** (1996), 169–219.

[Mu] Mumford, D., The red book of varieties and schemes, LNM 1358, 2nd edition, Springer Verlag 1999.

[N–Sch] Nakamura, H. and Schneps, L., *On a subgroup of the Grothendieck–Teichmüller group acting on the profinite Teichmüller modular group,* Invent. Math. **141** (2000), 503–560.

[Ne] Neukirch, J., *Über eine algebraische Kennzeichnung der Henselkörper,* J. reine angew. Math. **231** (1968), 75–81.

[P1] Pop, F., *Pro-l abelian-by-central Galois theory of Zariski prime divisors,* Israel J. Math. **180** (2010), 43–68.

[P2] Pop, F., *Inertia elements versus Frobenius elements,* Math. Annalen **438** (2010), 1005–1017.

[P3] Pop, F., *Recovering fields from their decomposition graphs,* in: Number theory, Analysis and Geometry –In memory of Serge Lang, Springer special Volume 2010; eds: Goldfeld, Jorgenson, Jones, Ramakrishnan, Ribet, Tate; pp. 519–594.

[P4] Pop, F., *On the birational anabelian program initiated by Bogomolov I,* Invent. Math. **187** (2012), 511–533.

[Ro] Roquette, P., *Zur Theorie der Konstantenreduktion algebraischer Mannigfaltigkeiten,* J. reine angew. Math. **200** (1958), 1–44.

[Sz] Szamuely, T., *Groupes de Galois de corps de type fini (d'après Pop),* Astérisque **294** (2004), 403–431.

[Uch] Uchida, K., *Isomorphisms of Galois groups of solvably closed Galois extensions,* Tôhoku Math. J. **31** (1979), 359–362.

[Z–S] Zariski, O. and Samuel, P., Commutative Algebra, Vol II, Springer-Verlag, New York, 1975.

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