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Inertia elements versus Frobenius elements

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Introduction

Recall the generalized Chebotarev's Density Theorem, see Serre [Se], which is one of the very fundamental facts in arithmetic geometry:

Theorem (Generalized Dirichlet Density).
Let $f : Y \to X$ be a generically finite and Galois morphism of integral separated schemes of finite type over $\mathbb{Z}$. Let K and L denote the function fields of X, respectively Y, and let $G := \mathrm{Gal}(L|K)$ be the group of (rational) automorphisms of Y over X. Then the following hold:

1) *There exists an open sub-scheme $U \subset X$ such that f is étale above U, and if $V = f^{-1}(U)$, then V is an open sub-scheme of Y, and $f : V \to V$ is an étale cover.*

2) *For every $\sigma \in G$, the set of all closed points $x \in U$ such that Frob_x is conjugated to σ has a Dirichlet density which equals $|\sigma^G|/|G|$.*

From this one gets a kind of "profinite variant" of the Chebotarev Density Theorem as follows: Let K be a finitely generated field (over its prime field). We consider normal models X of K, i.e., integral normal separated schemes of finite type over $\mathbb{Z}$ with function field equal to K. For such models X consider sets $\Sigma \subseteq X$ of closed points which have Dirichlet density equal to 1, which we call `Frobenius sets`. We remark that the set of all Frobenius sets is inductive, i.e., for given Frobenius sets $\Sigma'_{X'} \subseteq X'$ and $\Sigma''_{X''} \subseteq X''$ there exists a Frobenius set $\Sigma \subseteq X$ such that $\Sigma_X \subseteq \Sigma'_{X'} \cap \Sigma''_{X''}$. Indeed, since X' and X'' are birationally equivalent, there exist affine open subsets $U' \subseteq X'$ and $U'' \subseteq X''$ which are isomorphic. Note that U' and U'' are normal models of K, and since their complements have dimensions strictly less than $\dim(X) =: d := \dim(X')$, it follows by Serre [Se], that $\Sigma'_{U'} := \Sigma'_{X'} \cap U'$ and $\Sigma''_{U''} := \Sigma_{X''} \cap U''$ have Dirichlet

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density equal to 1. Hence $\Sigma'_{U'}$ and $\Sigma''_{U''}$ are Frobenius sets. Hence identifying U and U', say by setting $U' =: X := U''$, and denoting $\Sigma := \Sigma'_{U'} \cap \Sigma''_{U''} \subseteq X$, we get the desired result.

For every Galois extension $\tilde{K}|K$, and a given Frobenius set $\Sigma \subset X$, we define a `set of Frobenius lifts` $\mathfrak{Frob}_\Sigma(\tilde{K})$ in $\mathrm{Gal}(\tilde{K}|K)$ as follows: First let $\tilde{X} \to X$ be the normalization of X in the field extension $\tilde{K}|K$. For every point $x \in \Sigma$, let $(\mathcal{O}_x, \mathfrak{m}_x)$ be its local ring, and $\kappa_x = \mathcal{O}_x/\mathfrak{m}_x$ be the residue field at x, hence κ_x is a finite field. Let $\tilde{x}$ be a point of $\tilde{X}$ above x, and $T_{\tilde{x}|x} \subset Z_{\tilde{x}|x}$ the inertia, respectively decomposition, groups of $\tilde{x}|x$. Then one has a canonical exact sequence of profinite groups

$$1 \to T_{\tilde{x}|x} \to Z_{\tilde{x}|x} \to \mathrm{Gal}(\kappa_{\tilde{x}}|\kappa_x) \to 1.$$

We define a `Frobenius lift` at x to be any fixed preimage $\sigma_x \in Z_{\tilde{x}|x}$ of the Frobenius element of $\mathrm{Gal}(\kappa_{\tilde{x}}|\kappa_x)$. In particular, if σ_x is a Frobenius lift at x, then $\sigma_x T_{\tilde{x}|x}$ is the set of all the Frobenius lifts at x. Further, we define a `set of Frobenius lifts` of $\mathrm{Gal}(\tilde{K}|K)$ to be any subset $\mathfrak{Frob}_\Sigma(\tilde{K}) \subset \mathrm{Gal}(\tilde{K}|K)$ which is closed under conjugation and contains a Frobenius lift σ_x for each point $x \in \Sigma$. Then one has the following:

THEOREM. *Let K be a finitely generated field, and $\tilde{K}|K$ a Galois field extension. Then every set of Frobenius lifts $\mathfrak{Frob}_\Sigma(\tilde{K}) \subset \mathrm{Gal}(\tilde{K}|K)$ is a dense subset in $\mathrm{Gal}(\tilde{K}|K)$.*

Our aim in this note is to study the behavior of the set of all the inertia elements $\mathfrak{In}(\tilde{K})$, respectively of all the tame inertia elements $\mathfrak{In.tm}(\tilde{K})$, respectively of all the ramification elements $\mathfrak{Rm}(\tilde{K})$, in $\mathrm{Gal}(\tilde{K}|K)$, see below the precise definitions. It turns out that contrary to the sets $\mathfrak{Frob}_\Sigma(\tilde{K})$, which are dense in $\mathrm{Gal}(\tilde{K}|K)$, each of the sets $\mathfrak{In}(\tilde{K})$, $\mathfrak{In.tm}(\tilde{K})$, $\mathfrak{Rm}(\tilde{K})$ is closed in $\mathrm{Gal}(\tilde{K}|K)$. But in arithmetical/geometrical situations, the set of all the tame divisorial inertia elements $\mathfrak{In.tm.div}(\tilde{K})$ is dense in the set of all the tame inertia elements $\mathfrak{In.tm}(\tilde{K})$, thus $\mathfrak{In.tm.div}(\tilde{K})$ behaves with respect to $\mathfrak{In.tm}(\tilde{K})$ in the same way as $\mathfrak{Frob}_\Sigma(\tilde{K})$ do with respect to $\mathrm{Gal}(\tilde{K}|K)$.

As an application, it turns out that this last assertion is a key technical point in a strategy for detecting the so called `decomposition graphs` of function fields $K|k$ with $\mathrm{td}(K|k) > 1$ and k an algebraic closure of a finite field. Detecting the decomposition graphs is an essential technical step in tackling the Program (initiated by Bogomolov) for recovering function fields $K|k$ as above from their pro-ℓ *abelian-by-central* Galois theory, provided $\mathrm{td}(K|k) > 1$. See e.g. POP [P] for more about this.

Thus let us give definitions and explain the matter in detail. See [BOU], Ch VI, and [Z–S], Vol. 2, for basic valuation theoretical background. Let K be an arbitrary field, and $\tilde{K}|K$ be some Galois field extension with Galois group $\mathrm{Gal}(\tilde{K}|K)$. Let v be a valuation of K. For $\tilde{v}$ a prolongation of v to $\tilde{K}$, we denote by $V_{\tilde{v}} \subseteq T_{\tilde{v}} \subseteq Z_{\tilde{v}}$ the ramification, respectively the inertia, respectively the decomposition, groups of $\tilde{v}$ in $\mathrm{Gal}(\tilde{K}|K)$. We denote by $\tilde{K}\tilde{v}$ and Kv the residue fields of $\tilde{v}$, respectively v. As in the case of local rings of (closed) points, the residue field extension $\tilde{K}\tilde{v}\,|\,Kv$ is a normal algebraic field extension (but in general not Galois). We set $G_{\tilde{v}} := \mathrm{Aut}(\tilde{K}\tilde{v}|Kv)$ and recall that one has an exact sequence of profinite groups of the form:

$$1 \to T_{\tilde{v}} \to Z_{\tilde{v}} \to G_{\tilde{v}} \to 1.$$

Further, $V_{\tilde{v}}$ is trivial if the residual characteristic $p := \mathrm{char}(Kv)$ is zero, respectively $V_{\tilde{v}}$ is the unique Sylow p-group of $T_{\tilde{v}}$ otherwise. An element $g \in T_{\tilde{v}}$ is called a v-`inertia element`, or an `inertia element` at v. And an element $g \in V_{\tilde{v}}$ is called a v-`ramification element`, or a `ramification element` at v. An inertia element $g \in T_{\tilde{v}}$ is called a `tame inertia element`, if it satisfies the following equivalent conditions:

i) The order of g (as a super natural number) is prime to $p := \mathrm{char}(Kv)$.

ii) The closed subgroup generated by σ has trivial intersection with $V_{\tilde{v}}$.

To fix some notations, we denote by $\mathfrak{Rm}(\tilde{K})$, and $\mathfrak{In}(\tilde{K})$, and $\mathfrak{In}.\mathfrak{tm}(\tilde{K})$, the sets of all the ramification, respectively inertia, respectively tame inertia, elements in $\mathrm{Gal}(\tilde{K}|K)$ at all the valuations v of K. The first fact we announce is the following:

THEOREM A. *Let $\tilde{K}|K$ be an arbitrary Galois extension of fields. Then the following hold:*

1) *The sets $\mathfrak{Rm}(\tilde{K})$, $\mathfrak{In}(\tilde{K})$, and $\mathfrak{In}.\mathfrak{tm}(\tilde{K})$, are closed in* $\mathrm{Gal}(\tilde{K}|K)$.

2) *More precisely, the following hold: Let $\Delta \subset \mathrm{Gal}(\tilde{K}|K)$ be a closed subgroup such that for every finite Galois sub-extension $K_i|K$ of $\tilde{K}|K$, there exists a valuation v_i on K_i such that: $\Delta|_{K_i} \subseteq V_{v_i}$, respectively $\Delta|_{K_i} \subseteq T_{v_i}$, respectively $\Delta|_{K_i} \subseteq T_{v_i}\backslash V_{v_i}$. Then there exists a valuation $\tilde{v}$ of $\tilde{K}$ such that $\Delta \subseteq V_{\tilde{v}}$, respectively $\Delta \subseteq T_{\tilde{v}}$, respectively $\Delta \subseteq T_{\tilde{v}}\backslash V_{\tilde{v}}$.*

The result above is a kind of "general non-sense" type result, and is proved as follows: Let $\mathrm{Val}(\tilde{K})$ be the space of all the valuations of $\tilde{K}$ endowed with the `patch topology`, and $\mathrm{Sbg}(\mathrm{Gal}(\tilde{K}|K))$ the space of all the closed subgroups of $\mathrm{Gal}(\tilde{K}|K)$ endowed with the `étale topology`, see Section 1 for the definitions. Then the maps sending each $\tilde{v} \in \mathrm{Val}(\tilde{K})$ to either $T_{\tilde{v}}$ or $V_{\tilde{v}}$ is continuous, and it turns out that the above Theorem is a reinterpretation of this fact. The corresponding assertion for the decomposition groups is also true, but not interesting because the decomposition group of the trivial valuation is the whole $\mathrm{Gal}(\tilde{K}|K)$. This is in some sense the reason why the Chebotarev Density Theorem is *possible* in the first place, and moreover, hard and *interesting*!

The next result is much more subtle, and does not follow by "general nonsense" type arguments: Let K be either a *finitely generated field,* or a *function field* $K|k$ over some base field k. Recall that in the case K is finitely generated, a normal model for K is any separated integral normal scheme of finite type over $\mathbb{Z}$, whose function field equals K; and in the case $K|k$ is a function field, a normal model of K is any normal variety X over k with function field K. In both cases one defines a `prime divisor` of K to be any discrete valuation $\mathfrak{v}$ of K whose valuation ring is the local ring of (the generic point of) a Weil prime divisor $X_1 \subset X$ of some normal model X of K. Note that $K\mathfrak{v}$ is the function field of X_1 viewed as a scheme, and the following hold: If K is a finitely generated field, then $K\mathfrak{v}$ is a finitely generated field, whereas if $K|k$ is a function field, then $K\mathfrak{v}\,|\,k$ is a function field over k as well. And in particular, X_1 is a model for $K\mathfrak{v}$, and the normalization of X_1 is a normal model for the function field $K\mathfrak{v}|k$. Coming back to inertia elements, in the above context we make the following definition: Let $\tilde{K}|K$ be an arbitrary Galois extension. We say that $g \in \mathrm{Gal}(\tilde{K}|K)$ is a `divisorial inertia element`, if g is an inertia element at some prime divisor $\mathfrak{v}$ of K as defined above. Finally, we denote by $\mathfrak{In.tm.div}(\tilde{K})$ the set of all the divisorial tame inertia elements in $\mathrm{Gal}(\tilde{K}|K)$ in the case K is finitely generated; and in the case $K|k$ is a function field, we denote by $\mathfrak{In}(\tilde{K}|k)$, and $\mathfrak{In.tm}(\tilde{K}|k)$, the set of all the inertia, respectively tame inertia, elements at all the valuations of $\tilde{K}$ which are trivial on k; and by $\mathfrak{In.tm.div}(\tilde{K}|k)$ the set of divisorial inertia elements in $\mathrm{Gal}(\tilde{K}|K)$. The second result we announce is the following:

Theorem B. *Let K be either a finitely generated field or a function field over some base field k. Then in the above notations the following hold:*

1) *If K is finitely generated, then the set of all the divisorial inertia elements $\mathfrak{In.tm.div}(\tilde{K})$ is dense in $\mathfrak{In.tm}(\tilde{K})$.*

2) *If $K|k$ is a function field, then $\mathfrak{In}(\tilde{K}|k)$, and $\mathfrak{In.tm}(\tilde{K}|k)$ are closed subsets of $\mathrm{Gal}(\tilde{K}|K)$, and $\mathfrak{In.tm.div}(\tilde{K}|k)$ is dense in $\mathfrak{In.tm}(\tilde{K}|k)$.*

1. Proof of Theorem A

First let us recall the basics concerning the patch topology. Let Ω be an arbitrary field, and let $\mathrm{Val}(\Omega)$ be the set of all the valuation rings, thus equivalence classes of valuations, or of places, of Ω. One defines the `Zariski topology` on $\mathrm{Val}(\Omega)$ as being the topology τ^{Zar} which has as a basis all the sets U_A with $A \subset \Omega^\times$ finite and U_A defined by

$$U_A := \{v \in \mathrm{Val}(\Omega) \mid v(a) \le 0,\, a \in A\} = \{v \in \mathrm{Val}(\Omega) \mid v(a') \ge 0,\, 1/a' \in A\}.$$

It is easy to check that the trivial valuation (whose valuation ring is Ω itself) lies in all U_A, thus τ^{Zar} is not a Hausdorff topology. Further, τ^{Zar} is quasi-compact. The constructible, thus Hausdorff, topology generated by τ^{Zar} is called the `patch topology` on $\mathrm{Val}(\Omega)$, which we denote by τ^{pa}. A basis of this topology consists of all the sets of the form $U_{A,B}$ with $A, B \subset \Omega^\times$ finite and

$$U_{A,B} := \{v \in \mathrm{Val}(\Omega) \mid v(a) \le 0, v(b) < 0, a \in A, b \in B\}.$$

One of the basic facts concerning τ^{pa} is that this topology is Hausdorff and compact, and that the basic open subsets $U_{A,B}$ are actually open and closed. Thus $\mathrm{Val}(\Omega)$ endowed with the patch topology is a profinite topological space.

The Zariski topology and the patch topology behave nicely under field extensions as follows: Let $\Omega'|\Omega$ be a field extension. Then the canonical restriction map

$$\mathrm{res} : \mathrm{Val}(\Omega') \to \mathrm{Val}(\Omega), \quad v \mapsto v|_\Omega$$

is surjective (by Chavalley's theorem on the prolongation of places), and continuous in both the Zariski topology and the patch topology. Moreover, if $(\Omega_i)_{i\in J}$ is an inductive family of fields, and $\Omega = \lim_i \Omega_i$, then $\mathrm{Val}(\Omega_i)$, $i \in I$, endowed with the (surjective) restrictions $\mathrm{res}_{ji} : \mathrm{Val}(\Omega_j) \to \mathrm{Val}(\Omega_i)$, $i \le j$, is a projective system, and $\mathrm{Val}(\Omega)$ is in a canonical way the projective limit of this projective system.

Second, let G be a profinite group. Then the set of all the closed subgroups $\mathrm{Sbg}(G)$ of G carries in a canonical way the so called `étale topology` τ^{et}, which in some sense is similar to the Zariski topology on $\mathrm{Val}(\Omega)$. A basis of open subsets of τ^{et} is given by all the sets $U^{\mathrm{et}}_{G_1}$ with $G_1 \subseteq G$ is open and

$$U^{\mathrm{et}}_{G_1} := \{\Gamma \in \mathrm{Sbg}(G) \mid \Gamma \subseteq G_1\}.$$

Clearly, τ^{et} is quasi-compact and non-Hausdorff. The constructible topology on $\mathrm{Sbg}(G)$ is called the strict topology τ^{st}, and a basis of open subsets of this topology is given by all the sets $U^{\mathrm{st}}_{G_1,N}$ with $G_1, N \subseteq G$ open, N normal, and

$$U^{\mathrm{st}}_{G_1,N} := \{\Gamma \in \mathrm{Sbg}(G) \mid \Gamma N = G_1\}.$$

As above, it follows that τ^{st} is Hausdorff and compact, and that $U^{\mathrm{st}}_{G_1,N}$ are open and closed subsets of $\mathrm{Sbg}(G)$.

A special case of the above situation is when we consider a Galois extension of fields $\tilde{K}|K$, for which we fix notations as follows: Let $(K_i|K)_i$ be the family of all the finite Galois sub-extensions of $\tilde{K}|K$ inductively ordered by inclusion. For $K_i \subseteq K_j$, i.e., $i \le j$, we denote:

1) $\mathrm{pr}_i : \mathrm{Gal}(\tilde{K}|K) \to \mathrm{Gal}(K_i|K)$ and $\mathrm{pr}_{ji} : \mathrm{Gal}(K_j|K) \to \mathrm{Gal}(K_i|K)$ the canonical surjective projections.

2) $\mathrm{res}_i : \mathrm{Val}(\tilde{K}) \to \mathrm{Val}(K_i)$ and $\mathrm{res}_{ji} : \mathrm{Val}(K_j) \to \mathrm{Val}(K_i)$ the canonical surjective restriction maps.

For $\tilde{v} \in \mathrm{Val}(\tilde{K})$, let $V_{\tilde{v}} \subseteq T_{\tilde{v}} \subseteq Z_{\tilde{v}}$ be the ramification, respectively the inertia, respectively the decomposition, groups of $\tilde{v}$. Further, let $v_i := \tilde{v}|_{K_i}$ be the restriction of $\tilde{v}$ to K_i, and further let $V_{v_i} \subseteq T_{v_i} \subseteq Z_{v_i}$ be correspondingly defined. Then by Hilbert decomposition theory (for valuations), it follows that pr_i maps $V_{\tilde{v}} \subseteq T_{\tilde{v}} \subseteq Z_{\tilde{v}}$ onto $V_{v_i} \subseteq T_{v_i} \subseteq Z_{v_i}$. We conclude that there exist canonical maps

$$\psi^V, \psi^T, \psi^Z : \mathrm{Val}(\tilde{K}) \to \mathrm{Sbg}(\mathrm{Gal}(\tilde{K}|K))$$

defined by $\psi^V(\tilde{v}) := V_{\tilde{v}}$, $\psi^T(\tilde{v}) := T_{\tilde{v}}$, $\psi^Z(\tilde{v}) := Z_{\tilde{v}}$, and correspondingly for $K_i|K$, such that each pair of such maps fit into a commutative diagram

$$\begin{array}{ccc} \mathrm{Val}(\tilde{K}) & \xrightarrow{\psi^\bullet} & \mathrm{Sbg}(\mathrm{Gal}(\tilde{K}|K)) \\ \big\downarrow{\scriptstyle \mathrm{res}_i} & & \big\downarrow{\scriptstyle \mathrm{pr}_i} \\ \mathrm{Val}(K_i) & \xrightarrow{\psi_i^\bullet} & \mathrm{Sbg}(\mathrm{Gal}(K_i|K)) \end{array}$$

where $\bullet$ is any of the letters V, T, Z respectively. After this preparation we can announce the following:

THEOREM 1.1. *Let $\tilde{K}|K$ be a Galois field extension. Then in the above notations, the following hold:*

1) *The maps $\psi^V, \psi^T : \mathrm{Val}(\tilde{K}) \to \mathrm{Sbg}(\mathrm{Gal}(\tilde{K}|K))$ defined by $\psi^V(\tilde{v}) := V_{\tilde{v}}$ and $\psi^T(\tilde{v}) := T_{\tilde{v}}$ are continuous if we endow $\mathrm{Val}(\tilde{K})$ with the patch topology τ^{pa} and $\mathrm{Sbg}(\mathrm{Gal}(\tilde{K}|K))$ with the étale topology τ^{et}.*

2) *Let $\mathcal{V} \subseteq \mathrm{Val}(\tilde{K})$ be a τ^{pa}-closed subset. Then the sets $\mathfrak{Rm}_{\mathcal{V}}(\tilde{K})$, $\mathfrak{In}_{\mathcal{V}}(\tilde{K})$, and $\mathfrak{In}.\mathfrak{tm}_{\mathcal{V}}(\tilde{K})$ of all the ramification, respectively inertia, respectively tame ramification, elements at valuations $v \in \mathcal{V}$ are closed in $\mathrm{Gal}(\tilde{K}|K)$.*

3) *More precisely, in the situation above, let $\Delta \subseteq \mathrm{Gal}(\tilde{K}|K)$ be a closed subgroup such that for every $K_i|K$, there exists a valuation $v_i \in \mathrm{res}_i(\mathcal{V})$ such that* i) *$pr_i(\Delta) \subseteq V_{v_i}$, respectively* ii) *$pr_i(\Delta) \subseteq T_{v_i}$, respectively* iii) *$pr_i(\Delta) \subseteq T_{v_i} \backslash V_{v_i}$. Then there exists a valuation $\tilde{v} \in \mathcal{V}$ such that* i) *$\Delta \subseteq V_{\tilde{v}}$, respectively* ii) *$\Delta \subseteq T_{\tilde{v}}$, respectively* iii) *$\Delta \subseteq T_{\tilde{v}} \backslash V_{\tilde{v}}$.*

Proof. To 1): By the discussion before the Theorem, without loss of generality we can suppose that $\tilde{K}|K$ is finite. If so, then $\mathrm{Sbg}(\mathrm{Gal}(\tilde{K}|K))$ consists of all the subgroups of $\mathrm{Gal}(\tilde{K}|K)$. Further, one checks immediately that the sets of the form $B_\Delta := \{\Gamma \in \mathrm{Sbg}(\mathrm{Gal}(\tilde{K}|K)) \mid \Delta \subseteq \Gamma\}$, all $\Delta \subseteq \mathrm{Gal}(\tilde{K}|K)$, represent a basis for the τ^{et}-closed subsets in $\mathrm{Sbg}(\mathrm{Gal}(\tilde{K}|K))$. (Indeed: First, the complement of B_Δ is the union of all the basic open subsets U_{G_1} with $\Delta \nsubseteq G_1$, hence an τ^{et} open set. Second, the basic closed set which is the complement of U_{G_1} is exactly the union of all the subsets B_Δ with Δ all the subgroups $\Delta \not\subseteq G_1$.) Our strategy to prove that ψ^T and ψ^V are continuous, is to show that the preimages of τ^{et}-closed subsets of the form B_Δ are τ^{pa}-closed.

First let us show that ψ^T is continuous. For $g \in \mathrm{Gal}(\tilde K|K)$ and $x \in \tilde K$ let us consider

$$\mathcal{U}_{g,x} := \{\tilde v \in \mathrm{Val}\big(\tilde K\big) \mid \tilde v(x) \ge 0,\ \tilde v(gx - x) \le 0\}.$$

Then $\mathcal{U}_{g,x}$ is an τ^{pa}-open set, and if $\tilde v \in \mathcal{U}_{g,x}$, then $g \notin T_{\tilde v}$, by the definition of $T_{\tilde v}$. Therefore, $\mathcal{U}_g := \cup_x \mathcal{U}_{g,x}$ is τ^{pa}-open too, and its complement $\mathcal{V}_g$ in $\mathrm{Val}\big(\tilde K\big)$ is therefore closed. We now claim the following:

Claim. $\tilde v \in \mathcal{V}_g$ if and only if $g \in T_{\tilde v}$.

Indeed, if $\tilde v \in \mathcal{V}_g$, then for all $x \in K$ such that $\tilde v(x) \ge 0$, we must have $v(gx - x) > 0$. Equivalently, $g \in T_{\tilde v}$. The converse implication is obvious.

Now let $\Delta \subseteq \mathrm{Gal}(\tilde K|K)$ be a (closed) subgroup. Then $\mathcal{V}_\Delta := \cap_{g\in\Delta}\mathcal{V}_g$ is τ^{pa}-closed too, and by the Claim above one has: $v \in \mathcal{V}_\Delta$ if and only if $g \in T_{\tilde v}$ for all $g \in \Delta$; hence if and only if $\Delta \subset T_{\tilde v}$. Equivalently, $\mathcal{V}_\Delta$ is the preimage of B_Δ under ψ^T. We conclude that Ψ^T is continuous.

The continuity of ψ^V is proved in a similar way, but starting with τ^{pa}-open sets of the form

$$\mathcal{U}_{g,x} := \{\tilde v \in \mathrm{Val}\big(\tilde K\big) \mid \tilde v(x) \ge 0,\ \tilde v(gx - x) \le \tilde v(x)\}.$$

and the resulting claim that if $\mathcal{V}_g$ is the complement of $U_{g,x}$, then $\tilde v \in \mathcal{V}_g$ if and only if $g \in V_{\tilde v}$, etc.

To 2) and 3): It is clear that the closedness of the sets $\mathfrak{Rm}_{\mathcal{V}}\big(\tilde K\big)$, $\mathfrak{In}_{\mathcal{V}}\big(\tilde K\big)$, and $\mathfrak{In}.\mathfrak{tm}_{\mathcal{V}}\big(\tilde K\big)$ immediately follows from assertion 3). Thus we are left with proving assertion 3). We give the proof only in the case i), as the cases ii) and iii) are *mutatis mutandis* identical. Thus suppose that for every $K_i|K$ there exists $v_i \in \mathrm{res}_i(\mathcal{V})$ such that $\mathrm{pr}_i(\Delta) \subseteq T_{v_i}$. Hence in the notations from the proof of assertion 1), and taking into account the continuity of

$$\mathrm{pr}_i \circ \psi^T = \mathrm{res}_i \circ \psi^T_i : \mathcal{V} \to \mathrm{Sbg}\big(\mathrm{Gal}(K_i|K)\big),$$

it follows that the preimage $\mathcal{V}_i \subseteq \mathcal{V}$ of $\mathcal{V}_{\mathrm{pr}_i(\Delta)}$ under the continuous map above is closed and non-empty, by hypothesis. Thus $(\mathcal{V}_i)_i$ is a family of compact subsets of $\mathcal{V}$ which has the finite intersection property (as $\mathcal{V}_j \subseteq \mathcal{V}_i$ for $K_i \subseteq K_j$). Now take $\tilde v \in \cap_i \mathcal{V}_i$, and set $v_i := \tilde v|_{K_i}$. Then, by general Hilbert decomposition theory (for valuations), we have $T_{v_i} = \mathrm{pr}_i(T_{\tilde v})$, hence from $\mathrm{pr}_i(\Delta) \subseteq T_{v_i}$ we get $\mathrm{pr}_i(\Delta) \subseteq \mathrm{pr}_i(T_{\tilde v})$. This being true for all K_i, we finally have $\Delta \subseteq T_{\tilde v}$, as claimed. $\square$

2. Proof of Theorem B

First we remark that the assertion that $\mathfrak{In}(K|k)$ and $\mathfrak{In.tm}(K|k)$ are closed can be deduced from Theorem 1.1 above as follows: Let $\mathcal{V}$ be the set of all the valuations of $\tilde{K}$ which are trivial on k. Then $\tilde{v} \in \mathcal{V}$ if and only if $\forall\, x \in k^\times$ one has $v(x) = 0$. Hence $\mathcal{V} \subset \mathrm{Val}(\tilde{K})$ is the intersection of the closed and open basic subsets $U_{\{x\}}$, $x \in k^\times$, thus τ^{pa}-closed, etc.

Now let us prove that $\mathfrak{In.tm.div}(K)$ is dense in $\mathfrak{In.tm}(K)$, respectively that $\mathfrak{In.tm.div}(K|k)$ is dense in $\mathfrak{In.tm}(K|k)$. The proofs are *mutatis mutandis* the same, therefore we will make the proofs at the same time.

We first remark that it is sufficient to consider the case where $\tilde{K} = K^{\mathrm{s}}$ is the separable closure of K. Indeed, this follows immediately from the fact that for Galois field extensions $K \hookrightarrow L \hookrightarrow M$, and every valuation $v_M \in \mathrm{Val}(M)$ and its restriction v_L to L one has the following: The canonical projection $\mathrm{pr} : \mathrm{Gal}(M|K) \to \mathrm{Gal}(L|K)$ maps T_{v_M} onto T_{v_L}, and V_{v_M} onto V_{v_L}, etc.

Therefore, without loss of generality, we will suppose that $\tilde{K} = K^{\mathrm{s}}$, hence $\mathrm{Gal}(\tilde{K}|K) = G_K$ is the absolute Galois group of K, and we will denote the valuation $\tilde{v}$ by v.

We introduce notations which will be used throughout the proof as follows:

• $\sigma \in G_K$ is a fixed tame inertia element, and $\Sigma \subseteq G_K$ is the pro-cyclic closed subgroup of G_K generated by σ, and $|\Sigma| = |\sigma|$ denotes the order of Σ and of σ as a super natural number.

• $L|K$ is the fixed field of σ, hence of Σ, in $K^{\mathrm{s}}|K$. And for every finite Galois sub-extension $K_i|K$ of $K^{\mathrm{s}}|K$ we will denote by $L_i := L \cap K_i$ the fixed field of σ in K_i.

• $G_i := \mathrm{Gal}(K_i|L_i)$ is the cyclic group generated by $\sigma|_{K_i}$ in $\mathrm{Gal}(K_i|K)$. In particular, $\Sigma = G_L$ projects onto $G_i = \mathrm{Gal}(K_i|L_i)$, and so G_i is a finite quotient $\mathrm{pr}_i : \Sigma \to G_i$ of Σ.

Remarks 2.1. In the notations from above, we make the following more or less obvious remarks:

1) L and K_i are linearly disjoint over L_i, hence $[LK_i : L] =: n_i := [K_i : L_i]$, and the canonical projection below is an isomorphism:

$$\mathrm{Gal}(LK_i|L) \to \mathrm{Gal}(K_i|L_i) = \Sigma_i.$$

2) The assertion of Theorem B for σ is actually the following:

($*$) *For every finite Galois extension $K_i|K$, there exists some prime divisor $\mathfrak{v}_i$ of K_i such that $\sigma|_{K_i}$ is a tame inertia at $\mathfrak{v}_i$.*

3) Let $M|K$ be some finite field extension. Then $M^{\mathrm{s}} := MK^{\mathrm{s}}$ is a separable closure of M, and G_K contains G_M as an open subgroup in a canonical

way. Since σ generates the Galois group of G_L, we have: The compositum ML inside M^{s} is purely inseparable over L iff σ (viewed as element of G_M) fixes ML point-wise iff σ (viewed as element of G_M) acts trivially on M.

For finite extensions $M|K$ such that ML is purely inseparable over L, thus σ acts trivially on M, we consider finite cyclic extensions $N|M$ such that $\mathrm{Gal}(N|M) =: G$ has $\sigma_N := \sigma|_N$ as a generator. Then by the functoriality of Hilbert decomposition for valuations, we immediately deduce that the assertion $(*)$ above for a given $K_i|K$ follows from the following:

($\dagger$) *There exists some finite field extension $M|K$, and a finite cyclic extension $N|M$ satisfying the following:*
  i) *$ML\,|\,L$ is purely inseparable, and $K_i \subset NL$.*
  ii) *$\sigma_N := \sigma|_N$ is a tame inertia element at some prime divisor $\mathfrak{v}$ of N.*

4) Thus in order to prove assertion $(*)$ for some finite Galois sub-extension $K_i|K$ of $K^{\mathrm{s}}|K$, we will show that assertion ($\dagger$) above is satisfied for properly chosen finite extensions $M|K$ and finite cyclic extensions $N|M$ as above.

5) Suppose that σ is a non-trivial tame inertia element at the valuation v as above. Note that v is therefore non-trivial, and v is totally tamely ramified in $K^{\mathrm{s}}|L$. Hence v is also totally tamely ramified in the finite cyclic extensions $LK_i|L$ and $K_i|L_i$ too.

6) Since v is totally tamely ramified in $K^{\mathrm{s}}|L$, it follows that L contains the roots of unity $\mu_{|\Sigma|}$ of order $|\Sigma|$.

Step 1. *Getting started*

Let $K_i|K$ be a finite Galois sub-extension of $K^{\mathrm{s}}|K$, and let X be a proper model of K. By one of the main results of de Jong's alteration theory [dJ], Theorem 5.13, it follows that there exists an alteration $f : Y \to X$ of X such that the function field $N = \kappa(Y)$ of Y is a normal Galois extension of K, and the following are satisfied:

a) The group $\mathrm{Aut}(N|K)$ is contained in $\mathrm{Aut}(Y)$, and $\mathrm{Aut}(N|K)$ projects onto $\mathrm{Gal}(K_i|K)$ via the alteration $f : Y \to X$.

b) Y is regular.

Let $K'|K$ be the pure inseparable part of $N|K$. Then $K'^{\mathrm{s}} := K'K^{\mathrm{s}}$ is a separable closure of K', and one has a canonical identification $G_K = G_{K'}$, under which $L' := LK'$ is the fixed field of σ in K'^{s}. Further, v has a unique prolongation w to K'^{s}, and the groups $V_v \subseteq T_v$ are identified with $V_w \subseteq T_w$. Hence σ is a tame inertia element at w.

Let $M := N \cap L'$ be the fixed field of σ in N. Then $N|M$ is a cyclic extension with Galois group $G := <\sigma|_N>$, and $ML = K'L$ is purely inseparable over L, and $K_i \subset N$. Thus $N|M$ satisfies the condition i) from assertion (†) of Remark 2.1, 3), above. Therefore, in order to prove assertion $(*)$ for $K_i|K$, it is sufficient to prove that $N|M$ satisfies condition ii) from assertion (†) of Remark 2.1, 3), above.

Recall that w is totally tamely ramified in the finite cyclic extension $N|M$. Hence for all $g \in \Sigma$ one has: $g\mathcal{O}_w = \mathcal{O}_w$ and $g\,c - c \in \mathfrak{m}_w$ for all $c \in \mathcal{O}_w$. Since Y is proper, it follows by the valuation criterion of properness, that there exists a unique local ring $(\mathcal{O}, \mathfrak{m})$ of Y such that $(\mathcal{O}_w, \mathfrak{m}_w)$ dominates $(\mathcal{O}, \mathfrak{m})$. From the uniqueness of $(\mathcal{O}, \mathfrak{m})$, and by the discussion above, we get the following:

Fact 2.2. *N contains regular local rings $(\mathcal{O}, \mathfrak{m})$ such that $(\mathcal{O}, \mathfrak{m})$ is dominated by $(\mathcal{O}_w, \mathfrak{m}_w)$ and $N = \mathrm{Quot}(\mathcal{O})$. And for every such ring $(\mathcal{O}, \mathfrak{m})$ the following hold:*

1) *$G = \mathrm{Gal}(N|M) = \,<\sigma|_N>$ acts faithfully on $\mathcal{O}$, in particular, every $g \in G$ maps $\mathfrak{m}$ isomorphically onto itself.*

2) *For all $g \in G$ and all $c \in \mathcal{O}$ one has $g\,c - c \in \mathfrak{m}$. This means that the action of G on $\mathcal{O}$ is totally ramified.*

3) *The residue field $\kappa := \mathcal{O}/\mathfrak{m}$ is canonically embedable into the residue field $Nw := \mathcal{O}_w/\mathfrak{m}_w$ of w on N. In particular, $\mathrm{char}(\kappa) \neq \ell$, and G acts trivially on κ via the canonical projection $\mathcal{O} \to \kappa$.*

We will conclude the proof of Theorem B by using the fact that for a properly chosen `cyclotomic alteration` followed by a sequence of `local modifications` of the local rings $(\mathcal{O}, \mathfrak{m})$ from the Fact 2.2 above, one can reach a situation where *mutatis mutandis* the action of G on a properly chosen regular system of local parameters $(t_1, \dots, t_d)$ of $\mathcal{O}$, has very simple shape, namely:

- $g(t_k) = t_k$ for $k < d$.
- $g(t_k) = \zeta t_k$ for some primitive root of unity $\zeta \in \mu_{|G|}$.

If so, then the prime divisor $\mathfrak{v}$ of N defined by the t_d-adic valuation is totally tamely ramified in $N|M$, hence satisfies assertion (†).

Step 2. *Maximizing the decomposition groups*

Actually, the only alteration of the local rings as introduced in Fact 2.2 above, which is not a sequence of blowups, is the following `cyclotomic alteration`:

Lemma 2.3. *In the context and notations from Fact 2.2 above, denote $m := |G|$, and consider the cyclotomic extension $N_1 := N[\mu_m]$. Then letting $\mathcal{O} \hookrightarrow \mathcal{O}^{\mathrm{n}}$ be the normalization of $\mathcal{O}$ in the finite field extension $N_1|N$, the following hold:*

1) *The field* $M_1 := L' \cap N_1$ *is actually* $M_1 = M[\mu_m]$. *Hence the canonical restriction homomorphism* $\mathrm{Gal}(N_1|M_1) \to \mathrm{Gal}(N|M)$ *is an isomorphism, and* $N_1|M_1$ *satisfies condition* i) *from assertion* (†) *of Remark 2.1.*

2) *Let* $(\mathcal{O}_1, \mathfrak{m}_1)$ *be the unique localization of* $\mathcal{O}^{\mathrm{n}}$ *dominated by* $(\mathcal{O}_w, \mathfrak{m}_w)$. *Then* $(\mathcal{O}_1, \mathfrak{m}_1)$ *is a regular local ring, and* $N_1|M_1$ *endowed with* $(\mathcal{O}_1, \mathfrak{m}_1)$ *satisfy conditions* 1), 2), 3) *of Fact 2.2.*

Proof. The first assertion 1), follows from the fact that $\mu_m = \mu_{|\sigma_N|}$ are contained in L, by Remark 2.1, 6); hence we have $\mu_m \subset L' \cap N_1 =: M_1$.

To 2): Since $\mathrm{char}(\kappa_x) \neq \ell$, it follows that the ring extension $\mathcal{O}_y \hookrightarrow \mathcal{O}^{\mathrm{n}}$ is an étale ring extension. Hence $\mathcal{O}^{\mathrm{n}}$ is a semi-local regular ring, as being an étale Galois cover of the local regular ring $\mathcal{O}_y$. Finally, the valuation ring of v dominates one of the localizations of $\mathcal{O}^{\mathrm{n}}$, etc. □

LEMMA 2.4. *In the context of Fact 2.2 above, set* $m := |G|$, *and suppose that* $\mu_m \subset N$, *hence* $\mu_m \subset M = N \cap L$. *Then for a properly chosen regular system of parameters* $(t_1, \ldots, t_d)$ *of* $\mathcal{O}$, *the action of* G *on* $(t_1, \ldots, t_d)$ *is given by a system of characters* $\chi_k : G \to \mu_m$, $1 \leq k \leq d$, *of* G *as follows:*

$$g t_k = \chi_k(g) t_k, \quad g \in G,\ 1 \leq k \leq d.$$

Proof. First, consider $V := \mathfrak{m}/\mathfrak{m}^2$ as κ-vector space. Since G acts on $\mathcal{O}$ and maps $\mathfrak{m}$ isomorphically onto itself, it follows that G acts on V too. On the other hand, since by condition 3) of Fact 2.2, $\mathrm{char}(\kappa)$ does not divide $|G|$, the action of G on V is semi-simple. Recall that $G = <\sigma_N>$ is cyclic of order $|G| = m$, and $\mu_m \subset \mathcal{O}$, hence $\mu_m \subset \kappa$. Therefore, the minimal polynomial $P_{\sigma_N}(X) = X^m - 1$ of σ_N splits in linear factors over κ. This finally implies that the action of G on V is diagonalizable, i.e., there exist characters

$$\chi_k : G \to \mu_m, \quad 1 \leq k \leq d$$

and a κ-basis $(\overline{u}_1, \ldots, \overline{u}_d)$ of $V = \mathfrak{m}/\mathfrak{m}^2$ such that denoting by I_χ the diagonal matrix whose diagonal entries are the characters $\chi_1, \ldots, \chi_d$, one has:

$$g\,(\overline{u}_1, \ldots, \overline{u}_d) = (\overline{u}_1, \ldots, \overline{u}_d) \cdot I_\chi(g), \quad g \in G.$$

Now let $\underline{u} := (u_1, \ldots, u_d)$ be a preimage of $(\overline{u}_1, \ldots, \overline{u}_d)$ in $\mathcal{O}$. Then $\underline{u}$ is a regular system of local parameters of $\mathcal{O}$, and by the discussion above we have: For every $g \in G$ there exists some $\underline{u}'_g = (u'_{g1}, \ldots, u'_{gd})$ with $u'_{gk} \in \mathfrak{m}^2$ for all k such that:

$$g\,\underline{u} = \underline{u} \cdot I_\chi(g) + \underline{u}'_g, \quad g \in G.$$

We proceed by considering the I_χ-twisted G-action on the d-fold product $(N,+)^d := (N,+) \times \ldots \times (N,+)$ of the additive group of N, which is defined by $\widetilde{g}\,\underline{a} := g\,\underline{a}\,I_\chi(g^{-1})$, where $\underline{a} = (a_1, \ldots, a_d) \in (N,+)^d$ is arbitrary, and

$g(a_1, \dots, a_d) := (ga_1, \dots, ga_d)$ is the diagonal action of G on $(N,+)^d$. Then using the identity above, and setting $\underline{a}_g := \underline{u}' I_\chi(g^{-1})$ for all $g \in G$, we get:

$$\underline{a}_g = \widetilde{g}\,\underline{u} - \underline{u}, \quad g \in G.$$

In other words, $\{\underline{a}_g\}_g$ is a 1-cocycle of the twisted action of G with values in the additive subgroup $\mathfrak{m}^2 \times \dots \times \mathfrak{m}^2$ of $(N,+)^d$. Since $m := |G|$ is invertible in $\mathcal{O}$, it follows that the cocycle $\underline{a}_g$ is trivial. Actually, setting $\underline{a} := -\frac{1}{|G|}\sum_g \underline{a}_g$, we have $\underline{a} \in \mathfrak{m}^2 \times \dots \times \mathfrak{m}^2$ and $\underline{a}_g = \widetilde{g}\,\underline{a} - \underline{a}$. Therefore, setting $\underline{t} := \underline{u} - \underline{a}$, we have $\widetilde{g}\,\underline{t} = \underline{t}$. Equivalently,

$$g\,\underline{t} = \underline{t} \cdot I_\chi(g), \quad g \in G,$$

and note that $\underline{t} = (t_1, \dots, t_d)$ is a regular system of local parameters of $\mathcal{O}$, as $\underline{u} = (u_1, \dots, u_d)$ was so, and $\underline{a} = (a_1, \dots, a_d) \in \mathfrak{m}^2 \times \dots \times \mathfrak{m}^2$. □

Step 3. *Maximizing an inertia group*

Let $(t_1, \dots, t_d)$ be a system of regular local parameters of $\mathcal{O}$ as in the Lemma 2.6. We define a `local modification` of $\mathcal{O}$ to be a local regular ring in N obtained in two steps as follows:

1) First consider a blowup $\mathcal{Z} \to \operatorname{Spec}\mathcal{O}$ at any prime ideal of the form $\mathfrak{p}_{kl} = (t_k, t_l)$. Note that the zero set $V(\mathfrak{p}_{kl})$ is regular in $\operatorname{Spec}\mathcal{O}$, hence $\mathcal{Z}$ is regular, and the preimage of the closed point of $\mathcal{O}$ is the t_{kl}-projective line over κ, say with $t_{kl} := t_l/t_k$.

2) Second, replace $(\mathcal{O}, \mathfrak{m})$ by the local ring $(\mathcal{O}_{kl}, \mathfrak{m}_{kl})$ of $\mathcal{Z}$ defined by the zero $(t_{kl} = 0)$ of t_{kl}.

Remarks 2.5. Let $(\mathcal{O}', \mathfrak{m}') := (\mathcal{O}_{kl}, \mathfrak{m}_{kl})$ be a local modification of $(\mathcal{O}, \mathfrak{m})$ as above. Then the following hold:

1) The regular local ring $(\mathcal{O}_{kl}, \mathfrak{m}_{kl})$ is the localization of $\mathcal{O}[t_l/t_k]$ at its maximal ideal generated by $\mathfrak{m}$ and t_l/t_k, thus having $(t'_1, \dots, t'_d)$ with $t'_j = t_j$ for $j \neq l$, and $t'_l := t_l/t_k$, as a local system of regular parameters.

2) The regular local ring $(\mathcal{O}', \mathfrak{m}')$ dominates $(\mathcal{O}, \mathfrak{m})$, and the resulting embedding of residue fields $\kappa \hookrightarrow \kappa'$ is an isomorphism.

3) G acts on $(\mathcal{O}', \mathfrak{m}')$, as G maps both $\mathcal{O}[t_l/t_k]$, and its maximal ideal generated by $\mathfrak{m}$ and t_l/t_k, isomorphically onto themselves.

4) In particular, G acts trivially on κ', and therefore, G equals the inertia group of the action of G on $\mathcal{O}'$.

5) And the action of G on each t'_j is given by a character χ'_j of G such that $\chi'_j = \chi_j$ for $j \neq l$, and $\chi'_l = \chi_l \chi_k^{-1}$.

Lemma 2.6. *Let $N|M$ endowed with $(\mathcal{O}, \mathfrak{m})$ satisfy the conclusion of Lemma 2.4, There exists a character $\chi : G \to \mu_m$, and a finite sequence of local* *blowups as defined above such that the resulting regular local ring $(\mathcal{O}', \mathfrak{m}')$ has a regular system of local parameters $(t'_1, \dots, t'_d)$ on which the action of every $g \in G$ is given by:*

$$g(t'_k) = t'_k \quad \text{for } k < d, \quad \text{and} \quad g(t'_d) = \chi(g) t'_d.$$

Proof. Let $\chi_1, \dots, \chi_d$ be the characters of G defining the action of G on the given system of regular parameters $(t_1, \dots, t_d)$ of $(\mathcal{O}, \mathfrak{m})$. We fix some primitive character $\chi_0 : G \to \mu_m$. Then for every $k = 1, \dots, d$ there exists positive integers $e_k \le m$ such that $\chi_k = \chi_0^{e_k}$. We make induction on the number $n > 0$ of non-trivial characters $\chi_k \neq 1$.

If $n = 1$, then after renumbering the parameters $(t_1, \dots, u_d)$, without loss of generality we can suppose that $\chi := \chi_d$ is the unique non-trivial one, etc.

Now suppose that there exist at least two non-trivial characters χ_k, χ_l. As above, after renumbering the parameters, we can suppose that $\chi_1 = \chi^{e_1}$ and $\chi_2 = \chi^{e_2}$ are non-trivial.

Claim. Let $e := \text{g.c.d.}(e_1, e_2)$. There exists a sequence of local blowups of $\mathcal{O}$ such that the resulting $(\mathcal{O}', \mathfrak{m}')$ has a system of parameters $(t'_1, \dots, t'_d)$ satisfying the following condition:

i) $t'_k = t_k$, hence G acts by χ_k on t_k for $k > 2$.

ii) $g(t'_1) = t'_1$ for all $g \in G$.

ii) $g(t'_2) = \chi_0^e(g)\, t'_2$ for all $g \in G$.

Indeed, we make induction on $\tilde{e} := \max(e_1, e_2)$. Without loss of generality, after renumbering the parameters t_k, we can suppose that $e' = e_2 \ge e_1$.

Performing the local modification $(\mathcal{O}_{12}, \mathfrak{m}_{12})$, it follows by Remark 2.5, especially 5), that the resulting system of regular local parameters $(t'_1, \dots, t'_d)$ satisfies: $t'_k = t_k$ for $k \neq 2$, and $t'_2 = t_2/t_1$. Hence the action of G on t'_k is given by: $\chi'_k = \chi_k = \chi_0^{e_k}$, for $j \neq 2$; and $\chi'_2 = \chi_2 \chi_1^{-1} = \chi_0^{e_2 - e_1}$. Hence we can choose the exponents e'_k such that $\chi'_k = \chi_0^{e'_k}$ and they satisfy:

i) $e'_k = e_k$, for $k \neq 2$.

ii) $e'_2 = e_2 - e_1 < e_2$, and $e'_2 \ge 0$ by the hypothesis $e_2 \ge e_1$.

Hence we have both: First $\tilde{e}' := \max(e'_1, e'_2) > \max(e_1, e_2) = \tilde{e}$, and second, $\text{g.c.d.}(e_1, e_2) = \text{g.c.d.}(e'_1, e'_2)$. Hence we can conclude the proof of the Claim by induction, and this finally concludes the proof of Lemma 2.6 too. $\square$

Step 4. *Concluding the proof of Theorem B*

Let $(\mathcal{O}, \mathfrak{m})$ satisfying the conclusion of Lemma 2.6, and let $(t_1, \dots, t_d)$ be a system of local regular parameters of $\mathcal{O}$ such that G acts on t_k trivially for $k < d$, and acts on t_d via a character $\chi : G \to \mu_m$.

We first remark that since G acts faithfully on $N = \mathrm{Quot}(\mathcal{O})$, and its action on N factors through χ, it follows that the character χ must be a primitive character of G.

Second, we claim that G is contained in the inertia group of the principal ideal $\mathfrak{p} := t_d\mathcal{O}$. Indeed, the residue ring $\mathfrak{O} := \mathcal{O}/\mathfrak{p}$ is a regular local ring with maximal ideal $\mathfrak{n} = \mathfrak{m}/\mathfrak{p}$, having $(\bar{t}_1, \dots, \bar{t}_{d-1})$ as a regular system of local parameters, where $\bar{t}_k := t_k \pmod{\mathfrak{p}}$ for $k < d$, and $\mathfrak{O}/\mathfrak{n} = \mathcal{O}/\mathfrak{m} = \kappa$ as residue field. Since G acts trivially on t_k for $k < d$, it follows that G acts trivially on the local parameters $\bar{t}_k$ for $k = 1, \dots, d-1$. Since G acts trivially on the residue field $\kappa = \mathfrak{O}/\mathfrak{n}$ too, it follows that G acts trivially on $\mathfrak{O}$. Hence $\mathfrak{p}$ is totally ramified in $N|M$.

Therefore, the prime divisor $\mathfrak{v}$ defined by the t_d-adic valuation of N is totally tamely ramified in $N|M$.

This concludes the proof of Theorem B.

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